On a spin conformal invariant on open manifolds

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Introduction

In this thesis we want to study a conformal invariant on open Riemannian spin manifolds (M, g, σ).
Conformal invariants are quantities that are independent of the metric itself but only depend on its conformal class $[g] = \{f^2 g \mid 0 < f \in C^\infty(M)\}$.
The most prominent example of a conformal invariant of a Riemannian manifold (M^n, g) with dimension $n \geq 3$ is the Yamabe invariant Q, which is also known as Sobolev quotient. It is based on the conformal Laplacian

$$L_g = 4\frac{n-1}{n-2}\Delta_g + s_g$$

where Δ_g is the Laplacian and s_g the scalar curvature. With that operator the Yamabe invariant is defined as

$$Q(M, g) = \inf \left\{ \int_M v L_g v \mathrm{dvol}_g \;\middle|\; v \in C_c^\infty(M, \mathbb{R}),\ \|v\|_{L^p} = 1 \right\}$$

where $C_c^\infty(M, \mathbb{R})$ are the compactly supported smooth real-valued functions on M and $p = \frac{2n}{n-2}$.

This invariant was introduced by Yamabe in order to examine whether for a closed Riemannian manifold (M, g) there always exists a metric in the conformal class $[g]$ with constant scalar curvature. This can be interpreted as a generalization of the uniformization theorem to higher dimensions. The Yamabe problem is answered positively [46]. Actually, Yamabe's proof had a serious gap which was closed mainly in [11], [42] and [44]. An overview is given in [37] and [13].
On open manifolds the invariant can still be used to give an obstruction to conformal compactification [35].

The subject of this thesis is the study of an invariant that can be interpreted as a spin version of the Yamabe invariant on open manifolds. To this end, we restrict ourselves to Riemannian spin manifolds (M^n, g, σ), and the spin conformal invariant λ^+_{min} is defined using the Dirac operator D_g. The transformation behaviour under conformal changes of this operator is similar to that of the conformal Laplacian. That's why we can define λ^+_{min} similar to Q:

$$\lambda^+_{min}(M, g, \sigma) = \inf \left\{ \frac{\|D_g\phi\|^2_{L^q}}{(D_g\phi, \phi)} \;\middle|\; \phi \in C_c^\infty(M, S),\ (D_g\phi, \phi) > 0 \right\}$$

where $C_c^\infty(M,S)$ are the compactly supported smooth spinors on M and $q = \frac{2n}{n+1}$.

On closed manifolds the λ^+_{min}-invariant was studied e.g. in [3], [4] , [9] and [38]. Many statements concerning the Yamabe invariant have their counterparts for the λ^+_{min}-invariant although the technical details are often more involved since one has to deal with spinors and not with real-valued functions. For example, there is no maximum principle.

In the following we want to state some first result for the λ^+_{min}-invariant on closed manifolds. A more explicit overview will be given in Chapter 2.
In [9, Thm. 1.1] and for dimension $n = 2$ in [28] it is shown that on closed manifolds

$$\lambda^+_{min}(M, g, \sigma) \leq \lambda^+_{min}(S^n, g_{st}, \chi_{st})$$

where S^n is the standard sphere with its unique spin structure.
Furthermore, on closed manifolds

$$\lambda^+_{min}(M, g, \sigma) = \inf_{\overline{g}\in[g]} \lambda^+_1(\overline{g})\mathrm{vol}(M, \overline{g})^{\frac{1}{n}}$$

where $\lambda^+_1(\overline{g})$ is the first positive eigenvalue of the Dirac operator with respect to the conformal metric $\overline{g}$ [1, Prop. 2.6].

We examined the λ^+_{min}-invariant on open manifolds in [29]. This allows to consider open domains of closed manifolds. So we were able to give a new proof for the upper bound of λ^+_{min} in dimension $n \geq 2$ that is valid for both closed and open manifolds, see [29, Thm. 1.2] for $n > 2$ and Chapter 3 in general.
The role of the first positive Dirac eigenvalue is in general occupied by a Rayleigh-type quotient, cf. Lemma 2.0.1. On complete manifolds this turns out to be the infimum of the positive part of the Dirac spectrum, cf. Lemma 4.1.2.

Sometimes, e.g. in [7], the λ^+_{min}-invariant on closed manifolds is defined such that λ^+_1 is the Dirac eigenvalue with the smallest magnitude, i.e. also negative eigenvalues are considered. Many results are valid for both definitions, cf. Sections 4.5. However, this definition of this λ_{min}-invariant obviously sees the kernel of the Dirac operator, but with our definition we have

$$\lambda^+_{min}(M, g, \sigma) > 0$$

on all closed Riemannian spin manifolds.
We will restrict ourselves to this definition because this gives an easy obstruction to spin conformal compactification, i.e. conformal compactification that preserves the spin structure. To be precise:
If a Riemannian spin manifold has vanishing λ^+_{min}-invariant, it is not spin conformally compactifiable, cf. Remark 3.0.4.

Another obstruction is given by an inequality for the λ^+_{min}-invariant at infinity which generalizes the one of the corresponding Yamabe invariant of [35].

Theorem 3.0.1. *Let (M, g, σ) be a Riemannian spin manifold of dimension $n \geq 2$ with*

$$\lim_{r\to\infty} \lambda^+_{min}(M \setminus B_r(p), g, \sigma) < \lambda^+_{min}(S^n, g_{st}, \chi_{st})$$

for a fixed $p \in M$ and $B_r(p)$ a ball around p of radius r with respect to the metric g. Then (M, g) is not spin conformally compactifiable.

On closed manifolds there is a deep relation between the conformal Laplacian and the Dirac operator – the Hijazi inequality, which gives a lower estimate of the square of the Dirac eigenvalues in terms of the lowest eigenvalue of the conformal Laplacian. This inequality naturally gives rise to a conformal inequality that compares the spin invariant λ^+_{min} with the Yamabe invariant Q.
In this thesis we generalize this inequality to a class of open manifolds. On complete open Riemannian spin manifolds of finite volume we can compare – as in the Hijazi inequality – any eigenvalue (if present) with the infimum of the spectrum of the conformal Laplacian:

Theorem 4.0.5. *Let (M, g, σ) be a complete Riemannian spin manifold of finite volume and dimension $n > 2$. Moreover, let λ be an eigenvalue of its Dirac operator D and let μ be the infimum of the spectrum of the conformal Laplacian. Then we have*

$$\lambda^2 \geq \frac{n}{4(n-1)}\mu.$$

If equality holds, the manifold carries a real Killing spinor and has to be Einstein and closed.

This result enables us to prove the conformal Hijazi inequality on conformally parabolic manifolds.

Theorem 4.0.6. *Let (M, g, σ) be a conformally parabolic Riemannian spin manifold of dimension $n > 2$. If there exists a complete conformal metric $\overline{g}$ of finite volume and $0 \notin \sigma_{ess}(D_{\overline{g}})$, then the conformal Hijazi inequality is valid:*

$$\lambda^+_{min}(M, g, \sigma)^2 \geq \frac{n}{4(n-1)}Q(M, g).$$

We will also give a derivation of this result where we replace the condition that 0 is not contained in the essential spectrum of $D_{\overline{g}}$ by assumptions on the scalar curvature and the dimension of the manifold, cf. Corollary 4.3.4.

To obtain an estimate for λ^+_1 that holds for all metrics we will also generalize the Friedrich inequality to arbitrary open Riemannian spin manifolds, cf. Theorem 4.0.7.

The thesis is structured as follows:
First of all, we will give some preliminaries. Basic facts concerning spin manifold, the Dirac operator and its spectrum are stated. Furthermore, spinor bundles belonging to different metrics are compared.
Then in the second chapter we will define the spin conformal invariant λ^+_{min} and review its known properties.
In the third chapter we will extend some of these results to the case of surfaces, e.g for the obstruction of spin conformal compactification and the upper bound of the λ^+_{min}-invariant. The proofs will then hold for all dimensions $n \geq 2$. An important tool is an Aubin-type inequality on closed manifolds that we also prove in this chapter.
We will provide some examples of open surfaces with vanishing λ^+_{min}-invariant and generalize the Bär inequality to open spin surfaces homeomorphic to $\mathbb{R}^2$.
In the last chapter lower estimates for the λ^+_{min}-invariant on open manifolds will be given. Mainly, we will study the generalization of the Hijazi inequality to open conformally parabolic manifolds.

Chapter 1

Preliminaries

1.1 Spin manifolds and the Dirac operator

In this section we briefly introduce basic notions concerning spin manifolds and the Dirac operator. Details can be found in [27] and [36].

Let (M, g) be a connected and oriented Riemannian manifold of dimension $n \geq 2$ without boundary.
Furthermore, let $P_{\mathrm{SO}(n)}M_g$ be the $\mathrm{SO}(n)$-principal bundle over M of positively oriented orthonormal frames. A *spin structure* σ of (M, g) is a $\mathrm{Spin}(n)$-principal bundle $P_{\mathrm{Spin}(n)}M_g$ over M with a double covering $\eta : \; P_{\mathrm{Spin}(n)}M_g \to P_{\mathrm{SO}(n)}M_g$ such that the diagram

$$\begin{tikzcd}
\mathrm{Spin}(n) \times P_{\mathrm{Spin}(n)}M_g \arrow[r] \arrow[dd, "\Theta\times\eta"] & P_{\mathrm{Spin}(n)}M_g \arrow[dd, "\eta"] \arrow[dr] & \\
& & M \\
\mathrm{SO}(n) \times P_{\mathrm{SO}(n)}M_g \arrow[r] & P_{\mathrm{SO}(n)}M_g \arrow[ur] &
\end{tikzcd}$$

commutes where Θ is the double covering $\mathrm{Spin}(n) \to \mathrm{SO}(n)$ and the horizontal arrows denote the group actions. A Riemannian manifold that admits a spin structure is called a *Riemannian spin manifold.*

Remark 1.1.1.
i) Being spin is a topological property that is independent of the metric: An oriented manifold is spin if and only if its second Stiefel-Whitney class vanishes, see [36, Thm. II.2.1]. This is a global property. Locally a spin structure always exists. Whether these can be glued together to a global one depends on the following question: Let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of M and let $\varphi_{\beta\alpha} : U_\beta \cap U_\alpha \to \mathrm{SO}(n)$ be the transitions maps of $P_{\mathrm{SO}(n)}M_g$. A spin structure exists if these maps lift to

Spin(n)-transition function, i.e if there are maps

$$\widetilde{\varphi}_{\beta\alpha} : U_\beta \cap U_\alpha \to \mathrm{Spin}(n) \quad \text{with } \Theta \circ \widetilde{\varphi}_{\beta\alpha} = \varphi_{\beta\alpha}$$

for $\beta, \alpha \in A$ that fulfill the usual properties of transition functions: $\widetilde{\varphi}_{\alpha\alpha} = \mathrm{id}$ on U_α, $\widetilde{\varphi}_{\alpha\beta}\widetilde{\varphi}_{\beta\alpha} = \mathrm{id}$ on $U_\alpha \cap U_\beta$ and $\widetilde{\varphi}_{\alpha\beta}\widetilde{\varphi}_{\beta\gamma}\widetilde{\varphi}_{\gamma\alpha} = \mathrm{id}$ on $U_\alpha \cap U_\beta \cap U_\gamma$.
When lifting $\varphi_{\alpha\beta}$, the third relation, the cocycle condition, can cause problems since $\widetilde{\varphi}_{\alpha\beta}\widetilde{\varphi}_{\beta\gamma}$ and $\widetilde{\varphi}_{\alpha\gamma}$ may differ by a nontrivial deck transformation of the covering Θ, cf. [33, p. 74].
ii) A spin manifold can allow different spin structures. In the following, when talking about a Riemannian spin manifold (M, g, σ), the spin structure σ is already chosen and fixed.
iii) A simply connected Riemannian manifold is spin if and only if the fundamental group $\pi_1(P_{SO(n)}M_g) = \mathbb{Z}_2$. Then the spin structure is uniquely determined [27, p. 38].
iv) If M is a Riemannian spin manifold with boundary ∂M, then a spin structure on M induces a spin structure on the boundary, see [19, Sect. 3]. Note that the converse is not true: Not every spin structure on the boundary extends to a spin structure on M. A well-known example is the circle S^1 that possesses two spin structure, but only the one without harmonic spinors extends to the unique spin structure of the disk.

Let further $S_g = P_{\mathrm{Spin}(n)}M_g \times_\rho \Delta_n$ be the associated *spinor bundle* where $\Delta_n = \mathbb{C}^{2^{[\frac{n}{2}]}}$, and let $\rho : \mathrm{Spin}(n) \to End(\Delta_n)$ be the n-dimensional complex spinor representation. A section of S_g will be called a *spinor*.

The set of all spinors will be denoted by $\Gamma(S_g) = \Gamma(M, S) = \Gamma(S)$, the one of smooth spinors by $C^\infty(S)$ and the one of compactly supported smooth spinors by $C_c^\infty(M, S)$. S_g is a complex vector bundle over M and equipped fibrewise with a hermitian metric $\langle\ .\ ,\ .\ \rangle_{m,g}$ that depends smoothly on the base point $m \in M$ and with the Clifford multiplication

$$\mu : TM \otimes S_g \to S_g; \quad X \otimes \phi \mapsto \mu(X)\phi =: X \cdot \phi$$

such that for all $m \in M$, $X, Y \in TM$ and $\phi_1, \phi_2, \phi \in \Gamma(S_g)$

$$\langle X_m \cdot \phi_1(m), \phi_2(m)\rangle_{m,g} + \langle \phi_1(m), X_m \cdot \phi_2(m)\rangle_{m,g} = 0$$

and

$$X_m \cdot Y_m \cdot \phi(m) + Y_m \cdot X_m \cdot \phi(m) = -2g_m(X_m, Y_m) \cdot \phi(m).$$

Further, with this hermitian metric we can define an L^2-scalar product

$$(\phi, \psi)_{M,g} := \int_M \langle \phi(m), \psi(m)\rangle_{m,g} \mathrm{dvol}_g$$

for spinors ϕ, ψ of S_g. In the following, we will omit the base point m in the notation $< .,. >_{m,g}$ of the hermitian product.

Additionally, the Levi-Civita connection on $P_{\mathrm{SO}(n)}M_g$ induces a metric connection ∇ on the spinor bundle that is parallel with respect to the Clifford multiplication, that means

$$\nabla_X(Y \cdot \phi) = (\nabla_X Y) \cdot \phi + Y \cdot \nabla_X \phi \tag{1.1}$$

for all $X, Y \in \Gamma(TM)$ and $\phi \in \Gamma(S_g)$.
If $\nabla\phi \equiv 0$, i.e. $\nabla_X \phi = 0$ for all $X \in TM$, then ϕ is called a *parallel spinor*.

By composing the connection ∇ and the Clifford multiplication μ the *Dirac operator* is defined as

$$D_g = \mu \circ \nabla : \ \Gamma(S_g) \to \Gamma(T^*M \otimes S_g) \cong \Gamma(TM \otimes S_g) \to \Gamma(S_g),$$

cf. [27, p. 68]. Locally it is given by

$$D_g \psi = \sum_{i=1}^{n} e_i \cdot \nabla_{e_i} \psi$$

where $\{e_i\}_{i=1,\dots,n}$ is an orthonormal frame on M, see [33, p. 144].

Remark 1.1.2.
i) The Dirac operator is a first-order elliptic differential operator [36, p. 113].
ii) The Dirac operator is formally self-adjoint [36, p. 115], i.e. for all spinors ϕ, ψ, at least one of which is compactly supported on M, we have $(\phi, D\psi)_{M,g} = (D\phi, \psi)_{M,g}$. Together with the hermiticity of the scalar product this implies that $(\phi, D\phi)$ is real for all compactly supported spinors ϕ.
In particular, for $\phi \in C_c^\infty(M, S)$ the value of $(\phi, D\phi)$ only depends on $\phi_\perp$, the part of ϕ being (L^2-)perpendicular to the kernel $\ker D$, since for a spinor that decomposes as $\phi = \phi_\perp + \phi_{\ker}$ we have

$$(\phi, D\phi) = (\phi_\perp + \phi_{\ker}, D\phi_\perp) = (D(\phi_\perp + \phi_{\ker}), \phi_\perp) = (D\phi_\perp, \phi_\perp).$$

An important tool when examining the Dirac operator is the *Lichnerowicz formula*

$$D^2 = \Delta + \frac{s}{4},$$

where s is the scalar curvature, see [36, p. 160]. Often derivations from it are useful, e.g. for the *Friedrich connection* $\nabla^f_X \phi := \nabla_X \phi + \frac{f}{n} X \cdot \phi$ where $f \in C^\infty(M, \mathbb{C})$ one gets for the case of real-valued functions f that

$$(D - f)^2 \psi = \Delta^f \psi + \Big(\frac{s}{4} + \frac{n-1}{n} f^2\Big)\psi - \frac{n-1}{n}(2fD\psi + \operatorname{grad} f \cdot \psi), \tag{1.2}$$

where $\Delta^f = \nabla^{f*} \nabla^f$, cf. [31, (5.4)].

If $f = -n\lambda$ is a constant function, the Friedrich connection plays a somewhat special role. Then the parallel spinors of $\nabla^{-n\lambda}$ are exactly the (*real*) *Killing spinors* with Killing constant λ, i.e. those spinors ψ that fulfill

$$\nabla_X \psi = \lambda X \cdot \psi \quad \text{for all } X \in TM.$$

If such a spinor exists for an imaginary constant λ, it is correspondingly called *imaginary Killing spinor*.

The following result shows that the notion of a Killing spinor cannot be generalized to arbitrary functions:

Theorem 1.1.3. *[31, Cor. 3.6][39, Sect. 1] Let (M, g, σ) be a connected spin manifold and $f \in C^\infty(M, \mathbb{C})$. Moreover, let ψ be a parallel spinor of ∇^f, i.e. $\nabla^f \psi \equiv 0$. Then f is either a real constant or has the form $f = \mathrm{i}b$ with $b \in C^\infty(M, \mathbb{R})$.*

The existence of Killing spinors yields quite strong restrictions for the underlying manifold. It has to be Einstein, i.e. its Ricci tensor is proportional to the metric.

Theorem 1.1.4. *[27, p. 118] If a manifold possesses a real Killing spinor, it is an Einstein space of positive scalar curvature.*
If a manifold possesses a purely imaginary Killing spinor, it has to be an Einstein space of negative scalar curvature.
In both cases the scalar curvature is obtained from the Killing constant λ by

$$s = \frac{n}{4(n-1)} \lambda^2.$$

1.2 Spectrum

Let (M, g, σ) be a Riemannian spin manifold with Dirac operator $D_g = D$.

Definition 1.2.1. *[21, Sect. 8.2.1] A complex number λ is an* eigenvalue *of D if there exists a nonzero* eigenspinor *$\phi \in C^\infty(M, S) \cap L^2(M, S)$ with $D\phi = \lambda\phi$.*
In particular, a spinor that is an eigenspinor to the eigenvalue 0 is called harmonic.

Remark 1.2.2.
i) It is sufficient to require $\phi \in H_1^2(M, S)$ and $(D - \lambda)\phi = 0$ since by elliptic regularity theory [5, Lem. 2.1] ϕ is then automatically smooth.
ii) In general, an eigenspinor can have zeros. But the unique continuation property of the Dirac operator [10, Main Thm. and Rem. 3] states that there are no zeros of infinite order.
In [18, p. 189] Bär even showed that the zero set of an eigenspinor is contained in a countable union of $(n-2)$-dimensional submanifolds and has locally finite $(n-2)$-dimensional Hausdorff density. The proof of this theorem, cf. [18, p. 194], even gives that this union of submanifolds can be chosen in such a way that it has itself locally finite $(n-2)$-dimensional Hausdorff density.

Theorem 1.2.3. *[27, pp. 99, 102],[2, Prop. 4.30] Let M be closed. Then the Dirac operator D_g has a pure point spectrum, and there exists an orthonormal eigenbasis ψ_i of $L^2(S_g)$ $(i \in \mathbb{N})$ such that $D_g\psi_i = \lambda_i\psi_i$ with $\lambda_i \in \mathbb{R}$. Further, both $+\infty$ and $-\infty$ are accumulation points of the spectrum.*

On open manifolds there can exist a continuous part of the spectrum. In general the spectrum of the Dirac operator, denoted by $\sigma(D)$, is composed of the point, the continuous and the residual spectrum.

Theorem 1.2.4. *[27, pp. 96, 98] Let M be complete. Then the Dirac operator is essentially self-adjoint (on $C_c^\infty(M,S)$), the residual spectrum is empty and $\sigma(D) \subset \mathbb{R}$.*

Thus, for complete manifolds the spectrum can be divided into point and continuous spectrum. But often another decomposition of the spectrum is used – the one into discrete and essential spectrum.

Definition 1.2.5. *A complex number λ lies in the* essential spectrum *of D, denoted by $\sigma_{ess}(D)$, if there exists a sequence of smooth compactly supported spinors ϕ_i which are orthonormal with respect to the L^2-product and*

$$\|(D-\lambda)\phi_i\|_{L^2} \to 0.$$

The essential spectrum contains, for example, all eigenvalues of infinite multiplicity. In contrast, the discrete spectrum $\sigma_d(D) := \sigma_p(D) \setminus \sigma_{ess}(D)$ consists of all eigenvalues of finite multiplicity.

Moreover, the essential spectrum only depends on the manifold at infinity as can be seen by the next statement.

Theorem 1.2.6 (Decomposition principle)**.** *[21, Thm. 8.7] Let M_1 and M_2 be two complete Riemannian spin manifolds, let $K_i \subset M_i$ be compact. If there is a spin preserving isometry between $M_1 \setminus K_1$ and $M_2 \setminus K_2$, then the Dirac operators on M_1 and M_2 have the same essential spectrum.*

It is not possible to compute the Dirac spectrum in general. But for some manifolds – in particular the standard ones – the spectrum is known. We want to give some well-known examples:

Example 1.2.7.
i) [19, Sect. 3] The circle S^1 admits two spin structures, one – we will denote by σ_{nt} – that bounds the unique spin structure of the disk and one – σ_{tr} – that does not. The Dirac spectrum for σ_{tr} is $\{k \in \mathbb{Z}\}$ and for σ_{nt} it is $\{k+\frac{1}{2} \mid k \in \mathbb{Z}\}$. Thus, the circle is an example for the dependence of the Dirac spectrum on the spin structure.
ii) [16, Thm. 1] The standard sphere S^n admits only one spin structure for $n \geq 2$. Its Dirac spectrum is given by $\{\pm(\frac{n}{2}+k) \mid k \in \mathbb{Z}_{\geq 0}\}$.
iii) [21, Thm. 8.8] The Euclidean space $\mathbb{R}^n$ also admits exactly one spin structure.

Its Dirac spectrum is the whole real line and purely continuous.
iv) (essential spectrum: [23, Cor. 4.6], point spectrum: [14, Thm. 8.4]) The hyperbolic space $\mathbb{H}^n$ with its unique spin structure has a purely continuous spectrum consisting of all real numbers.
v) [19, Thm. 4.1] The torus $\mathbb{T}^n = \mathbb{R}^n/\Gamma$ where Γ is a lattice of $\mathbb{R}^n$ possesses 2^n different spin structures. We represent a spin structure by the n-tuple $(\delta_1, \dots, \delta_n)$ with $\delta_i \in \{0, 1\}$. Furthermore, let $b_1, \dots, b_n$ be a basis of Γ and $b_1^*, \dots, b_n^*$ be the corresponding dual basis.
Then the spectrum consists of the eigenvalues

$$\pm 2\pi \left| b^* + \frac{1}{2} \sum_{i=1}^{n} \delta_j b_j^* \right|$$

where b^* runs through the dual lattice Γ^*. Each b^* contributes to the multiplicity of the corresponding eigenvalue with $2^{[\frac{n}{2}]-1}$.

Sometimes, especially when considering the "zero-in-the-spectrum" question, it is more convenient to work with the square of the operator. This is based on the following result:

Lemma 1.2.8. *[27, p. 98] For a complete Riemannian spin manifold (M, g, σ) the kernels of D and D^2 coincide.*

Remark 1.2.9. On complete manifolds both D and D^2 are essentially self-adjoint [45, Thm. 6.1]. The proof uses the following estimate:

$$\|D\phi\|^2_{L^2} \leq t\|D^2\phi\|^2_{L^2} + \frac{1}{t}\|\phi\|^2_{L^2} \tag{1.3}$$

for any spinor ϕ in C^2 and any $0 < t \in \mathbb{R}$.
Furthermore, the domains of the operators satisfy $dom(D^2) \subset dom(D) = C_c^\infty(M, S)$. The inequality together with this inclusion then means that D is D^2-bounded with relative bound $\sqrt{t}$. For a general definition of relative boundedness, see [41, Sect. X.2].

Lemma 1.2.10. *Let (M, g, σ) be a complete Riemannian spin manifold and let $\lambda \in \mathbb{R}$. Then $(D - \lambda)$ and $(D - \lambda)^2$ are essentially self-adjoint.*

Proof. Since D is essentially self-adjoint, $(D - \lambda)$ and $-2\lambda D$ are also essentially self-adjoint. With inequality (1.3) we know that D is D^2-bounded with relative bound $\sqrt{t}$. Similarly, $-2\lambda D$ is D^2-bounded with relative bound $\frac{1}{2\lambda}\sqrt{t}$.
Then the Kato-Rellich Theorem [41, Thm. X.12] yields that $D^2 - 2\lambda D$ and, therefore, $(D - \lambda)^2$ is essentially self-adjoint. □

Lemma 1.2.11. *Let (M, g, σ) be a complete Riemannian spin manifold. Then 0 is in the essential spectrum of $D - \lambda$ if and only if 0 is in the essential spectrum of $(D - \lambda)^2$.*
If this is the case, then there is a normalized sequence $\phi_i \in C_c^\infty(M, S)$ such that ϕ_i converges L^2-weakly to 0 and $\|(D - \lambda)\phi_i\|_{L^2} \to 0$ and $\|(D - \lambda)^2\phi_i\|_{L^2} \to 0$.

Proof. Due to Lemma 1.2.10 both operators $A := D-\lambda$ and $A^2 = (D-\lambda)^2$ are essentially self-adjoint on $C_c^\infty(M,S)$. Furthermore, denote by E_A and E_{A^2} the projector-valued measure belonging to A and A^2, respectively. We have supp $E_{A^2} = [0,\infty)$ and $E_{A^2}([a,b]) = E_A([-\sqrt{b},-\sqrt{a}]) + E_A([\sqrt{a},\sqrt{b}])$ for $0 \leq a \leq b$ which follows from [22, Thm. 3.1]. Thus, if 0 is in the (not necessarily essential) spectrum of A, then it is also contained in the spectrum of A^2 and vice versa.
Let now $0 \in \sigma_{ess}(A)$. Then for every $\epsilon > 0$ we obtain for the dimension of the image space of the projector $E_A([-\epsilon,\epsilon])$ that $\dim E_A([-\epsilon,\epsilon])H = \infty$ where $H := L^2(M,S)$ and, thus, $\dim E_{A^2}([0,\epsilon^2])H = \infty$. Hence, we have $0 \in \sigma_{ess}(A^2)$. Analogously, it follows from $0 \in \sigma_{ess}(A^2)$, that $0 \in \sigma_{ess}(A)$.
Next, let $0 \in \sigma_{ess}((D-\lambda)^2)$. Due to the definition of σ_{ess} there is a normalized sequence $\phi_i \in C_c^\infty(M,S)$ such that ϕ_i converges L^2-weakly to 0 and $\|(D-\lambda)^2\phi_i\|_{L^2} \to 0$. Then, we have

$$\|(D-\lambda)\phi_i\|_{L^2}^2 = ((D-\lambda)^2\phi_i, \phi_i) \leq \|(D-\lambda)^2\phi_i\|_{L^2}\, \|\phi_i\|_{L^2} \to 0.$$

□

1.3 L^p-theory and Sobolev embeddings

In this section we want to fix some notations concerning L^p-theory and state the results for Sobolev embeddings needed in the following.

We denote the L^p-norm by $\|.\|_{L^p(g)}$ or shortly by $\|.\|_p$ if the metric g is fixed. The L^p-norm of a spinor ϕ is simply given as the L^p-norm of the corresponding function $|\phi|$.
On closed manifolds a smooth spinor ϕ is always in $L^p(M,S)$ for every p. For open manifolds this is no longer true. Hence, in the definition of the Dirac eigenvalue we had to require that the smooth eigenspinor is additionally L^2-integrable.
Since the L^2-norm occurs most often, we simply write $\|.\| := \|.\|_2$.

In general, an L^p-function does not have to be in any other L^r-space. But the Hölder inequality gives a possibility to compare different L^p-norms:

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

for all functions $f \in L^p$, $g \in L^q$ with $p \in [1,\infty]$ and $p^{-1} + q^{-1} = 1$ (with $q = 1$ for $p = \infty$ and vice versa).
In particular, if we have a manifold of finite volume, we obtain

$$\|f\|_m \leq (\mathrm{vol}(M,g))^{\frac{1}{m}-\frac{1}{l}} \|f\|_l$$

for $l \geq m \geq 1$.

Furthermore, we will need the Sobolev norm

$$\|\phi\|_{H_k^p} = \sum_{l=0}^{k} \|\underbrace{\nabla \cdots \nabla}_{l \text{ times}} \phi\|_p.$$

That gives rise to the Sobolev space $H_k^p(M,S)$ that is the closure of $C_c^\infty(M,S)$ in L^p with respect to the H_k^p norm. The next Lemma states that one can also define the Sobolev norm on spinors using the Dirac operator instead of the spin connection ∇.

Lemma 1.3.1. *[4, Cor. 3.2.4] Let (M,g,σ) be closed. For any $k \in \mathbb{N}$ the norm*

$$\phi \to \sum_{l=0}^{k} \|D^l \phi\|_p$$

and the H_k^p-norm are equivalent on $\Gamma(M,S)$.

We will also need the following Sobolev embedding theorem:

Theorem 1.3.2. *[4, Sect. 3.3.2] Let $k, s \in \mathbb{R}$, $k \geq s$ and $q, r \in (1,\infty)$ with*

$$\frac{1}{r} - \frac{s}{n} \geq \frac{1}{q} - \frac{k}{n},$$

then $H_k^q(M,S)$ is continuously embedded into $H_s^r(M,S)$.
If, additionally, both inequalities are strict, the inclusion $H_k^q(M,S) \hookrightarrow H_s^r(M,S)$ is a compact operator.

1.4 Spinor bundles of conformally related metrics

Let $\overline{g} = f^2 g$ with $0 < f \in C^\infty(M)$. Having fixed a spin structure σ on (M,g) with corresponding spinor bundle S_g, there always exists a spin structure $\overline{\sigma}$ with the corresponding spinor bundle $S_{\overline{g}}$ on $(M,\overline{g})$ and a vector bundle isomorphism $A: \ S_g \to S_{\overline{g}}$ that is fibrewise an isometry [31, sect. 4.1]. In the following, we will explain this isomorphism:
Firstly, we have an isometry G_f between the bundle of g-orthonormal frames $P_{\mathrm{SO}(n)}M_g$ and the bundle of $\overline{g}$-orthonormal frames $P_{\mathrm{SO}(n)}M_{\overline{g}}$ which is given by

$$\epsilon = (X_1, \dots, X_n) \xrightarrow{G_f} \overline{\epsilon} = (f^{-1}X_1, \dots, f^{-1}X_n).$$

Now there exists a spin structure $\overline{\sigma}$ on $(M,\overline{g})$ such that the diagram

$$\begin{tikzcd}
P_{\mathrm{Spin}(n)}M_g \arrow[r, "\overline{G}_f"] \arrow[d] & P_{\mathrm{Spin}(n)}M_{\overline{g}} \arrow[d] \\
P_{\mathrm{SO}(n)}M_g \arrow[r, "G_f"] & P_{\mathrm{SO}(n)}M_{\overline{g}}
\end{tikzcd}$$

commutes. This gives rise to an identification of the associated spinor bundles S_g and $S_{\overline{g}}$. This identification

$$S_g = P_{\mathrm{Spin}(n)}M_g \times_\rho \Delta_n \to S_{\overline{g}} = P_{\mathrm{Spin}(n)}M_{\overline{g}} \times_\rho \Delta_n$$

is given by

$$\psi = [s, \phi] \mapsto \overline{\psi} = [\overline{G}_f(s), \phi]$$

and, thus, fibrewise an isometry, i.e. $< \overline{\psi}, \overline{\psi} >_{\overline{g}} = < \psi, \psi >_g$.

Since being spin is independent on the metric, cf. Remark 1.1.1.i, and since a spin structure σ on (M, g) determines the spin structure $\overline{\sigma}$ on $(M, \overline{g})$, we will refer to both just as σ.

Using the above isometry it is possible to compare the corresponding Dirac operators D and $\overline{D}$ [31, Prop. 4.3.1]:

$$\overline{D}(f^{-\frac{n-1}{2}}\overline{\psi}) = f^{-\frac{n+1}{2}}\overline{D\psi}. \tag{1.4}$$

We, therefore, have the following conformally invariant properties:

$$\begin{aligned}(\overline{D}\,\overline{\phi}, \overline{\phi})_{\overline{g}} &= \int_M < \overline{D}\,\overline{\phi}, \overline{\phi} >_{\overline{g}} \mathrm{dvol}_{\overline{g}} \\ &= \int_M < f^{-\frac{n+1}{2}}\overline{D\psi}, f^{-\frac{n-1}{2}}\overline{\psi} >_{\overline{g}} f^n \mathrm{dvol}_g = (D\psi, \psi)_g,\end{aligned} \tag{1.5}$$

and

$$\| \overline{D}\,\overline{\phi} \|^q_{L^q(\overline{g})} = \int_M |\overline{D}\,\overline{\phi}|^q \mathrm{dvol}_{\overline{g}} = \| D\psi \|^q_{L^q(g)} \tag{1.6}$$

where $\phi = f^{-\frac{n-1}{2}}\psi$.

Remark 1.4.1. On closed manifolds the dimension of the kernel of the Dirac operator is also a conformal invariant since for each harmonic spinor ψ on (M, g, σ) the spinor $\overline{\phi} = f^{-\frac{n-1}{2}}\overline{\psi}$ is harmonic on $(M, f^2 g, \sigma)$, cf. [32, Sect. 1.4].

1.5 Bourguignon-Gauduchon-Trivialization

Whereas in the last section we identified spinor bundles associated to conformally related metrics, we now examine the relation of any spinor bundle to the spinor bundle associated to the standard Euclidean space. This only works locally and is known as the Bourguignon-Gauduchon-Trivialization:
The following can be found in [6, Sect. 3]: Let the exponential map $\exp_p$ at $p \in M$ be bijective on a neighbourhood $U \subset T_pM \cong \mathbb{R}^n$ with $\exp_p(U) =: V \subset M$ and let

$(x_1, \dots, x_n) = \exp_p^{-1}(m)$ denote the corresponding normal coordinates for a point $m \in V$. Furthermore, define the map

$$G: \ V \to S^2_+(n, \mathbb{R}); \quad m \mapsto G_m := (g_{ij}(m))_{ij},$$

where G_m is the matrix of the coefficients of the metric g at m in the basis $\partial_i := \frac{\partial}{\partial x^i}$. $S^2_+(n, \mathbb{R})$ is the set of all real, symmetric and positive-definite $n \times n$ matrices. Thus, there exists exactly one symmetric positive-definite matrix $B_m = (b_i^j(m))_{ij}$ with $B_m^2 = G_m^{-1}$.
For each $m \in V \subset M$ the matrix B_m gives rise to the isometry

$$B_m : (T_{\exp_p^{-1}(m)} U \cong \mathbb{R}^n, g_E) \to (T_m V, g_m); \quad (a^1, \dots, a^n) \mapsto \sum_{i,j} b_i^j(m) a^i \partial_j(m),$$

since $g_m(\sum_i b_k^i \partial_i, \sum_j b_l^j \partial_j) = \sum_{i,j} b_k^i b_l^j g_m(\partial_i, \partial_j) = \sum_{i,j} b_k^i b_l^j g_{ij} = \delta_{kl} = g_E(\partial_k, \partial_l)$.

This map is used to identify the $\mathrm{SO}(n)$-principal bundles $P_{\mathrm{SO}(n)} U_{g_E}$ and $P_{\mathrm{SO}(n)} V_g$ which lifts to an identification of the corresponding $\mathrm{Spin}(n)$-principal bundles and, thus, of the spinor bundles

$$S_{U,g_E} = P_{\mathrm{Spin}(n)} U_{g_E} \times_\rho \Delta_n \to S_{V,g} = P_{\mathrm{Spin}(n)} V_g \times_\rho \Delta_n; \quad \psi \mapsto \overline{\psi}.$$

Again this identification is fibrewise an isometry.

Furthermore, let ∇ and $\overline{\nabla}$ denote the Levi-Civita connections on (TU, g_E) and (TM, g) as well as the lifted connections on the spinor bundles S_{U,g_E} and $S_{V,g}$, respectively.

Next, the metric is developed in the geodesic normal coordinates $(x_1, \dots, x_n)$ in the neighbourhood $V \subset M$ around a fixed point $p \in M$. The derivation of the subsequent expressions can be found in [37, (5.4)]:

$$g_{ij}(p) = \delta_{ij} + \frac{1}{3} R_{i\alpha\beta j}(p) x^\alpha x^\beta + \frac{1}{6} R_{i\alpha\beta j;\gamma}(p) x^\alpha x^\beta x^\gamma + \mathcal{O}(r^4), \tag{1.7}$$

where the curvature is given by

$$R_{ijkl} = \langle \nabla_{e_j} \nabla_{e_i} e_k, e_l \rangle - \langle \nabla_{e_i} \nabla_{e_j} e_k, e_l \rangle - \langle \nabla_{[e_j, e_i]} e_k, e_l \rangle$$

for the orthonormal frame $(e_1, \dots, e_n)$ of (TV, g) with $e_i := b_i^j \partial_j$.

In the next step, we will compare the Dirac operators, cf. [6, Sect. 3 and 4]:
For this purpose, let D and $\overline{D}$ denote the Dirac operators acting on $\Gamma(S_{U,g_E})$ and $\Gamma(S_{V,g})$), respectively. Then we have

$$\overline{D}\,\overline{\psi} = \overline{D\psi} + \frac{1}{4} \sum_{ijk} \tilde{\Gamma}_{ij}^k e_i \cdot e_j \cdot e_k \cdot \overline{\psi} + \sum_{ij} (b_i^j - \delta_i^j) \overline{\partial_i \cdot \nabla_{\partial_j} \phi}, \tag{1.8}$$

where

$$\tilde{\Gamma}_{ij}^k := -\langle \overline{\nabla}_{e_i} e_j, e_k \rangle = \partial_i b_j^k - \frac{1}{3}(R_{ik\alpha j} + R_{i\alpha kj})x^\alpha + \mathcal{O}(r^2) \tag{1.9}$$

with

$$b_i^j = \delta_i^j - \frac{1}{6} R_{i\alpha\beta j} x^\alpha x^\beta + O(r^3). \tag{1.10}$$

Further

$$\overline{\nabla}_{e_i} \overline{\psi} = \overline{\nabla_{e_i} \psi} + \frac{1}{4} \sum \tilde{\Gamma}_{ij}^k e_j \cdot e_k \overline{\psi} \tag{1.11}$$

and

$$e_i \cdot \overline{\psi} = \overline{\partial_i \cdot \psi}. \tag{1.12}$$

Later on, when using the Bourguignon-Gauduchon-Trivialization we will leave out the bars on the spinors and identify the spinor bundles S_{U,g_E} and $S_{V,g}$. This can be done since the identification is fibrewise an isometry.

1.6 The conformal type of a manifold

Definition 1.6.1. *[34, Sect. 2] Let (M, g) be an n-dimensional Riemannian manifold, and let $C \subset M$ be a connected compact subset. Then the* conformal capacity *of (C, M) is defined as*

$$cap(C, M) = \inf \int_M |\nabla f|^n \mathrm{dvol}_g$$

where the infimum is taken over all compactly supported smooth functions with $0 \leq f \leq 1$ and $f \equiv 1$ on C.

Whether the conformal capacity is positive or zero is independent of C and depends only on the manifold at infinity. That's why, we can define the following: If $cap(C, M) = 0$, the manifold is called *conformally parabolic*, otherwise *conformally hyperbolic*.
The term conformally refers to the fact that $cap(C, M)$ only depends on the conformal class of g.

Example 1.6.2.
i) The Euclidean space is conformally parabolic. This can be seen by taking f to be a radial function with $f(x) = 1$ for $|x| < r_0$, $f(x) = 0$ for $|x| > r_1$ and in between $f(x) = \frac{\ln r_1 - \ln |x|}{\ln r_1 - \ln r_0}$. Then f is continuous, compactly supported, $f \in H_1^n$ and

$$\| \nabla f \|_n^n = \int_{B_{r_1} \setminus B_{r_0}} \frac{1}{(r \ln \frac{r_1}{r_0})^n} r^{n-1} \mathrm{dvol}_{g_E} = \mathrm{vol}(S^{n-1}, g_{st}) \ln^{1-n} \frac{r_1}{r_0} \to 0$$

for $r_1 \to \infty$. Since every function in H_1^n can be approximated by smooth functions such that also their H_1^n-norms converge, we obtain that $cap(B_{r_0}(0), \mathbb{R}^n) = 0$.
ii) The hyperbolic space is conformally equivalent to $(B_1(0) \subset \mathbb{R}^n, g_E)$ and, hence, conformally hyperbolic.

iii) [43, Cor. 5.2] Let (M, g) be a Riemannian manifold with a warped cylindrical end $(N \times [1, \infty),\ g_{end} = f(t)^2 g_N + dt^2)$. Then M is conformally parabolic if and only if $\int_1^\infty f(t)^{-1} dt = \infty$.

Finally, we state the following result which gives another description of being conformally parabolic:

Theorem 1.6.3. *[47, Sect. 3] A Riemannian manifold is conformally parabolic if and only if its conformal class contains a complete metric of finite volume.*

Example 1.6.4. From Example 1.6.2.i we know that $(\mathbb{R}^n, g_E)$ is conformally parabolic. We now want to give a metric conformal to g_E that illustrates Theorem 1.6.3, cf. [47, Sect. 3]:
Let $\mathbb{R}^n$ be equipped with a metric $f^2 g_E$ where $f(r) = (r \ln r)^{-1}$ for r being large enough. Since the integral

$$\int_c^\infty \frac{1}{r \ln r} dr = \lim_{R \to \infty} \ln \ln r|_c^R = \infty$$

diverges, the metric is complete and since

$$\int_c^\infty \frac{1}{r^n (\ln r)^n} r^{n-1} dr = \lim_{R \to \infty} \frac{-1}{(n-1)(\ln r)^{n-1}}\Big|_c^R < \infty,$$

the manifold is of finite volume.
Further, for the new metric and for dimension $n > 2$ the scalar curvature is bounded from below since for $h = f^{\frac{n-2}{2}}$ we can compute

$$\begin{aligned}
\overline{s} &= 4\frac{n-1}{n-2} h^{-\frac{n+2}{n-2}} \Delta h \\
&= -4\frac{n-1}{n-2} (r \ln r)^{\frac{n+2}{2}} \left(\partial_r^2 h + \frac{1}{r}(n-1)\partial_r h \right) \\
&= -4\frac{n-1}{n-2} (r \ln r)^{\frac{n+2}{2}} \Bigg(\frac{(n-2)n}{4} (r \ln r)^{-\frac{n+2}{2}} (\ln r + 1)^2 - \frac{n-2}{2} (r \ln r)^{-\frac{n}{2}} \frac{1}{r} \\
&\quad - \frac{1}{r} \frac{(n-1)(n-2)}{2} (r \ln r)^{-\frac{n}{2}} (\ln r + 1) \Bigg) \\
&= 4\frac{n-1}{n-2} \left(\frac{(n-2)^2}{4} \ln^2 r - \frac{(n-2)n}{4} \right) = (n-1)(n-2) \ln^2 r - n(n-1).
\end{aligned}$$

1.7 Refined Kato inequalities

The Kato inequality states that for any section ϕ of a Riemannian or Hermitian vector bundle E endowed with a metric connection ∇ on a Riemannian manifold (M, g) we have

$$2|\phi||d|\phi|| = |d|\phi|^2| = 2| < \nabla\phi, \phi > | \leq 2|\phi||\nabla\phi|, \tag{1.13}$$

i.e. $|d|\phi|| \leq |\nabla\phi|$ away from the zero set of ϕ. For this estimate it is used that $<\nabla_X\phi, \phi>\in \mathbb{R}$ for all $X \in TM$.
In [24] refined Kato inequalities were obtained for sections in the kernel of first-order differential operators. They have the form

$$|d|\phi|| \leq k_P|\nabla\phi|$$

where k_P depends on the operator P.

We now want to sketch the set-up used in [24]: Let E be an irreducible natural vector bundle E over an n-dimensional Riemannian (spin) manifold (M, g) with scalar product $< ., . >$ and a metric connection ∇. Irreducible natural means that the vector bundle is obtained either from the orthonormal frame bundle of M or from the spinor frame bundle with an irreducible representation of $\mathrm{SO}(n)$ or $\mathrm{Spin}(n)$ on a vector space V. We will denote this representation by λ. Furthermore, let τ be the standard representation of $\mathrm{SO}(n)$ or $\mathrm{Spin}(n)$ on $\mathbb{R}^n$. Then the real tensor product $\tau \otimes \lambda$ splits into irreducible components as

$$\tau \otimes \lambda = \bigoplus_{j=1}^{N} \mu^j, \qquad \mathbb{R}^n \otimes V = \bigoplus_{j=1}^{N} W_j.$$

This induces a decomposition of $T^*M \otimes E$ into irreducible subbundles F_j associated to μ^j. Furthermore, let Π_j denote the projection onto the jth summand of $\mathbb{R}^n \otimes V$ and $T^*M \otimes E$, respectively.

Let P be a first-order linear differential operator of the form $P = \sum_{i\in I} \Pi_i \circ \nabla$ where $I \subseteq \{1, \ldots, N\}$. Moreover, we denote $\Pi_I := \sum_{i\in I} \Pi_i$ and $\hat{I} := \{1, \ldots, N\} \setminus I$.

Following the ansatz for the refined Kato inequalities, we obtain the estimate:

Lemma 1.7.1. *Let P be an operator as defined above. Then we have away from the zero set of ϕ*

$$|d|\phi|| \leq |P\phi| + k_P|\nabla\phi|$$

where $k_P := \sup_{|\alpha|=|v|=1} |\Pi_{\hat{I}}(\alpha \otimes v)|$.

Proof. Let ϕ be a section of E. Then away from the zero set of ϕ we obtain

$$|d|\phi|| = \frac{|d|\phi|^2|}{2|\phi|} = \frac{|<\nabla\phi, \phi>|}{|\phi|}$$

Let now α_0 be a unit 1-form with $<\nabla\phi, \phi>= c\alpha_0$ for some $c \in \mathbb{R}$. Then we have

$$\begin{aligned} <\nabla\phi, \alpha_0 \otimes \phi> &= \sum_i <\nabla_{e_i}\phi, \alpha_0(e_i)\phi>= \sum_i \frac{1}{c} <\nabla_{e_i}\phi, \phi>^2 \\ &= \sum_i \frac{<\nabla_{e_i}\phi, \phi>^2}{|<\nabla\phi, \phi>|} = |<\nabla\phi, \phi>|. \end{aligned}$$

Thus, we obtain

$$\begin{aligned}
|d|\phi|| &= \frac{|<\nabla\phi, \alpha_0 \otimes \phi>|}{|\phi|} \\
&= \frac{|<(\Pi_I + \Pi_{\hat{I}})\nabla\phi, \alpha_0 \otimes \phi>|}{|\phi|} \\
&\leq \frac{|<P\phi, \alpha_0 \otimes \phi>|}{|\phi|} + \frac{|<\nabla\phi, \Pi_{\hat{I}}(\alpha_0 \otimes \phi)>|}{|\phi|} \\
&\leq |P\phi| + |\nabla\phi| \sup_{|\alpha|=|v|=1} |\Pi_{\hat{I}}(\alpha \otimes v)| = |P\phi| + k_P|\nabla\phi|.
\end{aligned}$$

□

The constant k_P is the same as the one in the refined Kato inequality. In [24] this constant is determined in terms of the conformal weights of the differential operator.

Example 1.7.2. [24, (3.9)] For the classical Dirac operator D and for $D - \lambda$, where $\lambda \in \mathbb{R}$, we have $k = \sqrt{\frac{n-1}{n}}$.

Chapter 2

The λ^+_{min}-invariant and an overview of known results

In this chapter we first define the λ^+_{min}-invariant, then we give an overview on what is known about this invariant and where can it be used.

Definition of λ^+_{min}.
Let (M, g, σ) be a Riemannian spin manifold of dimension $n \geq 2$ with Dirac operator D. We define

$$\lambda^+_{min}(M, g, \sigma) := \inf_{g_0 \in [g], \mathrm{vol}(M, g_0) < \infty} \lambda^+_1(M, g_0, \sigma)\mathrm{vol}(M, g_0)^{\frac{1}{n}}$$

where

$$\lambda^+_1(M, g, \sigma) := \inf \left\{ \frac{\|D\phi\|^2}{(D\phi, \phi)} \;\middle|\; 0 < (D\phi, \phi),\ \phi \in C^\infty_c(M, S) \right\}$$

and $[g]$ is the set of all metrics conformal to g. Furthermore, $C^\infty_c(M, S)$ denotes the compactly supported smooth spinors on (M, g, σ).

Note that in the conformal class of a metric there always exists a metric of finite volume, e.g. take $f^n(x) = (r^2 \mathrm{vol}(B_{r+1}(p) \setminus B_r(p), g))^{-1}$ for $r = dist(x, p)$ large enough and $p \in M$ fixed. Then, $\int f^n \mathrm{dvol}_g \leq \mathrm{vol}(B_j(p), f^2 g) + \sum_{i=j}^\infty \frac{1}{j^2} < \infty$. If f_i is a monotone approximation of f by smooth non-negative functions, we obtain metrics $f_i^2 g$ with finite volume.
If $\overline{g} = \kappa g$ with $\kappa \in \mathbb{R}_{>0}$, then from the transformation formula (1.4) we obtain $\lambda^+_1(\overline{g}) = \kappa^{-\frac{1}{2}} \lambda^+_1(g)$. Furthermore, we have $\mathrm{vol}(M, \kappa g) = \kappa^{\frac{n}{2}} \mathrm{vol}(M, g)$. Thus, λ^+_{min} could equally be defined as the infimum over all conformal metrics of unit volume.

In the case of closed manifolds λ^+_1 coincides with the smallest positive eigenvalue of the Dirac operator. That's why, λ^+_{min} is sometimes called the *first positive conformal eigenvalue* of D. For complete manifolds λ^+_1 is the infimum of the positive part of the Dirac spectrum, cf. Lemma 4.1.2.

As we will see in the next Lemma, the λ^+_{min}-invariant can also be defined by the following variational problem:

$$\Lambda^+_{min}(M,g,\sigma) := \inf\left\{ \frac{\|D\phi\|_q^2}{(D\phi,\phi)} \;\middle|\; 0<(D\phi,\phi),\ \phi\in C_c^\infty(M,S),\ q=\frac{2n}{n+1}\right\}. \quad (2.1)$$

The equations (1.5) and (1.6) make sure that this definition is conformally invariant, i.e. $\Lambda^+_{min}(M,g,\sigma)=\Lambda^+_{min}(M,\overline{g},\sigma)$ for all $\overline{g}\in[g]$.

Lemma 2.0.1. *The invariants defined above coincide:*

$$\lambda^+_{min}(M,g,\sigma)=\Lambda^+_{min}(M,g,\sigma).$$

Proof. This was proven in [1, Prop. 2.6] for closed manifolds, and a similar proof works for open manifolds:
We restrict to metrics g of unit volume. Since $q=\frac{2n}{n+1}<2$, the Hölder inequality implies $\|D\phi\|_q\le\|D\phi\|_2$ for all $\phi\in C_c^\infty(M,S)$. Hence, we obtain $\Lambda^+_{min}(g)\le\lambda_1^+(g)$ and, thus, $\Lambda^+_{min}\le\lambda^+_{min}$.

For the converse inequality let $\overline{g}=h^{\frac{4}{n+1}}g$ with $\mathrm{vol}(M,\overline{g})=1$. Setting $\overline{\phi}=h^{-\frac{n-1}{n+1}}\phi$, equation (1.5) implies $(\phi,D\phi)_g=(\overline{\phi},\overline{D}\,\overline{\phi})_{\overline{g}}$ for all $\phi\in C_c^\infty(M,S)$. We further restrict to spinors ϕ satisfying $\|D\phi\|_q=1$. By a small perturbation of ϕ we can always achieve that $|D\phi|\in C_c^\infty(M)$.

Now we choose the conformal factor $h\in C^\infty(M)$ such that $h=|D\phi|+\epsilon>0$. Then due to Lebesgue's dominated convergence theorem we have

$$\mathrm{vol}(M,\overline{g})=\int_M h^{\frac{2n}{n+1}}\mathrm{dvol}_g\to 1$$

and

$$\|\overline{D}\,\overline{\phi}\|^2_{L^2(\overline{g})}=\int_M|D\phi|^2h^{-\frac{2}{n+1}}\mathrm{dvol}_g\to\|D\phi\|_q^q=1$$

as $\epsilon\to0$. Therefore,

$$\lambda^+_{min}\le\lim_{\epsilon\to0}\lambda_1^+(\overline{g})\mathrm{vol}(\overline{g})^{\frac{1}{n}}\le\lim_{\epsilon\to0}\frac{\|\overline{D}\,\overline{\phi}\|^2_{L^2(\overline{g})}}{(\overline{\phi},\overline{D}\,\overline{\phi})_{\overline{g}}}\mathrm{vol}(\overline{g})^{\frac{1}{n}}=\frac{1}{(\overline{\phi},\overline{D}\,\overline{\phi})_{\overline{g}}}=\frac{\|D\phi\|^2_{L^q(g)}}{(\phi,D\phi)_g}$$

which yields the claim. □

Bounds for λ^+_{min}.
As in the Yamabe problem the standard sphere plays a special role. The λ^+_{min}-invariant of the standard sphere (S^n,g_{st}) with its unique spin structure χ_{st} is given by

$$\lambda^+_{min}(S^n,g_{st},\chi_{st})=\lambda_1^+(S^n,g_{st},\chi_{st})\mathrm{vol}(S^n,g_{st})^{\frac{1}{n}}=\frac{n}{2}\mathrm{vol}(S^n,g_{st})^{\frac{1}{n}}. \quad (2.2)$$

This is obtained by using the conformal Hijazi inequality or Bär's inequality and the Dirac spectrum of the standard sphere, cf. in [3, pp. 22, 32] and for $n = 2$ in [15, Thm. 2].
In the case of the standard sphere the infimum λ^+_{min} is really attained by a metric – the standard metric.

This value $\lambda^+_{min}(S^n)$ is the highest the λ^+_{min}-invariant can reach, cf. [3, Thm. 3.1], [6, Thm. 1.1] for closed manifolds of dimension $n > 2$, [28, Cor. 1.3] for closed surfaces and Chapter 3 for general manifolds. Thus, for all Riemannian spin manifolds (M, g, σ)

$$\lambda^+_{min}(M, g, \sigma) \leq \lambda^+_{min}(S^n, g_{st}, \chi_{st}).$$

A trivial lower bound is given by

$$\lambda^+_{min}(M, g, \sigma) \geq 0.$$

Theorem 2.0.2. *Let (M, g, σ) be a closed Riemannian spin manifold or an open Riemannian spin manifold that is spin conformally equivalent to an open and bounded subset with smooth boundary of any Riemannian spin manifold. Then we have*

$$\lambda^+_{min}(M, g, \sigma) > 0.$$

Proof. For closed manifolds this was shown in [4, Lem. 4.3.1].
Let now (M, g, σ) be spin conformally equivalent to an open and bounded subset Ω of a Riemannian spin manifold (N, h, χ) such that Ω has a smooth boundary. Without loss of generality we can assume N to be closed since we can always "double" N in such a way that Ω can still be viewed as a subset of the double and equipped with the original metric and spin structure (This procedure will be explained explicitly in the proof of Lemma 3.3.2). Then with Lemma 2.0.3 we obtain

$$\lambda^+_{min}(M, g, \sigma) = \lambda^+_{min}(\Omega, h, \chi) \geq \lambda^+_{min}(N, h, \chi) > 0.$$

□

On general open manifolds this is no longer true, the λ^+_{min}-invariant can vanish. In [29, Ex. 3.4], see also Lemma 3.4.1, we gave a first example, namely the cylinder $S^1 \times \mathbb{R}$ equipped with the metric $g = g_{S^1} + dt^2$ and with the spin structure whose restriction to S^1 has harmonic spinors – that is the one which does not bound to the spin structure of the disk. Further examples will be given in Section 3.4 and Example 4.1.5.ii.

λ^+_{min} for subsets and disjoint unions.

Lemma 2.0.3. *[29, Lem. 2.1.i] Let $\Omega_1 \subset \Omega_2 \subset M$ be open non-empty subsets of a Riemannian spin manifold (M, g, σ) equipped with the induced metric and the induced spin structure. Then*

$$\lambda^+_{min}(\Omega_1, g, \sigma) \geq \lambda^+_{min}(\Omega_2, g, \sigma).$$

This is easily seen from the definition of λ^+_{min}, cf. (2.1). In the same way we see immediately that for a compact exhaustion $\overline{M_i}$ of M, where M_i is open, we have

$$\lambda^+_{min}(M,g,\sigma) = \inf_i \lambda^+_{min}(M_i,g,\sigma).$$

We can say even more about open subsets of the standard sphere:

Lemma 2.0.4. *Let $\Omega \subset S^n$ be an open non-empty subset and $n \geq 2$. Then*

$$\lambda^+_{min}(\Omega, g_{st}, \chi_{st}) = \lambda^+_{min}(S^n, g_{st}, \chi_{st}).$$

This Lemma was proven in [29, Lem. 2.1.iii] for dimension $n > 2$. But it also holds for $n = 2$ what we will show in Remark 3.1.5.

Next we examine the behaviour of λ^+_{min} under disjoint unions:

Lemma 2.0.5. *Let (M,g,σ) be the disjoint union of Riemannian spin manifolds (M_i,g_i,σ_i) with $i = 1,\dots,m$. Then*

$$\lambda^+_{min}(M,g,\sigma) = \min_i \lambda^+_{min}(M_i,g_i,\sigma_i).$$

Proof. It is sufficient to prove the case $m = 2$.
Lemma 2.0.3 gives $\lambda^+_{min}(M) \leq \lambda^+_{min}(M_i)$ for all i. The converse inequality can be obtained as follows:
Let $\phi \in C^\infty_c(M,S)$ with $(D\phi,\phi) > 0$. Then we have $\phi = \phi_1 + \phi_2$ with $\phi_i := \phi_{|M_i} \in C^\infty_c(M_i,S)$. This implies

$$\begin{aligned}
\frac{\|D^M\phi\|_q^2}{(D^M\phi,\phi)} &= \frac{\|D^M\phi_1 + D^M\phi_2\|_q^2}{(D^{M_1}\phi_1,\phi_1) + (D^{M_2}\phi_2,\phi_2)} = \frac{(\|D^{M_1}\phi_1\|_q^q + \|D^{M_2}\phi_2\|_q^q)^{\frac{2}{q}}}{(D^{M_1}\phi_1,\phi_1) + (D^{M_2}\phi_2,\phi_2)} \\
&\geq \frac{\|D^{M_1}\phi_1\|_q^2 + \|D^{M_2}\phi_2\|_q^2}{(D^{M_1}\phi_1,\phi_1) + (D^{M_2}\phi_2,\phi_2)} \\
&\geq \begin{cases} \inf_i \dfrac{\|D^{M_i}\phi_i\|_q^2}{(D^{M_i}\phi_i,\phi_i)} & \text{if } (D^{M_i}\phi_i,\phi_i) > 0 \\ \dfrac{\|D^{M_i}\phi_i\|_q^2}{(D^{M_i}\phi_i,\phi_i)} & \text{if } (D^{M_j}\phi_j,\phi_j) \leq 0 \text{ for } i \neq j \end{cases} \\
&\geq \inf_i \lambda^+_{min}(M_i,g_i,\sigma_i).
\end{aligned}$$

The first inequality results from the monotonicity of the function

$$f:\ x \in \mathbb{R}_{>0} \to (a^x + b^x)^{\frac{2}{x}} = a^2\left(1 + \left(\frac{b}{a}\right)^x\right)^{\frac{1}{x}}$$

for positive a and b which is deduced from

$$\begin{aligned}
[(1+c^x)^{\frac{1}{x}}]' &= (1+c^x)^{\frac{1}{x}}\left(\frac{c^x \ln c}{(1+c^x)x} - \frac{\ln(1+c^x)}{x^2}\right) \\
&= \frac{(1+c^x)^{\frac{1}{x}}}{(1+c^x)x^2}(c^x \ln c^x - (1+c^x)\ln(1+c^x))
\end{aligned}$$

and from the fact that $x \in [1, \infty) \to x \ln x$ is monotonically increasing.
The second inequality is obtained from $\frac{a+b}{c+d} \geq \frac{a}{c}$ for $\frac{b}{d} \geq \frac{a}{c}$ and $c, d > 0$. Recall that at most one summand $(D^{M_j}\phi_j, \phi_j)$ is non-positive since its sum $(D^M\phi, \phi)$ is positive. □

Euler-Lagrange equation of the variational problem.
Since λ^+_{min} is also given by a variational problem, one can consider the corresponding Euler-Lagrange equation

$$D(\psi - (\lambda^+_{min})^{-1}|D\psi|^{q-2}D\psi) = 0, \quad \|D\psi\|_q = 1. \tag{2.3}$$

The following duality principle relates this equation to

$$D\phi = \lambda^+_{min}|\phi|^{p-2}\phi, \quad \|\phi\|_p = 1. \tag{2.4}$$

Lemma 2.0.6. *[4, Lem. 4.2.6]*
i) *If ψ satisfies* (2.3)*, then $\phi := |D\psi|^{p-2}D\psi$ satisfies* (2.4).
ii) *If ϕ satisfies* (2.4)*, then $\psi := (\lambda^+_{min})^{-1}\phi$ satisfies* (2.3).

This duality is quite useful since it is easier to work with equation (2.4), and in particular we get the following existence result.

Theorem 2.0.7. *[4, Thm. 4.2.2] If (M, g, σ) is closed and*

$$\lambda^+_{min}(M, g, \sigma) < \lambda^+_{min}(S^n, g_{st}, \chi_{st}),$$

then there is a spinor $\phi \in C^{1,\alpha}(M, S)$ that is smooth away from its zero set and fulfills (2.4).

Remark 2.0.8. Theorem 2.0.7 is also true if (M, g) is an open and smoothly bounded subset of a manifold (N, h) with $g = h_{|_M}$, cf. [29, Thm. 3.1]. But in contrast to the closed case the solution ϕ is not a minimizer of the variational problem since ϕ is then not compactly supported.

Thus, it would be interesting to know which Riemannian manifolds satisfy this strict inequality. For the Yamabe problem it is known that for a closed manifold the Yamabe invariant only attains the same value as the invariant of the standard sphere if it is itself conformally equivalent to the standard sphere. For the λ^+_{min}-invariant this is not known. But there are some results describing classes of manifolds which fulfill the strict inequality, cf. [9]. We want to sketch here the following:

Theorem 2.0.9. *[9, Thm. 1.2] Let (M, g, σ) be a conformally flat closed Riemannian spin manifold of dimension $n \geq 2$ with invertible Dirac operator. If the mass endomorphism (see below) possesses a positive eigenvalue, then*

$$\lambda^+_{min}(M, g, \sigma) < \lambda^+_{min}(S^n, g_{st}, \chi_{st}).$$

For defining the mass endomorphism we need the notion of a Green function:

Definition 2.0.10. *[9, Def. 2.1] Let $\pi_i : M \times M \to M$ be the projection to the i-th component. A smooth section*

$$G_D : M \times M \setminus \{(p,p) \mid p \in M\} \to \pi_1^*(\Gamma(M,S)) \otimes (\pi_2^*(\Gamma(M,S))^*$$

that is locally integrable on $M \times M$ is called a Green function *for the Dirac operator D if for any $y \in M$, $\psi_0 \in \Gamma_y(M,S)$ and $\phi \in \Gamma(M,S)$ we have*

$$\int < G_D(x,y)\psi_0, D^*\phi(x) > dx =< \psi_0, \phi(y) >,$$

$\Gamma_y(M,S)$ denotes the set of values of a spinor at $y \in M$.

On the Euclidean space the Green function is given by

$$G_{\text{eucl}}(x,y) = -\frac{1}{\omega_{n-1}} \frac{x-y}{|x-y|^n}.$$

In general, there is the following result

Theorem 2.0.11. *[9, Prop. 2.3] For a metric which is flat near $y \in M$ the Green function G_D exists. Furthermore, in the trivialization given by the local comparison of g and the Euclidean metric (cf. Section 1.4) the Green function expands for x tending to y as follows:*

$$G_D(x,y)\psi_0 = G_{\text{eucl}}(x,y)\psi_0 + \frac{1}{\omega_{n-1}} v(x,y)\psi_0 \tag{2.5}$$

where $v(x,y) : \Gamma_y(M,S) \to \Gamma_x(M,S)$ is a homomorphism such that $\partial_x v(x,y)(\psi_0) = 0$ on a neighbourhood of y.

With these preparations the mass endomorphism can now be defined:

Definition 2.0.12. *[9, Def. 2.10] Let (M,g,σ) be closed and conformally flat on a neighbourhood of $y \in M$. Choose a metric $\overline{g} \in [g]$ that is flat on a neighbourhood of y and fulfills $\overline{g}_y = g_y$. Let G_D be the Green function for D. Then the* mass endomorphism *is defined as*

$$\alpha_y : \Gamma_y(M,S) \to \Gamma_y(M,S), \;\; \psi_0 \mapsto v(y,y)(\psi_0)$$

where v is the map that appears in the expansion (2.5) *with respect to $\overline{g}$.*

Since for two conformal metrics $g_1 = f^2 g_2$ that are both flat in a neighbourhood of y the map $v(y,y)$ coincides [9, Prop. 2.9], this definition is independent on the choice of $\overline{g}$.

Applications for λ^+_{min} and other occurrences.
The first obvious application is that λ^+_{min} is a conformal lower bound for the positive part of the Dirac spectrum if the manifold has finite volume. Thus, it is interesting

to get some estimates for λ^+_{min}. The most prominent is probably the Hijazi inequality we will also deal with in Chapter 4.
Moreover, on closed surfaces a solution of (2.4) allows to prove the existence of a constant mean curvature immersion, cf. [1, Prop. 10.2].
On open manifolds the λ^+_{min}-invariant at infinity gives an obstruction to spin conformal compactification, see Theorem 3.0.1.

There are further investigations of the λ^+_{min}-invariant in different directions.
Firstly, Raulot [40] considered in his PhD-thesis the λ^+_{min}-invariant for manifolds with boundary with different boundary conditions.
Additionally, another quantity that can be considered is the spin version of the σ-invariant, that is the supremum of the Yamabe invariant of a manifold taken over all conformal classes. This is called the τ-invariant

$$\tau(M,\sigma) = \sup_{[g]\in\mathcal{C}} \lambda^+_{min}(M,g,\sigma)$$

where $\mathcal{C}$ is the set of all conformal classes on M [7]. This is now not only a conformal invariant but depends only on the manifold and its spin structure.
In [7, Thm. 1.1] the behaviour of the τ-invariant under 0-dimensional surgery was studied. This consideration also allowed to compute the τ-invariant on closed spin surfaces, cf. [7, Thm. 1.3].

There still are a lot of open questions that would be interesting. One of the most interesting ones might be: When does a Riemannian spin manifold fulfills $\lambda^+_{min}(M,g,\sigma) = \lambda^+_{min}(S^n)$?

Chapter 3

The λ^+_{min}-invariant on surfaces

In [29] we examined the behaviour of the λ^+_{min}-invariant on open spin manifolds. However, most of the results were only proven for dimension $n > 2$. The main goal of this chapter is to explore the two-dimensional case, although most of the results given here will be proven for all dimensions – as long as they are still valid then.

Firstly, we give the obstruction to spin conformal compactification of [29, Thm. 1.4] now also for dimension 2.

Theorem 3.0.1. *Let (M, g, σ) be an open complete Riemannian spin manifold of dimension $n \geq 2$ with $\overline{\lambda^+_{min}(M, g, \sigma)} < \lambda^+_{min}(S^n, g_{st}, \chi_{st})$. Then (M, g, σ) is not spin conformal to an open subset of a closed Riemannian spin n-manifold.*

Definition 3.0.2. *The λ^+_{min}-*invariant at infinity *$\overline{\lambda^+_{min}}$ of an open Riemannian spin manifold (M, g, σ) is given by*

$$\overline{\lambda^+_{min}(M, g, \sigma)} := \lim_{r\to\infty} \lambda^+_{min}(M \setminus B_r(p), g, \sigma)$$

where $B_r(p)$ is a ball of radius r around a fixed $p \in M$ with respect to the metric g.

The existence of the limit follows from Lemma 2.0.3 and Theorem 3.0.6. The definition is independent of the chosen point p.

Here we give an example illustrating the two-dimensional case:

Example 3.0.3. Consider the cylinder $S^1 \times \mathbb{R}$ with metric $g = g_{S^1} + dt^2$, see [29, Ex. 3.4] and Example 3.4.2. Due to Example 1.6.2.iii this cylinder is conformally parabolic, and, furthermore, it is conformally compactifiable to the standard sphere. But for the spin structure that is trivial on S^1 we have $\overline{\lambda^+_{min}(S^1 \times \mathbb{R}, g, \sigma_{tr})} = \lambda^+_{min}(S^1 \times \mathbb{R}, g, \sigma_{tr}) = 0$ and, thus, due to the Theorem above the cylinder is in this case not spin conformally compactifiable. In contrast, the cylinder equipped with the spin structure that is nontrivial on S^1 is conformally spin compactifiable by two points and, hence, with Lemma 3.1.1 and Lemma 2.0.4 we have $\lambda^+_{min}(S^1 \times \mathbb{R}, g, \sigma_{nt}) = \overline{\lambda^+_{min}(S^1 \times \mathbb{R}, g, \sigma_{nt})} = \lambda^+_{min}(S^2, g_{st}, \chi_{st})$.

Remark 3.0.4. Theorem 2.0.2 gives another obstruction to spin conformal compactification:
If $\lambda^+_{min}(M, g, \sigma) = 0$, it cannot be spin conformally compactified. That follows from Lemma 2.0.3 since as a subset of a closed manifold its invariant would be greater than zero.

To prove Theorem 3.0.1 we use an Aubin-type inequality that is an analog to the original Aubin inequality [12, Thm. 9].

Theorem 3.0.5. *Let (M, g, σ) be a closed Riemannian spin manifold of dimension $n \geq 2$. Set $q = \frac{2n}{n+1}$ and $\lambda_n = \lambda^+_{min}(S^n)^{-1}$. Then for every $\epsilon > 0$ there exists a constant $c(\epsilon)$ such that for all $\phi \in C^\infty(M, S)$ we have*

$$(D^M\phi, \phi) \leq (1+\epsilon)\lambda_n \parallel D^M\phi \parallel_q^2 + c(\epsilon) \parallel \phi \parallel_q^2 .$$

Moreover, we will also prove the existence of the upper bound of λ^+_{min} for two-dimensional manifolds. So we will have for dimensions $n \geq 2$ and all manifolds, closed or open:

Theorem 3.0.6. *For any Riemannian spin manifold (M, g, σ) with dimension $n \geq 2$ it holds*

$$\lambda^+_{min}(M, g, \sigma) \leq \lambda^+_{min}(S^n, g_{st}, \chi_{st}).$$

For closed manifolds this was proven for $n > 2$ in [3, Thm. 3.1] and [6, Thm. 1.1] and for $n = 2$ in [28, Cor. 1.3]. For open manifolds with $n > 2$ we proved this in [29, Thm. 1.2].

Furthermore, we collect some other properties for λ^+_{min}. We examine the behaviour of λ^+_{min} when subsets of lower dimensions are removed. Moreover, in two dimensions we state the Bär inequality for open manifolds.

3.1 Stability of λ^+_{min} when removing subsets

In this section we examine the question which sets can be cut out of a manifold of dimension $n \geq 2$ without changing its λ^+_{min}-invariant or even without changing λ^+_1. The easiest case, cutting out a point, was considered in [29] for dimensions bigger than 2. Before considering subsets of higher dimensions we will extend this result to surfaces.

Lemma 3.1.1. *Let (M, g, σ) be a Riemannian spin manifold of dimension $n \geq 2$. Fix a point $p \in M$. Then*

$$\lambda^+_1(M \setminus \{p\}, g, \sigma) = \lim_{\beta \to 0} \lambda^+_1(M \setminus B_\beta(p), g, \sigma) = \lambda^+_1(M, g, \sigma)$$

and

$$\lambda^+_{min}(M \setminus \{p\}, g, \sigma) = \lim_{\beta \to 0} \lambda^+_{min}(M \setminus B_\beta(p), g, \sigma) = \lambda^+_{min}(M, g, \sigma)$$

with $B_\beta(p)$ being a closed ball around p of radius β with respect to the metric g.

Proof. The proof is based on the idea of B. Ammann to use a logarithmic cut-off function. The structure of the proof remains the same as in [29, Lemma 2.1]: Lemma 2.0.3 shows that $\lambda^+_{min}(M \setminus B_\beta(p), g, \sigma) \geq \lambda^+_{min}(M, g, \sigma)$ for all β and, analogously, the inequality holds with λ^+_1.

Now we show that for each spinor $\phi \in C^\infty_c(M, S)$ the quotient $\frac{\|D\phi\|^2}{(D\phi,\phi)}$ can be approximated by spinors from $C^\infty_c(M \setminus \{p\}, S)$ that are obtained from ϕ by a cut-off near p. As cut-off function ρ we now use

$$\rho_{a,\epsilon}(x) = \begin{cases} 0 & \text{for } r < a\epsilon \\ 1 - \delta \ln \frac{\epsilon}{r} & \text{for } a\epsilon \leq r \leq \epsilon \\ 1 & \text{for } \epsilon < r \end{cases}$$

where $r := d(x, p)$ is the distance from x to p with respect to the metric g. The constant a fulfills $a < 1$, and δ is chosen such that $\rho(a\epsilon) = 0$, i.e. $a = e^{-\frac{1}{\delta}}$. Then ρ is continuous, constant outside a compact set and, thus, Lipschitz. Hence, for $\phi \in C^\infty_c(M, S)$ the spinor $\rho_{a,\epsilon}\phi$ is contained in $H^r_1(M, S)$ for all $1 \leq r \leq \infty$.

For every $\phi \in C^\infty_c(M, S)$ we define $\phi_{a,\epsilon} := \rho_{a,\epsilon}\phi \in H^r_1(M, S)$ for every $1 < r < \infty$. This spinor is compactly supported in $M \setminus B_{a\epsilon}(p)$. Since $\rho_{a,\epsilon} \nearrow 1$, we have

$$\| \phi_{a,\epsilon} - \phi \| = \| (\rho_{a,\epsilon} - 1)\phi \| \to 0$$

for $a \to 0$. Furthermore,

$$\begin{aligned} \| D\phi_{a,\epsilon} - D\phi \| &\leq \| (\rho_{a,\epsilon} - 1)D\phi \| + \| \nabla\rho_{a,\epsilon} \cdot \phi \| \\ &\leq \| (\rho_{a,\epsilon} - 1)D\phi \| + \sup_{B_\epsilon(p)} |\phi| \;\; \| \nabla\rho_{a,\epsilon} \| . \end{aligned}$$

Provided that $\| \nabla\rho_{a,\epsilon} \| \to 0$ for $a \to 0$, we obtain that $D\phi_{a,\epsilon} \to D\phi$ in L^2 for $a \to 0$. Using the Hölder inequality we then have

$$|(D\phi_{a,\epsilon}, \phi_{a,\epsilon}) - (D\phi, \phi)| \leq \| D\phi_{a,\epsilon} - D\phi \| \| \phi_{a,\epsilon} \| + \| D\phi \| \| \phi_{a,\epsilon} - \phi \| \to 0$$

for $a \to 0$ and, therefore, $\lambda^+_1(M \setminus B_\beta(p), g, \sigma) \to \lambda^+_1(M, g, \sigma)$.

Thus, it remains to consider the term including $\nabla\rho_{a,\epsilon}$:

$$\begin{aligned} \int_{B_\epsilon \setminus B_{a\epsilon}} |\nabla\rho_{a,\epsilon}|^2 \mathrm{dvol}_g &\leq c\omega_{n-1} \int_{a\epsilon}^{\epsilon} \frac{\delta^2}{r^2} r^{n-1} dr \\ &= \begin{cases} c'\delta^2(\ln\epsilon - \ln(a\epsilon)) = -c'\delta^2 \ln a = c'\delta & \text{for } n = 2 \\ c\omega_{n-1}\delta^2 \int_{a\epsilon}^{\epsilon} r^{n-3} dr = c'' r^{n-2}|^{\epsilon}_{a\epsilon} & \text{else} \end{cases} \to 0 \end{aligned}$$

for $\delta \to 0$ (i.e. $a \to 0$) where ω_{n-1} is the volume of the standard sphere S^{n-1} and c, c', c'' are positive constants. The constant c in the first inequality arises when dvol_g is compared with the volume element of the Euclidean space via the exponential map $B_\epsilon(p) \to \mathbb{R}^n$.

For the proof of the second equality we observe that the first equality already implies that $\lambda^+_{min}(M \setminus \{p\}, g, \sigma) \leq \lambda^+_{min}(M, g, \sigma)$. The converse inequality is obtained from Lemma 2.0.3. $\square$

The following is a generalization of Lemma 3.1.1 to higher codimensions.

Lemma 3.1.2. *Let (M, g, σ) be a Riemannian spin manifold of dimension $n \geq 2$. Fix a closed and bounded subset $\Omega \subset M$ that is contained in a countable union of m-dimensional submanifolds $(m \leq n-2)$ that has locally finite m-dimensional Hausdorff measure. Then*

$$\lambda^+_1(M \setminus \Omega, g, \sigma) = \lambda^+_1(M, g, \sigma)$$

and

$$\lambda^+_{min}(M \setminus \Omega, g, \sigma) = \lambda^+_{min}(M, g, \sigma).$$

Definition 3.1.3. *[18, p. 189] A subset of a manifold is called* countably m-C^∞-rectifiable *if it is contained in a countable union of smooth m-dimensional submanifold.*

Proof of Lemma 3.1.2: We can assume Ω to be itself this countable union of closed smooth $(n-2)$-dimensional submanifolds described in the assumptions, since for every $\Omega' \subset \Omega$ we have with Lemma 2.0.3 $\lambda^+_1(M) \leq \lambda^+_1(M \setminus \Omega') \leq \lambda^+_1(M \setminus \Omega)$ and $\lambda^+_{min}(M) \leq \lambda^+_{min}(M \setminus \Omega') \leq \lambda^+_{min}(M \setminus \Omega)$.
Furthermore, we define $\rho_{a,\epsilon}$ as in Lemma 3.1.1 but with $r := d(x, \Omega)$ being the distance from x to the set Ω. With $B_\epsilon := \{x \in M \mid d(x, \Omega) \leq \epsilon\}$ we set $B^2_\epsilon(p) := \{x \in B_\epsilon \mid d(x, p) = d(x, \Omega)\}$ for a point $p \in \Omega$.
For sufficiently small ϵ each $B^2_\epsilon(p)$ can be identified with a subset of $B_\epsilon(0) \subset \mathbb{R}^2$.
Then we can calculate

$$\begin{aligned}
\int_{B_\epsilon} |\nabla \rho_{a,\epsilon}|^2 \mathrm{dvol}_g &\leq \mathrm{vol}_{n-2}(\Omega) \sup_{x \in \Omega} \int_{B^2_\epsilon(x) \setminus B^2_{a\epsilon}(x)} |\nabla \rho_{a,\epsilon}|^2 \mathrm{dvol}_{g_2} \\
&\leq c \mathrm{vol}_{n-2}(\Omega) \int_{B_\epsilon(0) \setminus B_{a\epsilon}(0)} |\nabla \rho_{a,\epsilon}|^2 \mathrm{dvol}_{g_E} \\
&\leq c' \int_{a\epsilon}^{\epsilon} \frac{\delta^2}{r^2} r dr \to 0 \quad \text{for } a \to 0
\end{aligned}$$

where vol_{n-2} denotes the $(n-2)$-dimensional volume, g_2 is the metric on $B^2_\epsilon(x)$ induced from g. The positive constants c, c' arise from $\mathrm{vol}_{n-2}(\Omega)$ and the comparison of dvol_{g_2} with the volume element of the Euclidean metric.
The rest of the proof is the same as the one of Lemma 3.1.1, only the balls $B_r(p)$ around p are replaced by neighbourhoods $B_r(\Omega)$ of Ω. $\square$

Remark 3.1.4.
i) We expect Lemma 3.1.2 to be wrong for codimension one although we have no counterexample yet. It clearly does not hold for unbounded subsets of codimension

one since it already fails for a real line embedded into the standard cylinder with vanishing λ^+_{min}-invariant, see Example 3.4.2. When leaving out a whole real line we obtain $\mathbb{R}^2$ with $\lambda^+_{min}(\mathbb{R}^2) = \lambda^+_{min}(S^2)$, cf. Example 3.1.6.
ii) The analog of Lemma 3.1.2 for the Yamabe invariant on Riemannian (not necessarily spin) manifolds can be proven in the same way.

Remark 3.1.5. Since we proved Lemma 3.1.1 now also for $n = 2$, the proof of Lemma 2.0.4 [29, Lem. 2.1.iii] carries over, too:
Let Ω be an open domain of the standard sphere S^n ($n \geq 2$). For all $\epsilon > 0$ and fixed $p \in S^n$ there always exists a spin conformal map $\Phi_\epsilon : (S^n, g_{st}) \to (S^n, g_{st})$ with $S^n \setminus B_\epsilon(p) \subset \Phi_\epsilon(\Omega)$. Thus, we have then $\lambda^+_{min}(\Omega) \geq \lambda^+_{min}(S^n)$ and for $\epsilon \to 0$

$$\lambda^+_{min}(\Omega) = \lambda^+_{min}(\Phi_\epsilon(\Omega)) \leq \lambda^+_{min}(S^n \setminus B_\epsilon(p)) \to \lambda^+_{min}(S^n).$$

Hence, we obtained

$$\lambda^+_{min}(\Omega, g_{st}, \chi_{st}) = \lambda^+_{min}(S^n, g_{st}, \chi_{st}).$$

Example 3.1.6. The Euclidean space and the hyperbolic space each admit exactly one spin structure, cf. Remark 1.1.1. $(\mathbb{R}^n, g_E)$ is spin conformally equivalent to $(S^n \setminus \{*\}, g_{st})$. Hence, Lemma 3.1.1 ensures that $\lambda^+_{min}(\mathbb{R}^n, g_E) = \lambda^+_{min}(S^n, g_{st})$. Moreover, there is a spin conformal map $(\mathbb{H}^n, g_{\mathbb{H}}) \to (B_1(*) \subset S^n, g_{st})$. Thus, Lemma 2.0.4 and Remark 3.1.5 imply $\lambda^+_{min}(\mathbb{H}^n, g_{\mathbb{H}}) = \lambda^+_{min}(S^n, g_{st})$.

While $\lambda^+_{min}(S^n, g_{st})$ for $n > 2$ was computed using the Hijazi inequality, the value of $\lambda^+_{min}(S^2, g_{st})$ was obtained from Bär's inequality:
Bär proved in [15, Thm. 2] that for any Riemannian spin manifold homeomorphic to S^2 each Dirac eigenvalue λ satisfies

$$\lambda^2 area(M) \geq 4\pi.$$

Since for the standard metric on S^2 equality is attained, this implies $\lambda^+_{min}(S^2) = 2\sqrt{\pi}$, cf. (2.2).
Next, we prove this inequality for the corresponding case of open manifolds:

Theorem 3.1.7. *Let (M, g, σ) be a Riemannian spin surface of finite area that is homeomorphic to $\mathbb{R}^2$. Then*

$$\lambda_1^+(g)^2 area(M, g) \geq 4\pi.$$

Proof. By the uniformization theorem (M, g) is either conformally equivalent to the standard Euclidean or hyperbolic space. In both cases we have $\lambda^+_{min}(M, g, \sigma) = \lambda^+_{min}(S^2, g_{st}) = 2\sqrt{\pi}$. That implies $\lambda_1^+(M, g, \sigma)^2 area(M, g) \geq 4\pi$. □

Remark 3.1.8. Let (M, g) be $(\mathbb{R}^2, f^2 g_E)$. Then the inverse stereographic projection π^{-1} is a (spin-structure preserving) isometry from $(\mathbb{R}^2, f^2 g_E)$ to $(S^2 \setminus \{*\}, F^2 g_{st})$ with $f^2 = \frac{4}{(|x|^2+1)^2)}(F^2 \circ \pi^{-1})$. Thus, $\lambda_1^+(\mathbb{R}^2, f^2 g_E) = \lambda_1^+(S^2 \setminus \{*\}, F^2 g_{st})$. If F

can be continued through $\{*\}$ to the whole sphere, we get from Lemma 3.1.1 that $\lambda_1^+(\mathbb{R}^2, f^2 g_E) = \lambda_1^+(S^2, F^2 g_{st})$. With Bär's inequality we see that in this case equality can only be achieved for $F \equiv 1$, i.e. for $f = \frac{2}{|x|^2+1}$.
If equality can also be achieved for a function F with an irremovable singularity at $\{*\}$ or for $(\mathbb{H}^2, f^2 g_{\mathbb{H}})$, we do not know yet.

3.2 An Aubin-type inequality

In this section we want to prove a spin version of an inequality of Aubin which we state now for the sake of comparison:

Theorem 3.2.1. *[12, Thm. 9] Let (M, g) be a closed Riemannian manifold of dimension $n \geq 3$. Furthermore, let $p' = \frac{2n}{n-2}$ and $\sigma_n = Q(S^n, g_{st})^{-1}$. Then for all $\epsilon > 0$ there exists a constant $c(\epsilon)$ such that for all $v \in C^\infty(M)$*

$$\|v\|_{p'}^2 \leq (1+\epsilon)\sigma_n \|\nabla v\|_2^2 + c(\epsilon)\|v\|_2^2.$$

To prove the spin analog we start by considering the Euclidean space: On $\mathbb{R}^n$ we have

$$(D\phi, \phi) \leq \lambda_n \|D\phi\|_q^2 \tag{3.1}$$

for all $\phi \in C_c^\infty(\mathbb{R}^n, S)$ and with $\lambda_n \lambda_{min}^+(S^n) = 1$. This follows from the description of λ_{min}^+ by the variational problem (2.1) and the equality $\lambda_{min}^+(\mathbb{R}^n, g_E) = \lambda_{min}^+(S^n, g_{st})$, cf. Example 3.1.6.
But before proving the generalization of this statement, Theorem 3.0.5, we want to mention the following Lemma where $p = \frac{2n}{n-1}$ is the conjugate exponent of q:

Lemma 3.2.2. *Let (M, g, σ) be a closed Riemannian spin manifold. Then there exists a constant $C > 0$ such that for all $\psi \in C^\infty(M, S) \cap (\ker D^M)^\perp$ the following inequalities hold*
i) $\|\nabla^M \psi\|_q \leq C\|D^M \psi\|_q$
ii) $\|\psi\|_q \leq C\|D^M \psi\|_q$
iii) $\|\psi\|_p \leq C\|D^M \psi\|_q$.

This Lemma was stated in [7, Prop. 2.4] for invertible Dirac operators and in [8, Lem. 5.1] for the special case of the two-dimensional torus. The proof of ii), we'll give, follows mainly the arguments of [8] where the condition that the spinor is (L^2-)perpendicular to $\ker D^M$ replaces the invertibility of the Dirac operator assumed there.

Proof. ii) We give a proof by contradiction. Assume that there is a sequence of spinors $\phi_i \in C^\infty(M, S) \cap (\ker D^M)^\perp$ with $\|\phi_i\|_q = 1$ but $\|D^M \phi_i\|_q \leq \frac{1}{i}$. Then the sequence $\{\phi_i\}$ is bounded in H_1^q. Since H_1^q is reflexive, there exists a spinor ϕ that is the weak limit of (a subsequence of) ϕ_i in H_1^q. The inclusion $H_1^q \hookrightarrow L^q$ is a compact operator, see Theorem 1.3.2. Hence, $\phi_i \to \phi$ converges even strongly in L^q. Thus,

$\|\phi\|_q = 1$ and, in particular, $\phi \neq 0$. Together with the weak convergence of ϕ_i in H_1^q this implies

$$\|D^M\phi\|_q \leq \liminf \|D^M\phi_i\|_q = 0.$$

Thus, ϕ is a harmonic spinor and, therefore, due to elliptic regularity theory $\phi \in C^\infty$, cf. Remark 1.2.2.i. Then the Hölder inequality implies for all $\psi \in \ker D^M$ that

$$(\phi, \psi) = (\phi - \phi_i, \psi) \leq \|\phi - \phi_i\|_q \|\psi\|_p \to 0.$$

So we obtain $\phi \perp \ker D^M$. But this is a contradiction to ϕ being harmonic.
i) Since the Sobolev norms with respect to D^M and ∇^M are equivalent, cf. Lemma 1.3.1, there exists a constant $c > 0$ such that for all $\phi \in H_1^q(M, S)$ we have

$$\|\phi\|_q + \|\nabla^M\phi\|_q \leq c(\|\phi\|_q + \|D^M\phi\|_q).$$

If $c \leq 1$, we are done. For $c > 1$ we get the claimed inequality by using ii).
iii) Since $H_1^q \to L^p$ is a continuous Sobolev embedding, there exists a constant $c > 0$ with

$$\|\psi\|_p \leq c(\|\psi\|_q + \|D^M\psi\|_q)$$

for all $\psi \in C^\infty(M, S)$. With i) and ii) the claim is obtained. □

Remark 3.2.3. i) and ii) of Lemma 3.2.2 hold for every $q > 1$, iii) for all $p \leq \frac{qn}{n-q}$. The proof is the same, and the inequality is only used to assert the existence of the Sobolev embedding $H_1^q \to L^p$.

Now we can prove the Aubin-type inequality:

Proof of Theorem 3.0.5. The proof uses a similar ansatz as the proof of the original Aubin inequality for functions [37, Thm. 2.3] – the idea of a covering on which all operations can be compared to the Euclidean case.

First we observe that it is sufficient to prove the inequality for spinors perpendicular to $\ker D^M$ since the part of the spinor parallel to the kernel only enlarges the right handside.
Next we fix $\epsilon > 0$ sufficiently small. For every point $x \in M$ we choose $U_x := \exp_x^{-1}(B_\epsilon(0))$ equipped with geodesic normal coordinates. In order to compare the metric and resulting quantities, we use the development of the metric in these coordinates, see Section 1.5. Then, $\mathrm{dvol}_g = f \mathrm{dvol}_{g_E}$ $(=: f dx)$ with $f \in C^\infty(M)$ and $(1+\epsilon)^{-1} < f < (1+\epsilon)$. Since M is closed, we can fix a finite subcover $\{U_i\}$ of $\{U_x\}_{x \in M}$ and a subordinate partition of unity $\{\alpha_i^2\}$ with $\alpha_i \in C_c^\infty(U_i)$ and $\sum \alpha_i^2 = 1$.

In the course of this proof the L^r-norms $\|.\|_r$ are always taken with respect to the metric g.

$$
\begin{aligned}
(D^M\phi,\phi) &= \int_M \sum_l \alpha_l^2 < D^M\phi,\phi > \mathrm{dvol}_g \\
&= \sum_l \int_{U_l} < D^M(\alpha_l\phi) - \nabla\alpha_l\cdot\phi, \alpha_l\phi > \mathrm{dvol}_g \\
&= \sum_l \int_{U_l} < D^M(\alpha_l\phi)_\perp, (\alpha_l\phi)_\perp > \mathrm{dvol}_g \\
&= \sum_l \int_{U_l} \mathrm{Re} < D^M(\alpha_l\phi)_\perp, (\alpha_l\phi)_\perp > f dx \\
&\le \sum_l \int_{U_l} \mathrm{Re} < D^M(\alpha_l\phi)_\perp, (\alpha_l\phi)_\perp > dx + \epsilon \sum_l \int_{U_l} | < D^M(\alpha_l\phi)_\perp, (\alpha_l\phi))_\perp > | dx \\
&\le \sum_l \int_{U_l} \mathrm{Re} < D^M(\alpha_l\phi)_\perp, (\alpha_l\phi)_\perp > dx + \epsilon(1+\epsilon) \sum_l \|D^M(\alpha_l\phi)_\perp\|_q \|(\alpha_l\phi)_\perp\|_p \\
&\le \sum_l \int_{U_l} \mathrm{Re} < D^M(\alpha_l\phi)_\perp, (\alpha_l\phi)_\perp > dx + \epsilon(1+\epsilon) C \sum_l \|D^M(\alpha_l\phi)_\perp\|_q^2.
\end{aligned}
$$

The third equality follows since $< \nabla f\cdot\psi,\psi > \in i\mathbb{R}$ [31, Lem. 3.1] for all $f \in C^\infty(M)$ but $(D^M\phi,\phi) \in \mathbb{R}$. Furthermore, $\psi_\perp$ denotes the part of the spinor ψ that is perpendicular to $\ker D^M$. Moreover to obtain the first inequality, we used that for a real-valued function $h \in C^\infty(M)$

$$
\begin{aligned}
\int_{U_l} hf dx &\le (1+\epsilon)^{-1} \int_{\{h<0\}\cap U_l} h dx + (1+\epsilon) \int_{\{h>0\}\cap U_l} h dx \\
&\le (1-\epsilon) \int_{\{h<0\}\cap U_l} h dx + (1+\epsilon) \int_{\{h>0\}\cap U_l} h dx = \int_{U_l} h dx + \epsilon \int_{U_l} |h| dx.
\end{aligned}
$$

The second last inequality was obtained from the Hölder inequality. The factor $(1+\epsilon)$ occurs since the L^r-norms refer to the volume form $f dx$. The last inequality is deduced from Lemma 3.2.2.iii.
All constants C, C_i or c_i arising here in this proof are positive.

Recall from (1.8) and (1.12) that the Bourguignon-Gauduchon-Trivialization reads

$$
D^M\psi = \underbrace{D^{\mathbb{R}^n}\psi}_{A} + \underbrace{\frac{1}{4}\sum \tilde{\Gamma}_{ij}^k \partial_i \cdot \partial_j \cdot \partial_k \psi}_{B} + \underbrace{\sum (b_i^j - \delta_i^j)\partial_i \cdot \nabla_{\partial_j}\psi}_{C},
$$

where we already identify spinors in $\Gamma(M,S)$ with spinors in $\Gamma(\mathbb{R}^n, S)$, see Section 1.5.
In the development of the geodesic normal coordinates of U we can use (1.9) and (1.10) to estimate

$$
\left| \sum \tilde{\Gamma}_{ij}^k \partial_i \partial_j \partial_k \right| \le \epsilon a_1 \text{ and } |b_i^j - \delta_i^j| \le \epsilon^2 a_2 \tag{3.2}
$$

where a_1, a_2 are positive constants arising from the curvature. Thus, these constants only depend on (M, g) not on ϵ.

We start to estimate the summands arising from A. From (3.1) we find

$$\begin{aligned}\sum_l \int_{U_l} < D^{\mathbb{R}^n}(\alpha_l\phi), (\alpha_l\phi) > dx &\leq \lambda_n \sum_l \left(\int_{U_l} |D^{\mathbb{R}^n}(\alpha_l\phi)|^q dx\right)^{\frac{2}{q}} \\ &\leq \lambda_n(1+\epsilon)^{\frac{2}{q}} \sum_l \|D^{\mathbb{R}^n}(\alpha_l\phi)\|_q^2.\end{aligned}$$

Thus, we have

$$\begin{aligned}&\sum_l \int_{U_l} < D^{\mathbb{R}^n}(\alpha_l\phi)_\perp, (\alpha_l\phi)_\perp > dx \\ &\quad= \sum_l \int_{U_l} [< D^{\mathbb{R}^n}(\alpha_l\phi), (\alpha_l\phi) > -2\mathrm{Re} < D^{\mathbb{R}^n}(\alpha_l\phi)_{ker}, (\alpha_l\phi)_\perp > \\ &\qquad - < D^{\mathbb{R}^n}(\alpha_l\phi)_{ker}, (\alpha_l\phi)_{ker} >]dx \\ &\quad\leq \lambda_n \sum_l \left(\int_{U_l} |D^{\mathbb{R}^n}(\alpha_l\phi)|^q dx\right)^{\frac{2}{q}} + \sum_l c\Big|\int_{U_l} < D^{\mathbb{R}^n}(\alpha_l\phi)_{ker}, \alpha_l\phi > dx\Big| \\ &\quad\leq \lambda_n(1+\epsilon)^{\frac{2}{q}} \sum_l \|D^{\mathbb{R}^n}(\alpha_l\phi)\|_q^2 + \sum_l c\|D^{\mathbb{R}^n}(\alpha_l\phi)_{ker}\|_2\|\alpha_l\phi\|_2,\end{aligned}$$

where ψ_{ker} is the part of a spinor ψ that belongs to the kernel of D^M and c is a positive constant. Equation (3.1) is not applicable on the perpendicular part of $\alpha_l\phi$ since this spinor is in general not compactly supported.

Furthermore,

$$\begin{aligned}\|D^{\mathbb{R}^n}(\alpha_l\phi)_\perp\|_q &= \|D^M(\alpha_l\phi)_\perp - \frac{1}{4}\sum \tilde{\Gamma}_{ij}^k \partial_i \cdot \partial_j \cdot \partial_k(\alpha_l\phi)_\perp \\ &\quad - \sum_{i,j}(b_i^j - \delta_i^j)\partial_i \cdot \nabla_{\partial_j}(\alpha_l\phi)_\perp\|_q \\ &\leq \|D^M(\alpha_l\phi)_\perp\|_q + \|\frac{1}{4}\sum \tilde{\Gamma}_{ij}^k \partial_i \cdot \partial_j \cdot \partial_k(\alpha_l\phi)_\perp\|_q \\ &\quad + \|\sum_{i,j}(b_i^j - \delta_i^j)\partial_i \cdot \nabla_{\partial_j}(\alpha_l\phi)_\perp\|_q.\end{aligned}$$

Using (3.2) we get

$$\|\frac{1}{4}\sum \tilde{\Gamma}_{ij}^k \partial_i \cdot \partial_j \cdot \partial_k(\alpha_l\phi)_\perp\|_q \leq \epsilon c_1\|(\alpha_l\phi)_\perp\|_q$$

and with $b_j^a \nabla^M_{\partial_a}\phi = b_j^a \nabla_{\partial_a}\phi + \frac{1}{4}\sum \tilde{\Gamma}_{jk}^m \partial_k \cdot \partial_m \cdot \phi$, cf. (1.11),

$$
\begin{aligned}
\| \sum_{i,j} (b_i^j - \delta_i^j)\partial_i \cdot \nabla_{\partial_j}(\alpha_l \phi)_\perp \|_q &\leq \epsilon^2 c_2 \|\nabla_{\partial_j}(\alpha_l\phi)_\perp\|_q \\
&= \epsilon^2 c_2 \|\nabla^M_{\partial_j}(\alpha_l\phi)_\perp - \frac{1}{4}\sum (b_j^a)^{-1}\tilde{\Gamma}^m_{ak}\partial_k \cdot \partial_m \cdot (\alpha_l\phi)_\perp\|_q \\
&\leq \epsilon^2 c_2 \|\nabla^M_{\partial_j}(\alpha_l\phi)_\perp\|_q + \epsilon^2 c_2 \|\frac{1}{4}\sum (b_j^a)^{-1}\tilde{\Gamma}^m_{ak}\partial_k \cdot \partial_m \cdot (\alpha_l\phi)_\perp\|_q \\
&\leq \epsilon^2 c_2 \|\nabla^M (\alpha_l\phi)_\perp\|_q + \epsilon^3 c_3 \|(\alpha_l\phi)_\perp\|_q \\
&\leq \epsilon^2 C_2 \|D^M (\alpha_l\phi)_\perp\|_q + \epsilon^3 c_3 \|(\alpha_l\phi)_\perp\|_q
\end{aligned}
$$

where the fourth line is again obtained from (3.2) and (1.10) and the last follows from Lemma 3.2.2.i. Hence, we find

$$\|D^{\mathbb{R}^n}(\alpha_l\phi)_\perp\|_q \leq (1+\epsilon^2 C_2)\|D^M(\alpha_l\phi)_\perp\|_q + \epsilon C_1 \|(\alpha_l\phi)_\perp\|_q.$$

Similarly and bearing in mind that $D^M(\alpha_l\phi)_{ker} = 0$ and $\|\psi\|_q + \|\nabla^M\psi\|_q \leq c(\|\psi\|_q + \|D^M\psi\|_q)$, we obtain

$$\|D^{\mathbb{R}^n}(\alpha_l\phi)_{ker}\|_q \leq \epsilon C_1 \|(\alpha_l\phi)_{ker}\|_q,$$

and, analogously,

$$\|D^{\mathbb{R}^n}(\alpha_l\phi)_{ker}\|_2 \leq \epsilon C_1' \|(\alpha_l\phi)_{ker}\|_2.$$

Thus, for ϵ sufficiently small we have

$$
\begin{aligned}
&\sum_l \int_{U_l} < D^{\mathbb{R}^n}(\alpha_l\phi)_\perp, (\alpha_l\phi)_\perp > dx \\
&\quad \leq \lambda_n (1+\epsilon)^{\frac{2}{q}} \sum_l \left(\|D^{\mathbb{R}^n}(\alpha_l\phi)_\perp\|_q + \|D^{\mathbb{R}^n}(\alpha_l\phi)_{ker}\|_q \right)^2 \\
&\qquad + \sum_l c \|D^{\mathbb{R}^n}(\alpha_l\phi)_{ker}\|_2 \|\alpha_l\phi\|_2 \\
&\quad \leq \lambda_n (1+\epsilon)^{\frac{2}{q}} \sum_l \left((1+\epsilon^2 C_2)\|D^M(\alpha_l\phi)_\perp\|_q + \epsilon C\|(\alpha_l\phi)\|_q \right)^2 + \sum_l \epsilon c' \|\alpha_l\phi\|_2^2 \\
&\quad \leq \lambda_n (1+\epsilon)^{\frac{2}{q}+1}(1+\epsilon^2 C_2)^2 \sum_l \|D^M(\alpha_l\phi)\|_q^2 + c_3(\epsilon)\sum_l \|\alpha_l\phi\|_q^2 + \epsilon c'\|\phi\|_2^2 \\
&\quad \leq \lambda_n (1+\epsilon)^{\frac{2}{q}+1}(1+\epsilon^2 C_2)^2 \sum_l \|D^M(\alpha_l\phi)\|_q^2 + c_3(\epsilon)\sum_l \|\alpha_l^2|\phi|^2\|_{\frac{q}{2}} + \epsilon c'\|\phi\|_p^2 \\
&\quad \leq \lambda_n (1+\epsilon)^{\frac{2}{q}+1}(1+\epsilon^2 C_2)^2 \sum_l \|D^M(\alpha_l\phi)\|_q^2 + c_3(\epsilon)\|\sum_l \alpha_l^2|\phi|^2\|_{\frac{q}{2}} + \epsilon c''\|D^M\phi\|_q^2.
\end{aligned}
$$

The third inequality is obtained by using $(a+b)^2 \leq (1+\epsilon)a^2 + (1+\epsilon^{-1})b^2$. The $c_i(\epsilon)$'s are positive constants depending only on ϵ. The last line uses the Minkowski inequality for the quasi-norm $\|.\|_{\frac{q}{2}}$ $(0 < \frac{q}{2} < 1)$ of positive functions [30, Thm. 198] and again Lemma 3.2.2.iii.

With $\sum_l |\nabla\alpha_l|^2 \leq c_4(\epsilon)$ we have

$$\begin{aligned}
\sum_l \|D^M(\alpha_l\phi)\|_q^2 &= \sum_l \|\alpha_l D^M\phi + \nabla\alpha_l\cdot\phi\|_q^2 \\
&\leq (1+\epsilon)\sum_l \|\alpha_l D^M\phi\|_q^2 + (1+\epsilon^{-1})\sum_l \|\nabla\alpha_l\phi\|_q^2 \\
&\leq (1+\epsilon)\sum_l \|\alpha_l^2|D^M\phi|^2\|_{\frac{q}{2}} + (1+\epsilon^{-1})\sum_l \||\nabla\alpha_l|^2|\phi|^2\|_{\frac{q}{2}} \\
&\leq (1+\epsilon)\|\sum_l \alpha_l^2|D^M\phi|^2\|_{\frac{q}{2}} + (1+\epsilon^{-1})\|\sum_l |\nabla\alpha_l|^2|\phi|^2\|_{\frac{q}{2}} \\
&\leq (1+\epsilon)\|D^M\phi\|_q^2 + c_5(\epsilon)\|\phi\|_q^2. \qquad (3.3)
\end{aligned}$$

After a rescaling of ϵ we finally obtain an estimate for the summand A:

$$\sum_l \int_{U_l} < D^{\mathbb{R}^n}(\alpha_l\phi)_\perp, (\alpha_l\phi)_\perp > dx \leq \lambda_n(1+\epsilon)\|D^M\phi\|_q^2 + c_6(\epsilon)\|\phi\|_q^2.$$

Next, we approximate the terms coming from B and C:

$$\begin{aligned}
&(1+\epsilon)\mathrm{Re}\sum_l \int_{U_l} < \sum_{i,j,k}\frac{1}{4}\tilde{\Gamma}_{ij}^k\partial_i\cdot\partial_j\cdot\partial_k\cdot(\alpha_l\phi)_\perp, (\alpha_l\phi)_\perp > dx \\
&\leq (1+\epsilon)^2\sum_l \int_M |\sum_{i,j,k}\frac{1}{4}\tilde{\Gamma}_{ij}^k\partial_i\cdot\partial_j\cdot\partial_k||(\alpha_l\phi)_\perp|^2\mathrm{dvol}_g \\
&\leq (1+\epsilon)^2\sum_l \int_M |\sum_{i,j,k}\frac{1}{4}\tilde{\Gamma}_{ij}^k\partial_i\cdot\partial_j\cdot\partial_k||\alpha_l\phi|^2\mathrm{dvol}_g \\
&\leq \epsilon c_7\int_M |\phi|^2\mathrm{dvol}_g \leq \epsilon c_7\|\phi\|_q\|\phi\|_p \leq \epsilon c_7 C^2\|D^M\phi\|_q^2
\end{aligned}$$

where the first inequality uses Cauchy-Schwarz, the fourth one is the Hölder inequality and the last inequality is obtained from Lemma 3.2.2.ii and .iii. Further,

$$\begin{aligned}
&(1+\epsilon)\mathrm{Re}\sum_l \int_{U_l}\sum_{i,j} < (b_i^j - \delta_i^j)\partial_i\cdot\nabla_{\partial_j}(\alpha_l\phi)_\perp, (\alpha_l\phi)_\perp > dx \\
&\leq (1+\epsilon)\mathrm{Re}\sum_l \int_{U_l}\sum_{i,j} < (b_i^j - \delta_i^j)\partial_i\cdot\Big(\nabla^M_{\partial_j}(\alpha_l\phi)_\perp \\
&\quad -\frac{1}{4}\sum (b_j^a)^{-1}\tilde{\Gamma}_{jk}^m\partial_k\cdot\partial_m\cdot(\alpha_l\phi)_\perp\Big), (\alpha_l\phi)_\perp > dx \\
&\leq (1+\epsilon)\mathrm{Re}\sum_l \int_{U_l}\sum_{i,j} < (b_i^j - \delta_i^j)\partial_i\cdot\nabla^M_{\partial_j}(\alpha_l\phi)_\perp, (\alpha_l\phi)_\perp > dx + \epsilon^3 c_7\|\phi\|_q\|\phi\|_p \\
&\leq (1+\epsilon)\sum_l \int_{U_l}\sum_{i,j} |b_i^j - \delta_i^j||\nabla^M(\alpha\phi)_\perp||(\alpha_l\phi)_\perp|dx + \epsilon^3 c_7\|\phi\|_q\|\phi\|_p
\end{aligned}$$

$$
\begin{aligned}
&\leq \epsilon^2 c_8 \sum_l \|\nabla^M(\alpha\phi)_\perp\|_q \|(\alpha\phi)_\perp\|_p + \epsilon^3 c_7 \|\phi\|_q \|\phi\|_p \\
&\leq \epsilon^2 c_8 C \sum_l \|D^M(\alpha_l\phi)_\perp\|_q^2 + \epsilon^3 c_9 \|\phi\|_p^2 \leq \epsilon^2 c_{10} \|D^M\phi\|_q^2 + c(\epsilon)\|\phi\|_q^2
\end{aligned}
$$

where the first inequality arises from equation (1.11). The other inequalities are obtained by combinations of the Hölder inequality, Lemma 3.2.2 and the estimates (3.2) and (3.3).

Collecting all the terms estimated above we get

$$
\begin{aligned}
(D^M\phi,\phi) &= \sum_l \int_{U_l} \mathrm{Re} < D^M(\alpha_l\phi)_\perp, (\alpha_l\phi)_\perp > dx + \epsilon(1+\epsilon)C \sum_l \|D^M(\alpha_l\phi)_\perp\|_q^2 \\
&\leq \sum_l \int_{U_l} < D^{\mathbb{R}^n}(\alpha_l\phi)_\perp, (\alpha_l\phi)_\perp > dx + \epsilon c_{11}\|D^M\phi\|_q^2 + c_{12}(\epsilon)\|\phi\|_q^2 \\
&\leq (1+\epsilon)\lambda_n \|D^M\phi\|_q^2 + c(\epsilon)\|\phi\|_q^2
\end{aligned}
$$

where the last inequality is obtained by rescaling ϵ. □

3.3 Proof of Theorem 3.0.1 and 3.0.6

The proofs of these theorems follow the ideas in [29] – even for $n = 2$. One only has to make sure that the key lemmas are valid for all $n \geq 2$. For the key lemma to the proof of Theorem 3.0.6 [29, Lem. 2.3.] nothing changes, but since we know now that $\lambda^+_{min}(\mathbb{R}^2) = \lambda^+_{min}(S^2)$, see Example 3.1.6, it is also valid for dimension two:

Lemma 3.3.1. *Let (M, g, σ) be a Riemannian spin manifold of dimension $n \geq 2$ and $p \in M$ fixed. Then*

$$\lim_{\epsilon \to 0} \lambda^+_{min}(B_\epsilon(p), g, \sigma) = \lambda^+_{min}(S^n, g_{st})$$

where $B_\epsilon(p)$ is an open ball around p and radius ϵ.

Proof. We define rescaled geodesic normal coordinates by

$$\sigma_\epsilon : T_pM \cong \mathbb{R}^n \to M, \quad \sigma_\epsilon(x) = \exp_p(\epsilon x).$$

Let $B \subset T_pM$ be an open ball around 0 with respect to the Euclidean metric g_E such that the exponential map $\exp_p$ restricted to B is a diffeomorphism and we can use the Bourguignon-Gauduchon-Trivialization. Then, $U_\epsilon := \sigma_\epsilon(B) \to \{p\}$ as $\epsilon \to 0$. Furthermore, we define the spinor $\psi_\epsilon := \epsilon^{-\frac{n-1}{2}}\psi \circ \sigma_\epsilon$ for any $\psi \in C^\infty(B, S)$ and the metric $g_\epsilon := \epsilon^{-2}\sigma_\epsilon^*(g)$, i.e. locally $(g_\epsilon)_{ij} = (g_{ij} \circ M_\epsilon)dx^i dx^j$ where M_ϵ denotes multiplication by ϵ. Then conformal invariance implies

$$\frac{(D_g\psi_\epsilon, \psi_\epsilon)_{U_\epsilon, g}}{\| D_g\psi_\epsilon \|^2_{L^q(U_\epsilon, g)}} = \frac{(D_\epsilon\psi, \psi)_{B, g_\epsilon}}{\| D_\epsilon\psi \|^2_{L^q(B, g_\epsilon)}}$$

where D_ϵ is the Dirac operator with respect to g_ϵ.
Next, we show

$$\frac{\|D_\epsilon\psi\|^2_{L^q(B,g_\epsilon)}}{(D_\epsilon\psi,\psi)_{g_\epsilon}} - \frac{\|D^{\mathbb{R}^n}\psi\|^2_{L^q(B,g_E)}}{(D^{\mathbb{R}^n}\psi,\psi)_{g_E}} \to 0 \quad \text{for } \epsilon \to 0. \tag{3.4}$$

At first, with the Bourguignon-Gauduchon-Trivialization we have

$$|D_\epsilon\psi - D^{\mathbb{R}^n}\psi| \le \frac{1}{4}\left|\sum \tilde{\Gamma}^k_{ij}(g_\epsilon)\partial_i\cdot\partial_j\cdot\partial_k\cdot\psi\right| + \left|\sum(b^j_i - \delta^j_i)\partial_j\cdot\nabla_{\partial_i}\psi\right|$$

Using $(g_\epsilon)_{ij} = g_{ij}\circ M_\epsilon$ and the expansions (1.7), (1.8) and (1.10), we obtain $|b^j_i - \delta^j_i| = \mathcal{O}(\epsilon^2)$ and $|\tilde{\Gamma}^k_{ij}(g_\epsilon)| = \mathcal{O}(\epsilon)$ which implies

$$|D_\epsilon\psi - D^{\mathbb{R}^n}\psi| \to 0 \text{ for } \epsilon \to 0.$$

Furthermore,

$$|\|D_\epsilon\psi\|_{L^q(B,g_\epsilon)} - \|D_\epsilon\psi\|_{L^q(B,g_E)}| \le \int_B |D_\epsilon\psi|^q(\mathrm{dvol}_{g_\epsilon} - \mathrm{dvol}_{g_E}) \to 0$$

since $\mathrm{dvol}_{g_\epsilon} = (1+\mathcal{O}(\epsilon))\mathrm{dvol}_{g_E}$. Summarizing we have

$$\begin{aligned}
&|\|D_\epsilon\psi\|_{L^q(B,g_\epsilon)} - \|D^{\mathbb{R}^n}\psi\|_{L^q(B,g_E)}| \\
&\le |\|D_\epsilon\psi\|_{L^q(B,g_\epsilon)} - \|D_\epsilon\psi\|_{L^q(B,g_E)}| + |\|D_\epsilon\psi\|_{L^q(B,g_E)} - \|D^{\mathbb{R}^n}\psi\|_{L^q(B,g_E)}| \\
&\to 0 \text{ for } \epsilon \to 0
\end{aligned}$$

For the denominators of (3.4) we get

$$|(D_\epsilon\psi,\psi)_{g_\epsilon} - (D^{\mathbb{R}^n}\psi,\psi)_{g_E}| \le |(D_\epsilon\psi,\psi)_{g_\epsilon}) - (D_\epsilon\psi,\psi)_{g_E})| + |((D_\epsilon - D^{\mathbb{R}^n})\psi,\psi)_{g_E}| \to 0$$

and, thus, (3.4) is shown. Hence, there exist sequences $\psi^i \in C^\infty_c(B,S)$ and $\delta_i \in \mathbb{R}_{>0}$ such that $\delta_i \to 0$ and

$$\lambda^+_{min}(S^n)^{-1} - \delta_i = \frac{(D^{\mathbb{R}^n}\psi^i,\psi^i)}{\|D^{\mathbb{R}^n}\psi^i\|^2_{L^q(B,g_E)}} = \lim_{\epsilon\to 0}\frac{(D_g\psi^i_\epsilon,\psi^i_\epsilon)}{\|D_g\psi^i_\epsilon\|^2_{L^q(U_\epsilon,g)}} \le \liminf_{\epsilon\to 0}\lambda^+_{min}(U_\epsilon,g)^{-1}.$$

Here we used $\lambda^+_{min}(S^n,g_{st}) = \lambda^+_{min}(B,g_{st})$, cf. Remark 3.1.5.
Furthermore, there exist sequences $\phi^i_\epsilon \in C^\infty_c(U_\epsilon,S)$ and $\beta_i \in \mathbb{R}_{>0}$ with $\beta_i \to 0$ such that

$$\lim_{\epsilon\to 0}(\lambda^+_{min}(U_\epsilon,g)^{-1} - \beta_i) = \lim_{\epsilon\to 0}\frac{(D_g\phi^i_\epsilon,\phi^i_\epsilon)}{\|D_g\phi^i_\epsilon\|^2_{L^q(U_\epsilon,g)}} = \frac{(D^{\mathbb{R}^n}\phi^i,\phi^i)}{\|D^{\mathbb{R}^n}\phi^i\|^2_{L^q(B,g_E)}} \le \lambda^+_{min}(S^n)^{-1}.$$

For i tending to infinity we obtain from these estimates that $\lambda^+_{min}(U_\epsilon,g)$ tends to $\lambda^+_{min}(S^n,g_{st})$. Since for each $\epsilon > 0$ there exists an $\epsilon' > 0$ with $B_{\epsilon'}(p) \subset U_\epsilon$, we obtain the claim. $\square$

Proof of Theorem 3.0.6. Let $p \in M$ be fixed and for $\epsilon > 0$ let B_ϵ be the ball around p with radius ϵ with respect to g. Then, with Lemma 3.3.1 we have $\lambda^+_{min}(M, g, \sigma) \leq \lambda^+_{min}(B_\epsilon, g, \sigma) \to \lambda^+_{min}(S^n, g_{st}, \chi_{st})$ for $\epsilon \to 0$. □

The next Lemma generalizes Lemma 3.3.1. But we needed Lemma 3.3.1 to prove first Theorem 3.0.6 since that will be used in the following proof.

Lemma 3.3.2. *Let (M, g, σ) be a Riemannian spin manifold of dimension $n \geq 2$. Assume that there exists a sequence $\{\Gamma_i\}$ of smoothly bounded open subsets of (M, g) with $|\Gamma_i| := \mathrm{vol}(\Gamma_i, g) \to 0$ and $\Gamma_i \subset \Gamma_1$ for $i \in \mathbb{N}$. Then*

$$\lim_{i\to\infty} \lambda^+_{min}(\Gamma_i, g, \sigma) = \lambda^+_{min}(S^n, g_{st}, \chi_{st}).$$

Remark 3.3.3. This Lemma was stated in [29, Lem. 3.2]. Unfortunately, in the proof we used the wrong statement of [29, Lem. 2.2.]. Here, we will give a new proof using the Aubin-type inequality.

Proof of Lemma 3.3.2. Set $\lambda_i := \lambda^+_{min}(\Gamma_i, g, \sigma)$. We prove the statement by contradiction. Due to Theorem 3.0.6 we have $\lambda_i \leq \lambda^+_{min}(S^n)$ for all i. Thus, we assume that there exists a constant k such that $\lambda_i \leq \lambda^+_{min}(S^n) - k$ for all i.
Due to the definition of λ^+_{min} there exists a sequence $\phi_i \in C^\infty_c(M, S)$ with supp $\phi_i \subset \Gamma_i$, $(D^M\phi_i, \phi_i) = 1$ and

$$\lambda_i \leq \|D^M\phi_i\|^2_q \leq \lambda_i + \frac{1}{i}.$$

Let Γ be a doubling of Γ_1, i.e. we choose an open subset $\hat{\Gamma}$ with $\overline{\Gamma_1} \subset \hat{\Gamma} \subset M$ and perturb the metric in such a way that it is unchanged on Γ_1 and has product structure near the boundary $\partial\hat{\Gamma}$ (That is only necessary if M is open. If M is closed, we can take $\Gamma = M$.) Then the perturbed metric gives a metric on the double Γ of $\hat{\Gamma}$, such that the original and the perturbed metric coincide on $\Gamma_1 \subset \hat{\Gamma}$. With the double of Γ we mean the manifold that is obtained by glueing two copies of Γ with different orientations along their boundaries by means of the identity. Then the double is also oriented.
Furthermore, Γ is equipped with a spin structure that restricted to Γ_1 is the original one: This can be seen when considering a sufficiently fine open cover $\{U_\alpha\}_{\alpha\in A}$ of Γ that is obtained by doubling an open cover of $\hat{\Gamma}$ and by adding the following open sets: Let $\{V_\alpha\}_{\alpha\in B}$ be an open cover of $\partial\hat{\Gamma}$. Then all U_α that intersect $\partial\hat{\Gamma}$ are chosen to have the structure $U_\alpha = V_\alpha \times (-\epsilon, \epsilon)$ and $B \subset A$.
Let further be $\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{SO}(n)$ the transition functions for $P_{\mathrm{SO}(n)}\Gamma$. Then for $\beta, \alpha \in B$ the maps $\varphi_{\alpha\beta} : (V_\alpha \cap V_\beta) \times (-\epsilon, \epsilon) \to \mathrm{SO}(n)$ are constant along $(-\epsilon, \epsilon)$ since the metric has product structure near $\partial\hat{\Gamma}$. When trying to lift the $\varphi_{\alpha\beta}$'s to obtain transition functions that map into $\mathrm{Spin}(n)$ – as described in Remark 1.1.1.i – only the sets U_α where $\alpha \in B$ could cause problems since $\hat{\Gamma} \subset M$ is already assumed to be spin. Thus, this breaks down to the question whether the lifts $\widetilde{\varphi}_{\alpha\beta}$ fulfill the cocycle condition for $U_\alpha \cap U_\beta \cap U_\gamma$ with $\alpha, \beta, \gamma \in B$. This is true since $\partial\hat{\Gamma}$ possesses a spin structure, see 1.1.1.iv, and, therefore, fulfills the cocycle condition for the

V_α's. As a lift, $\widetilde{\varphi}_{\alpha\beta}$ is also constant along $(-\epsilon, \epsilon)$. Thus, the cocylce condition for V_α extends to U_α.

Then the ϕ_i's can be viewed as elements of $C_c^\infty(\Gamma, S)$ with $D^M\phi_i = D^\Gamma\phi_i =: D\phi_i$. Let ψ_i be the part of ϕ_i that is perpendicular to $\ker D^\Gamma$. Then $D\psi_i = D\phi_i$ and, thus, $(D\psi_i, \psi_i) = 1$. We apply Theorem 3.0.5 and obtain

$$\begin{aligned} 1 &\leq (1+\epsilon)\lambda_n \|D\psi_i\|_q^2 + c(\epsilon)\|\psi_i\|_q^2 \\ &\leq (1+\epsilon)\lambda_n \left(\lambda_i + \frac{1}{i}\right) + c(\epsilon)\|\psi_i\|_q^2. \end{aligned}$$

The Hölder inequality yields

$$\|\psi_i\|_q \leq \|\psi_i\|_p |\Gamma_i|^{\frac{1}{n}}$$

where $p = \frac{2n}{n-1}$. Due to the assumption we can choose ϵ small enough such that

$$1 - (1+\epsilon)\lambda_n \left(\lambda_i + \frac{1}{i}\right) \geq k' > 0$$

for a constant k' and for all i being large enough. Hence,

$$k' \leq c(\epsilon)\|\psi_i\|_p^2 |\Gamma_i|^{\frac{2}{n}}.$$

Since $|\Gamma_i| \to 0$ for $i \to \infty$, the L^p-norm $\|\psi_i\|_p$ has to diverge for $i \to \infty$. With Lemma 3.2.2.iii this contradicts $\|D\psi_i\|_q = 1$. $\square$

Proof of Theorem 3.0.1. The statement is proven by contradiction in complete analogy to the proof for the Yamabe invariant in [35]. Assume that (M, g) is conformal to a subdomain $(M, u^{\frac{4}{n-1}}g)$ of a closed Riemannian spin manifold (K, h), where $u \in C^\infty(M, S)$. Take smooth compact domains X_i in M with $X_i \subset \overline{X_i} \subset X_{i+1}$ such that $\mathrm{vol}(M \setminus X_i, u^{\frac{4}{n-1}}g) \to 0$ for $i \to \infty$. Since $\overline{\lambda^+_{min}(M, g)} < \lambda^+_{min}(S^n)$ is assumed, there exist spinor fields $\phi_i \in C_c^\infty(M \setminus X_i, S)$ with

$$\frac{\|D_g\phi_i\|_q^2}{(D_g\phi_i, \phi_i)_g} \leq \lambda^+_{min}(S^n) - c$$

for a positive constant c and for all $i \in \mathbb{N}$. We take smoothly bounded open subsets Y_i of M with $X_i \subset \overline{X_i} \subset Y_i$ and $\operatorname{supp} \phi_i \subset Y_i \setminus X_i \subset M \setminus X_i$. The only use of the Y_i's is to get smoothly bounded subsets $Y_i \setminus \overline{X_i}$ to which we will apply Lemma 3.3.2. By the conformal invariance of λ^+_{min} it follows

$$\lambda^+_{min}(Y_i \setminus \overline{X_i}, h) = \lambda^+_{min}(Y_i \setminus \overline{X_i}, g) \leq \frac{\|D_g\phi_i\|_q^2}{(D_g\phi_i, \phi_i)_g} \leq \lambda^+_{min}(S^n) - c.$$

Since the volume $\mathrm{vol}(Y_i \setminus \overline{X_i}, u^{\frac{4}{n-1}}g) \to 0$, this contradicts Lemma 3.3.2. $\square$

3.4 Surfaces with cusps

Let (M, g, σ) be an open complete spin surface with cusps. This means that M is the union of a compact surface M_0 and finitely many ends $U_i = S^1 \times (0, \infty)$ each equipped with a warped product metric $g_i = f_i(t)^2 d\phi^2 + dt^2$.
Since we have

$$\lambda^+_{min}(M, g, \sigma) \leq \overline{\lambda^+_{min}(M, g, \sigma)} = \overline{\lambda^+_{min}(\sqcup_i U_i, g, \sigma)}$$

and from Lemma 2.0.5

$$\overline{\lambda^+_{min}(\sqcup_i U_i, g, \sigma)} = \min_i \overline{\lambda^+_{min}(U_i, g_i, \sigma)},$$

we now want to obtain upper bounds for $\lambda^+_{min}(U_i)$ and $\overline{\lambda^+_{min}(U_i)}$, i.e. for cylinders with warped product metrics.

Lemma 3.4.1. *Let the cylinder $M = S^1 \times \mathbb{R}$ be equipped with the warped product metric $g = f(t)^2 g_{S^1} + dt^2$ and the spin structure σ_{tr} whose restriction to S^1 admits harmonic spinors. Then we have*

$$\lambda^+_{min}(S^1 \times \mathbb{R}, f(t)^2 g_{S^1} + dt^2, \sigma_{tr}) \leq 4 \inf_{\epsilon > 0} (\epsilon^{-\frac{1}{2}} \max_{t \in [0, 5\epsilon]} f(t)^{\frac{1}{2}})$$

and

$$\overline{\lambda^+_{min}(S^1 \times \mathbb{R}, f(t)^2 g_{S^1} + dt^2, \sigma_{tr})} \leq 4 \liminf_{c \to \infty} \inf_{\epsilon > 0} (\epsilon^{-\frac{1}{2}} \max_{t \in [c, c+5\epsilon]} f(t)^{\frac{1}{2}}).$$

Proof. The Dirac operator on M is given by

$$D^M \phi = \partial_t \cdot \begin{pmatrix} D^{S^1} & 0 \\ 0 & -D^{S^1} \end{pmatrix} \phi + \partial_t \cdot \partial_t \phi + \frac{\dot{f}}{2f} \partial_t \cdot \phi,$$

see [17, Prop. 2.2] and [14, Sect. 1]. The operator acts on spinors $\phi = \binom{\phi_1}{\phi_2} : M \to \mathbb{C}^2$ where ∂_t is the unit vector field along $\mathbb{R}$ that is normal to S^1 with $\partial_t \cdot \binom{\phi_1}{\phi_2} = \binom{-\phi_2}{\phi_1}$. As a test function we choose $\phi = \psi \binom{\kappa_1}{\kappa_2}$ where $\psi : S^1 \to \mathbb{C}$ with $D^{S^1} \psi = 0$ and κ_i are compactly supported real-valued Lipschitz functions on $\mathbb{R}$.
We obtain

$$D^M \phi = \partial_t \cdot \psi \left(\begin{pmatrix} \partial_t \kappa_1 \\ \partial_t \kappa_2 \end{pmatrix} + \frac{\dot{f}}{2f} \begin{pmatrix} \kappa_1 \\ \kappa_2 \end{pmatrix} \right).$$

Thus,

$$(D^M \phi, \phi) = \|\psi\|_2^2 \int_{\mathbb{R}} (\partial_t \kappa_1 \, \kappa_2 - \partial_t \kappa_2 \, \kappa_1) f dt$$

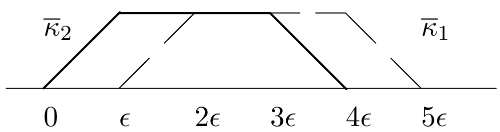
and

$$\|D^M \phi\|_q^q = \|\psi\|_q^q \int_{\mathbb{R}} \Big(\sum_i \Big| \partial_t \kappa_i + \frac{\dot{f}}{2f} \kappa_i \Big|^2 \Big)^{\frac{q}{2}} f dt.$$

Setting $\overline{\kappa}_i = f^{\frac{1}{2}}\kappa_i$ we have

$$\frac{\|D^M\phi\|_q^2}{(D^M\phi,\phi)} = \frac{\|\psi\|_q^2}{\|\psi\|_2^2}\frac{\left[\int_{\mathbb{R}}\left(\sum_i |\partial_t\overline{\kappa}_i|^2\right)^{\frac{q}{2}} f^{1-\frac{q}{2}}dt\right]^{\frac{2}{q}}}{2\int_{\mathbb{R}}\partial_t\overline{\kappa}_1\,\overline{\kappa}_2 dt}.$$

We choose $\overline{\kappa}_2$ to be compactly supported on $[0,4\epsilon]$ for $\epsilon>0$ such that $\overline{\kappa}_2(2\epsilon-t) = \overline{\kappa}_2(2\epsilon+t)$. Further, $\overline{\kappa}_2$ is chosen to be 1 on $(\epsilon,3\epsilon)$. On $(0,\epsilon)$ and on $(3\epsilon,4\epsilon)$ we choose the cut-off function such that it satisfies $|\partial_t\overline{\kappa}_2(t)| = \frac{1}{\epsilon}$ and such that it is continuous on $\mathbb{R}$. Moreover, $\overline{\kappa}_1(t) := \overline{\kappa}_2(t-\epsilon)$.



Then we have with $q = \frac{2n}{n+1} = \frac{4}{3}$

$$\begin{aligned}\frac{\|D^M\phi\|_q^2}{(D^M\phi,\phi)} &\le \max_{t\in[0,5\epsilon]} f(t)^{\frac{2}{q}-1}\frac{\left[\int_{\mathbb{R}}\left(\sum_i |\partial_t\overline{\kappa}_i|^2\right)^{\frac{q}{2}} dt\right]^{\frac{2}{q}}}{2}\\ &\le \max_{t\in[0,5\epsilon]} f(t)^{\frac{2}{q}-1}\frac{\frac{1}{\epsilon^2}(4\epsilon)^{\frac{2}{q}}}{2}\\ &\le \max_{t\in[0,5\epsilon]} f(t)^{\frac{1}{2}}4\epsilon^{-\frac{1}{2}}.\end{aligned}$$

Thus, we get

$$\lambda^+_{min}(S^1\times\mathbb{R}, f(t)^2g_{S^1}+dt^2,\sigma_{tr}) \le 4\inf_{\epsilon>0}(\epsilon^{-\frac{1}{2}}\max_{t\in[0,5\epsilon]} f(t)^{\frac{1}{2}}).$$

Analogously, shifting $\overline{\kappa}_2$ such that it is compactly supported on $(c, c+4\epsilon)$ we obtain the claimed estimate for $\overline{\lambda^+_{min}}$. □

Example 3.4.2. If $f(t) < kt$ for a constant $k>0$ and all large enough t, we have $\overline{\lambda^+_{min}} = 0$. With Lemma 2.0.3 this already implies $\lambda^+_{min} = 0$.
In particular, this includes the example of the cylinder with product metric and trivial spin structure we gave in [29, Ex. 3.4].

In a similar way we can estimate the λ^+_{min}-invariant if the spin structure along S^1 does not admit harmonic spinors.

Lemma 3.4.3. *Let the cylinder $M = S^1\times\mathbb{R}$ be equipped with the warped product metric $g = f(t)^2g_{S^1}+dt^2$ and the non-trivial spin structure σ_{nt}. Then the following estimate holds for any $c\in\mathbb{R}$ and $\epsilon>0$:*

$$\lambda^+_{min}(S^1\times(c,\infty), f(t)^2g_{S^1}+dt^2,\sigma_{nt}) \le 4\epsilon^{\frac{3}{2}}(2^{-2}5^{\frac{3}{4}}+\epsilon^{-1})^2\max_{[c,c+5\epsilon]} f(t)^{\frac{1}{2}}.$$

Proof. We only need to slightly modify the proof of Lemma 3.4.1. As a test function we choose $\phi = \psi\binom{\kappa_1}{\kappa_2}$ where the κ_i's and $\overline{\kappa}_i$'s are defined as above but ψ is now an eigenspinor satisfying $D^{S^1}\psi = \frac{1}{2}\psi$. Recall that $\frac{1}{2}$ is the lowest positive eigenvalue of D^{S^1} if the spin structure is nontrivial, see Example 1.2.7.i. We obtain

$$D^M\phi = \partial_t \cdot \psi \left(\begin{pmatrix} \partial_t\kappa_1 \\ \partial_t\kappa_2 \end{pmatrix} + \left(\frac{\dot f}{2f} + \frac{1}{2} \right) \begin{pmatrix} \kappa_1 \\ \kappa_2 \end{pmatrix} \right)$$

and, therefore, with $q = \frac{4}{3}$ we have

$$\begin{aligned}
\frac{\|D^M\phi\|_q^2}{(D^M\phi,\phi)} &= \frac{\|\psi\|_q^2}{\|\psi\|_2^2} \frac{\left[\int_{\mathbb{R}} \left(\sum_i |\partial_t\overline{\kappa}_i + \frac{1}{2}\overline{\kappa}_i|^2 \right)^{\frac{q}{2}} f^{1-\frac{q}{2}} dt \right]^{\frac{2}{q}}}{2\int_{\mathbb{R}} \partial_t\overline{\kappa}_1\, \overline{\kappa}_2 dt} \\
&\leq \frac{1}{2} \left(\left\| f^{-\frac{1}{2}}\partial_t\overline{\kappa}_i \right\|_q + \left\| \frac{1}{2} f^{-\frac{1}{2}}\overline{\kappa}_i \right\|_q \right)^2 \\
&\leq \frac{1}{2} \max_{[c,c+5\epsilon]} f(t)^{\frac{1}{2}} \left(4^{\frac{1}{q}}\epsilon^{\frac{1}{q}-1} + \frac{1}{2} 2^{\frac{1}{2}} 5^{\frac{1}{q}} \epsilon^{\frac{1}{q}} \right)^2 \\
&\leq 4\epsilon^{\frac{3}{2}} \max_{[c,c+5\epsilon]} f(t)^{\frac{1}{2}} \left(\epsilon^{-1} + 2^{-2}5^{\frac{3}{4}} \right)^2 .
\end{aligned}$$

The denominator is given by $\int_{\mathbb{R}} \partial_t\overline{\kappa}_1\, \overline{\kappa}_2 dt = 1$ and, thus, the first inequality is simply obtained by triangle inequality. The L^q-norm is taken with respect to the volume element $f dt$. □

Example 3.4.4. Let $f(t) < kt^{-r}$ for some constant $k > 0$, t large enough and $r > 3$. If we apply Lemma 3.4.3, we get

$$\overline{\lambda^+_{min}(S^1 \times \mathbb{R}, f(t)^2 g_{S^1} + dt^2, \sigma)} = 0.$$

Even if these estimates might be quite rough and probably more functions f will lead to vanishing λ^+_{min}-invariant, it is clear that we cannot expect this for all functions from Example 3.4.2. Already $f(t) \equiv 1$ gives $\lambda^+_{min} = \lambda^+_{min}(S^2)$ since the cylinder with product metric is conformally compactifiable.

Chapter 4

Estimates of λ_1^+ and λ_{min}^+

Naturally the question arises whether results for the λ_{min}^+-invariant on closed manifolds can be carried over to the open case. In the next sections we will examine this problem for the Hijazi and the Friedrich inequality.

It was shown by Hijazi in [31] that on closed Riemannian spin manifolds (M, g, σ) of dimension $n > 2$ the smallest eigenvalue μ of the conformal Laplacian

$$L = \frac{4(n-1)}{n-2}\Delta + s,$$

where s is the scalar curvature, gives a lower bound for the magnitude of a Dirac eigenvalue λ by

$$\lambda^2 \geq \frac{n}{4(n-1)}\mu.$$

In terms of the corresponding conformal invariants the Yamabe invariant $Q(M, g)$ and the λ_{min}^+-invariant, this reads

$$\lambda_{min}^+(M, g, \sigma)^2 \geq \frac{n}{4(n-1)} Q(M, g)$$

which in the following will be referred to as *conformal Hijazi inequality.*

The Yamabe invariant is usually defined by the variational problem

$$Q(M, g) = \inf \left\{ \int_M vLv\,\mathrm{dvol}_g \;\middle|\; \|v\|_{\frac{2n}{n-2}} = 1, v \in C_c^\infty(M) \right\}$$

and it can be shown that for $Q \geq 0$

$$Q(M, g) = \inf_{g_0 \in [g], \mathrm{vol}(M, g_0) < \infty} \mu(g_0)\mathrm{vol}(M, g_0)^{\frac{2}{n}} \tag{4.1}$$

where μ is the infimum of the spectrum of L.
Whereas on closed manifolds L is always bounded from below, since s is bounded

and Δ is a positive operator, this is not true on open manifolds. For example for a complete Riemannian spin manifold with finite volume and with scalar curvature unbounded from below L is unbounded from below.

Since the conformal Laplacian and the Dirac operator on open manifolds usually do not have pure point spectrum, we will restrict to the case where there is still an eigenvalue present, cf. Remark 4.2.1.

Theorem 4.0.5. *Let (M, g, σ) be a complete Riemannian spin manifold of finite volume and dimension $n > 2$. Moreover, let λ be an eigenvalue of its Dirac operator D, and let μ be the infimum of the spectrum of the conformal Laplacian. Then the following inequality holds:*

$$\lambda^2 \geq \frac{n}{4(n-1)}\mu.$$

If equality is attained, the manifold admits a real Killing spinor and has to be Einstein and closed.

Using this result we will be able to prove the conformal Hijazi inequality for conformally parabolic Riemannian spin manifolds.

Theorem 4.0.6. *Let (M, g, σ) be a conformally parabolic Riemannian spin manifold of dimension $n > 2$. If, additionally, there is a complete conformal metric $\overline{g}$ of finite volume such that $0 \notin \sigma_{ess}(D_g)$, then the conformal Hijazi inequality holds:*

$$\lambda^+_{min}(M, g, \sigma)^2 \geq \frac{n}{4(n-1)} Q(M, g).$$

We will further provide a version of Theorem 4.0.5 in case λ is in the essential spectrum of the Dirac operator. To this end, we assume additionally that the scalar curvature is bounded from below and that the dimension $n \geq 5$, cf. Theorem 4.2.2. With these requirements we can replace the assumption $0 \notin \sigma_{ess}(D_g)$ in Theorem 4.0.6, see Corollary 4.3.4.

Finally, we prove the Friedrich inequality [26, Thm. A] for open manifolds.

Theorem 4.0.7. *Let (M, g, σ) be a Riemannian spin manifold. Let further $\phi_i \in C_c^\infty(M, S)$ be a minimizing sequence of λ_1^+ with $\|\phi_i\| = 1$. Then*

$$\frac{4(n-1)}{n}\lambda_1^+(g)^2 \geq \limsup_i \int_M s|\phi_i|^2 \mathrm{dvol}_g \geq \inf_M s.$$

Moreover, if (M, g) is complete and $\lambda_1^+(g)^2 = \frac{n}{4(n-1)} \inf_M s$, one of the following cases occurs:

(a) λ_1^+ is a positive eigenvalue. This implies that (M, g) is already closed, Einstein and possesses a real Killing spinor.

(b) λ_1^+ is in the essential spectrum.
If, additionally, (M, g) has finite volume, then $0 \in \sigma_{ess}(D)$ and $0 \leq s(x)$.

4.1 Properties of λ_1^+

The aim of this section is to prove some properties of λ_1^+.

Lemma 4.1.1.
i) *If $\lambda_1^+(M,g,\sigma)=0$ and $\mathrm{vol}(M,g)<\infty$, then $\lambda_{min}^+(M,g,\sigma)=0$.*
ii) *If $\lambda_1^+(M,g,\sigma)=0$ and if $f\in C^\infty(M)$ is bounded from below by a positive constant, then $\lambda_1^+(M,f^2g,\sigma)=0$.*
iii) *If (M,g) is complete and $\lambda>0$ is an eigenvalue of D or an element of its essential spectrum, then $\lambda_1^+(M,g,\sigma)\le\lambda$.*
iv) *Any complete spin manifold of finite volume for which there exists $\lambda>0$ in the essential spectrum of its Dirac operator satisfies $\overline{\lambda_{min}^+(M,g,\sigma)}=\lambda_{min}^+(M,g,\sigma)=0$.*

Proof. i) is obvious.
ii) We have the estimate

$$\lambda_1^+(\overline{g}=f^2g)\le\frac{\|\overline{D}\,\overline{\phi}\|_{\overline{g}}^2}{(\overline{D}\,\overline{\phi},\overline{\phi})_{\overline{g}}}=\frac{\|f^{-\frac{1}{2}}D\psi\|_g^2}{(D\psi,\psi)_g}\le(\inf_M f)^{-1}\frac{\|D\psi\|_g^2}{(D\psi,\psi)_g}$$

for all $\psi\in C_c(M,S)$ and $\phi=f^{-\frac{n-1}{2}}\psi$. Thus, $\lambda_1^+(\overline{g})\le(\inf_M f)^{-1}\lambda_1^+(g)$. With the assumptions $\lambda_1^+(g)=0$ and $\inf_M f>0$ this gives the claim.
iii) There exists a sequence $\phi_i\in C_c^\infty(M,S)$ with $\|D\phi_i-\lambda\phi_i\|\to 0$ and $\|\phi_i\|\to 1$: If λ is in the essential spectrum, we just choose the sequence ϕ_i as in the Definition 1.2.5. If λ is an eigenvalue with eigenspinor $\phi\in C^\infty(M,S)\cap L^2(M,S)$, we choose $\phi_i=\eta_i\phi$ where η_i is a smooth cut-off function such that $\eta_i\equiv 1$ on $B_i(p)$ ($p\in M$ fixed), $\eta_i\equiv 0$ on $M\setminus B_{2i}(p)$ and in between $|\nabla\eta_i|\le\frac{2}{i}$. This is always possible since (M,g) is complete. Then ϕ_i is the sequence in demand since $\|(D-\lambda)\phi_i\|=\|\nabla\eta_i\cdot\phi\|\le\frac{2}{i}\|\phi\|$. Thus, in both cases

$$\frac{\|D\phi_i\|^2}{(D\phi_i,\phi_i)}\to\lambda$$

which proves the claim.
iv) Since the essential spectrum is a property of the manifold at infinity, see Theorem 1.2.6, there is a sequence $\phi_i\in C_c^\infty(M\setminus B_i(p),S)$ ($p\in M$ fixed) with $\|(D-\lambda)\phi_i\|\to 0$ and $\|\phi_i\|=1$. Thus, from iii) we find

$$\lambda_{min}^+(M\setminus B_r(p),g,\sigma)\le\lambda\,\mathrm{vol}(M\setminus B_r(p),g)\to 0$$

for $r\to\infty$. Hence, $\lambda_{min}^+(M,g,\sigma)\le\overline{\lambda_{min}^+(M,g,\sigma)}=0$. □

For complete manifolds λ_1^+ is closely related to the Dirac spectrum:

Lemma 4.1.2. *Let (M,g,σ) be a complete Riemannian spin manifold. Then*

$$\lambda_1^+(M,g,\sigma)=\inf\{\sigma(D)\cap(0,\infty)\}$$

where $\sigma(D)$ denotes the Dirac spectrum.

Proof. Since M is complete, D is essentially self-adjoint and has no residual spectrum, cf. Theorem 1.2.4. By the spectral theorem for unbounded self-adjoint operators we obtain for all $\phi \in C_c^\infty(M,S)$ with $(D\phi,\phi) > 0$ that

$$\frac{\|D\phi\|^2}{(D\phi,\phi)} = \frac{\int_{\sigma(D)} \lambda^2 \, d < E_\lambda \phi, \phi >}{\int_{\sigma(D)} \lambda \, d < E_\lambda \phi, \phi >} \geq \frac{\int_{\sigma(D)\cap(0,\infty)} \lambda^2 \, d < E_\lambda \phi, \phi >}{\int_{\sigma(D)\cap(0,\infty)} \lambda \, d < E_\lambda \phi, \phi >}$$
$$\geq \frac{\lambda_0 \int_{\sigma(D)\cap(0,\infty)} \lambda \, d < E_\lambda \phi, \phi >}{\int_{\sigma(D)\cap(0,\infty)} \lambda \, d < E_\lambda \phi, \phi >} = \lambda_0$$

where $\lambda_0 = \inf\{\sigma(D) \cap (0,\infty)\}$. Note that the denominator $\int_{\sigma(D)\cap(0,\infty)} \lambda \, d < E_\lambda \phi, \phi >$ is always positive since $(D\phi,\phi) > 0$. Hence, we have $\lambda_1^+ \geq \inf\{\sigma(D) \cap (0,\infty)\}$.
The converse inequality is obtained by Lemma 4.1.1.iii. □

In particular, if D has pure point spectrum (e.g. if M is closed), λ_1^+ is the first positive Dirac eigenvalue.

Remark 4.1.3. The above relation of λ_1^+ to the spectrum is no longer true if the manifold is not complete. One example is an open subset $\Omega \subset S^n$ with the induced standard metric. Let ϕ be an eigenspinor of D^{S^n} with eigenvalue $\frac{n}{2}$. Then $\phi_{|\Omega}$ is an eigenspinor of D^Ω with eigenvalue $\frac{n}{2}$, but with Lemma 2.0.4 we have for $0 < \mathrm{vol}(\Omega, g_{st}) < \mathrm{vol}(S^n, g_{st})$ that

$$\lambda_1^+(\Omega, g_{st}) \geq \frac{\lambda_{min}^+(\Omega, g_{st})}{\mathrm{vol}(\Omega, g_{st})^{\frac{1}{n}}} = \frac{\lambda_{min}^+(S^n, g_{st})}{\mathrm{vol}(\Omega, g_{st})^{\frac{1}{n}}} = \frac{n}{2}\left(\frac{\mathrm{vol}(S^n, g_{st})}{\mathrm{vol}(\Omega, g_{st})}\right)^{\frac{1}{n}} > \frac{n}{2}.$$

Corollary 4.1.4. *Let (M,g,σ) be a complete Riemannian spin manifold of finite volume with $\lambda_{min}^+ > 0$. Then $\sigma(D) \cap (0,\infty)$ consists only of eigenvalues.*

Proof. This follows immediately from Lemma 4.1.1.iv and Lemma 4.1.2. □

Example 4.1.5.
i) The spectrum of the Dirac operator on the Euclidean space $\mathbb{R}^n$ and the standard hyperbolic space $\mathbb{H}^n$, respectively, consists of all real numbers, see Example 1.2.7. Thus, from Lemma 4.1.1.iii we have $\lambda_1^+(\mathbb{R}^n) = \lambda_1^+(\mathbb{H}^n) = 0$ for $n \geq 2$. But in Example 3.1.6 we showed $\lambda_{min}^+(\mathbb{R}^n) = \lambda_{min}^+(\mathbb{H}^n) = \lambda_{min}^+(S^n)$.
ii) One class of manifolds satisfying the conditions of Lemma 4.1.1.iv are the complete hyperbolic manifolds of finite volume that have trivial spin structure along at least one cusp [20, Thm. 1].

4.2 Proof of Theorem 4.0.5

In this section we will prove Theorem 4.0.5. Then we will consider some additional assumptions to replace the condition that zero is not in the essential spectrum of the Dirac operator.

Proof of Theorem 4.0.5. Let $\psi \in C^\infty(M,S) \cap L^2(M,S)$ be an eigenspinor satisfying $D\psi = \lambda\psi$ and $\|\psi\| = 1$. Its zero-set Ω is closed and contained in a closed countable union of smooth $(n-2)$-dimensional submanifolds which has locally finite $(n-2)$-dimensional Hausdorff measure, cf. Remark 1.2.2.ii.

We fix a point $p \in M$. Since M is complete, there exists a cut-off function $\eta_i : M \to [0,1]$ which is zero on $M \setminus B_{2i}(p)$ and one on $B_i(p)$. In between the function is chosen such that $|\nabla\eta_i| \leq \frac{4}{i}$ and $\eta_i \in C_c^\infty(M)$.
While η_i cuts off ψ at infinity, we define another cut-off near the zeros of ψ. For this purpose, we can assume without loss of generality that Ω is itself the countable union of $(n-2)$-submanifolds described above.

Let now $\rho_{a,\epsilon}$ as in Lemma 3.1.1 be defined as

$$\rho_{a,\epsilon}(x) = \begin{cases} 0 & \text{for } r < a\epsilon \\ 1 - \delta \ln \frac{\epsilon}{r} & \text{for } a\epsilon \leq r \leq \epsilon \\ 1 & \text{for } \epsilon < r \end{cases}$$

where $r = d(x, \Omega)$ is the distance from x to Ω. The constant $a < 1$ is chosen such that $\rho_{a,\epsilon}(a\epsilon) = 0$, i.e. $a = e^{-\frac{1}{\delta}}$. Then $\rho_{a,\epsilon}$ is continuous, constant outside a compact set and Lipschitz. Hence, for $\phi \in C^\infty(M,S)$ the spinor $\rho_{a,\epsilon}\phi$ is an element in $H_1^r(M,S)$ for all $1 \leq r \leq \infty$.

Now let $\psi_{ia} := \eta_i \rho_{a,\epsilon}\psi \in H_1^r(M,S)$ be defined. These spinors are compactly supported in $M \setminus \Omega$. Furthermore, $\overline{g} = e^{2u}g = h^{\frac{4}{n-2}}g$ with $h = |\psi|^{\frac{n-2}{n-1}}$ is a metric on $M \setminus \Omega$. Setting $\overline{\phi_{ia}} := e^{-\frac{n-1}{2}u}\overline{\psi_{ia}}$ ($\phi = e^{-\frac{n-1}{2}u}\psi$), the Lichnerowicz-type formula (1.2) implies

$$\begin{aligned}
\|(\overline{D} - \lambda e^{-u})\overline{\phi_{ia}}\|_{\overline{g}}^2 &= \|\overline{\nabla}^{\lambda e^{-u}}\overline{\phi_{ia}}\|_{\overline{g}}^2 + \int_{M\setminus\Omega} \left(\frac{\overline{s}}{4} - \frac{n-1}{n}\lambda^2 e^{-2u}\right) |\overline{\phi_{ia}}|^2 \mathrm{dvol}_{\overline{g}} \\
&\quad - \frac{n-1}{n}(2\lambda e^{-u}(\overline{D} - \lambda e^{-u})\overline{\phi_{ia}} + \lambda e^{-u}\overline{\mathrm{grad}\, e^{-u} \cdot \phi_{ia}}, \overline{\phi_{ia}})_{\overline{g}} \\
&= \|\overline{\nabla}^{\lambda e^{-u}}\overline{\phi_{ia}}\|_{\overline{g}}^2 + \int_M \left(\frac{\overline{s}}{4} - \frac{n-1}{n}\lambda^2 e^{-2u}\right) e^u |\psi_{ia}|^2 \mathrm{dvol}_g \\
&\quad - 2\frac{n-1}{n}((D-\lambda)\psi_{ia}, \lambda e^{-u}\psi_{ia})_g \\
&= \|\overline{\nabla}^{\lambda e^{-u}}\overline{\phi_{ia}}\|_{\overline{g}}^2 + \frac{1}{4}\int_M h^{-1} Lh\, e^{-u}|\psi_{ia}|^2 \mathrm{dvol}_g \\
&\quad - \frac{n-1}{n}\lambda^2 \int_M e^{-u}|\psi_{ia}|^2 \mathrm{dvol}_g - 2\frac{n-1}{n}((D-\lambda)\psi_{ia}, \lambda e^{-u}\psi_{ia})_g,
\end{aligned}$$

where $\nabla_X^f \phi := \nabla_X \phi + \frac{f}{n} X \cdot \phi$ for $f = \lambda e^{-u} \in C^\infty(M)$ is the Friedrich connection. For the second line we used $|\overline{\phi_{ia}}|^2 \mathrm{dvol}_{\overline{g}} = e^u |\psi_{ia}|^2 \mathrm{dvol}_g$, and the term $(\lambda e^{-u}\overline{\mathrm{grad} e^{-u} \cdot \phi_{ia}}, \overline{\phi_{ia}})_{\overline{g}}$ vanishes since $< \nabla f \cdot \phi, \phi > \in \mathrm{i}\mathbb{R}$, cf. [31, Lem. 3.1]. The last line is obtained by replacing $\overline{s}e^{2u} = h^{-1}Lh$.

With $D\psi = \lambda\psi$ and $< \nabla f \cdot \phi, \phi > \in \mathrm{i}\mathbb{R}$ we obtain

$$((D-\lambda)\psi_{ia}, \lambda e^{-u}\psi_{ia})_g = (\nabla(\eta_i\rho_{a,\epsilon})\psi, \lambda e^{-u}\eta_i\rho_{a,\epsilon}\psi)_g = 0.$$

Inserting this result, $\overline{D}\,\overline{\phi} = \lambda e^{-u}\overline{\phi}$ and $\|\overline{\nabla}^{\lambda e^{-u}}\overline{\phi_{ia}}\|_{\overline{g}}^2 \geq 0$ into the formula from above we further have

$$\|\overline{\nabla}(\eta_i\rho_{a,\epsilon})\overline{\phi}\|_{\overline{g}}^2 \geq \frac{1}{4}\int_M \eta_i^2\rho_{a,\epsilon}^2|\psi|^{\frac{n-2}{n-1}}L|\psi|^{\frac{n-2}{n-1}}\mathrm{dvol}_g - \frac{n-1}{n}\lambda^2\int_M \eta_i^2\rho_{a,\epsilon}^2|\psi|^{2\frac{n-2}{n-1}}\mathrm{dvol}_g.$$

Moreover, we have

$$\begin{aligned}\int_M |\overline{\nabla}(\eta_i\rho_{a,\epsilon})\overline{\phi}|^2\mathrm{dvol}_{\overline{g}} &= \int_M |e^{-u}\overline{\nabla(\eta_i\rho_{a,\epsilon})\cdot\phi}|^2\mathrm{dvol}_{\overline{g}} \\ &= \int_M |\nabla(\eta_i\rho_{a,\epsilon})\cdot\psi|^2 e^{-u}\mathrm{dvol}_g\end{aligned}$$

Thus, with $e^u = |\psi|^{\frac{2}{n-1}}$ the above inequality reads

$$\begin{aligned}\int_M |\nabla(\eta_i\rho_{a,\epsilon})|^2|\psi|^{2\frac{n-2}{n-1}}\mathrm{dvol}_g &\geq \frac{1}{4}\int_M \eta_i\rho_{a,\epsilon}|\psi|^{\frac{n-2}{n-1}}L(\eta_i\rho_{a,\epsilon}|\psi|^{\frac{n-2}{n-1}})\mathrm{dvol}_g \\ &- \frac{n-1}{n-2}\int_M |\nabla(\eta_i\rho_{a,\epsilon})|^2|\psi|^{2\frac{n-2}{n-1}}\mathrm{dvol}_g - \frac{n-1}{n}\lambda^2\int_M \eta_i^2\rho_{a,\epsilon}^2|\psi|^{2\frac{n-2}{n-1}}\mathrm{dvol}_g.\end{aligned}$$

Hence, we obtain

$$\frac{2n-3}{n-2}\int_M |\nabla(\eta_i\rho_{a,\epsilon})|^2|\psi|^{2\frac{n-2}{n-1}}\mathrm{dvol}_g \geq \left(\frac{\mu}{4} - \frac{n-1}{n}\lambda^2\right)\int_M \eta_i^2\rho_{a,\epsilon}^2|\psi|^{2\frac{n-2}{n-1}}\mathrm{dvol}_g,$$

where μ is the infimum of the spectrum of the conformal Laplacian. With $(a+b)^2 \leq 2a^2 + 2b^2$ we have

$$k\int_M (\eta_i^2|\nabla\rho_{a,\epsilon}|^2 + \rho_{a,\epsilon}^2|\nabla\eta_i|^2)|\psi|^{2\frac{n-2}{n-1}}\mathrm{dvol}_g \geq \left(\frac{\mu}{4} - \frac{n-1}{n}\lambda^2\right)\int_M \eta_i^2\rho_{a,\epsilon}^2|\psi|^{2\frac{n-2}{n-1}}\mathrm{dvol}_g.$$

where $k = 2\frac{2n-3}{n-2}$.

Next, we want a tend to zero:
Recall that $\Omega \cap \overline{B_{2i}(p)}$ is bounded, closed, $(n-2)$-C^∞-rectifiable and has still locally finite $(n-2)$-dimensional Hausdorff measure. For fixed i we estimate

$$\int_M |\nabla\rho_{a,\epsilon}|^2\eta_i^2|\psi|^{2\frac{n-2}{n-1}}\mathrm{dvol}_g \leq \sup_{B_{2i}(p)}|\psi|^{2\frac{n-2}{n-1}}\int_{B_{2i}(p)}|\nabla\rho_{a,\epsilon}|^2\mathrm{dvol}_g.$$

Further, we set $B^2_\epsilon(p) := \{x \in B_\epsilon \mid d(x,p) = d(x,\Omega)\}$ with $B_\epsilon := \{x \in M \mid d(x,\Omega) \leq \epsilon\}$. For ϵ sufficiently small each $B^2_\epsilon(p)$ is star shaped. Moreover, there is an inclusion $B^2_\epsilon(p) \hookrightarrow B_\epsilon(0) \subset \mathbb{R}^2$ via the normal exponential map. Then we can calculate

$$\begin{aligned}
\int_{B_\epsilon \cap B_{2i}(p)} |\nabla \rho_{a,\epsilon}|^2 \mathrm{dvol}_g &\leq \mathrm{vol}_{n-2}(\Omega \cap B_{2i}(p)) \sup_{x\in\Omega\cap B_{2i}(p)} \int_{B^2_\epsilon(x)\setminus B^2_{a\epsilon}(x)} |\nabla\rho_{a,\epsilon}|^2 \mathrm{dvol}_{g_2} \\
&\leq c\mathrm{vol}_{n-2}(\Omega \cap B_{2i}(p)) \int_{B_\epsilon(0)\setminus B_{a\epsilon}(0)} |\nabla\rho_{a,\epsilon}|^2 \mathrm{dvol}_{g_E} \\
&\leq c' \int_{a\epsilon}^{\epsilon} \frac{\delta^2}{r} dr = -c'\delta^2 \ln a = c'\delta \to 0 \quad \text{for } a \to 0
\end{aligned}$$

where $\mathrm{vol}_{(n-2)}$ denotes the $(n-2)$-dimensional volume and $g_2 = g_{|_{B^2_\epsilon(p)}}$. The positive constants c and c' arise from $\mathrm{vol}_{n-2}(\Omega \cap B_{2i}(p))$ and the comparison of dvol_{g_2} with the volume element of the Euclidean metric.

Furthermore, for any compact set $K \subset M$ and $0 < f \in C^\infty(M)$ it holds $\rho^2_{a,\epsilon} f \nearrow f$ and, thus, by the monotone convergence theorem

$$\int_K \rho^2_{a,\epsilon} f \mathrm{dvol}_g \to \int_K f \mathrm{dvol}_g$$

as $a \to 0$.

Since η_i has compact support, we finally have for $a \to 0$ that

$$k \int_M |\nabla\eta_i|^2 |\psi|^{2\frac{n-2}{n-1}} \mathrm{dvol}_g \geq \left(\frac{\mu}{4} - \frac{n-1}{n}\lambda^2\right) \int_M \eta_i^2 |\psi|^{2\frac{n-2}{n-1}} \mathrm{dvol}_g.$$

Next we want to establish the limit for $i \to \infty$:
Since M has finite volume and $\|\psi\| = 1$, the Hölder inequality ensures that $\int_M |\psi|^{2\frac{n-2}{n-1}} \mathrm{dvol}_g$ is bounded. With $|\nabla\eta_i| \leq \frac{4}{i}$ we get

$$\lambda^2 \geq \frac{n}{4(n-1)}\mu.$$

Equality is attained if and only if $\|\overline{\nabla}^{\lambda e^{-u}} \overline{\phi_{ia}}\|^2_{\overline{g}} \to 0$ for $i \to \infty$ and $a \to 0$. We have

$$\begin{aligned}
0 \leftarrow \|\overline{\nabla}^{\lambda e^{-u}} \overline{\phi_{ia}}\|_{\overline{g}} &= \|\eta_i\rho_{a,\epsilon}\overline{\nabla}^{\lambda e^{-u}}\overline{\phi} + \overline{\nabla}(\eta_i\rho_{a,\epsilon})\overline{\phi}\|_{\overline{g}} \\
&\geq \|\eta_i\rho_{a,\epsilon}\overline{\nabla}^{\lambda e^{-u}}\overline{\phi}\|_{\overline{g}} - \|\overline{\nabla}(\eta_i\rho_{a,\epsilon})\overline{\phi}\|_{\overline{g}}.
\end{aligned}$$

With $\|\overline{\nabla}(\eta_i\rho_{a,\epsilon})\overline{\phi}\|_{\overline{g}} \to 0$, see above, $\overline{\nabla}^{\lambda e^{-u}}\overline{\phi}$ has to vanish on $M \setminus \Omega$. By Theorem 1.1.3 this implies that e^{-u} is constant. Thus, by Theorem 1.1.4 (M, g) is Einstein and possesses a real Killing spinor. Furthermore, its Einstein constant is positive. Thus, the Ricci curvature is a positive constant and, hence, due to the Theorem of Bonnet-Myers M is already closed.

□

Remark 4.2.1.
i) Allowing an infinite volume and taking λ_1^+ instead of an eigenvalue the Hijazi inequality is not fulfilled. A counterexample is given by the standard hyperbolic space. There $\lambda_1^+(\mathbb{H}^n) = 0$, cf. Example 4.1.5.i. But μ is positive: $\mu = 4\frac{n-1}{n-2}\mu(\Delta) - n(n-1) = \frac{n-1}{n-2}$ where $\mu(\Delta) = \frac{(n-1)^2}{4}$ is the infimum of the spectrum of the Laplacian on $\mathbb{H}^n$ [25, Prop. 7.2].
ii) There is still the hope that Theorem 4.0.5 also holds if λ is in the essential spectrum of a complete Riemannian spin manifold of finite volume. The next Theorem gives a first partial result.

Theorem 4.2.2. *Let (M, g, σ) be a complete Riemannian spin manifold of dimension $n \geq 5$ with finite volume. Furthermore, let the scalar curvature of M be bounded from below. If λ is in the essential spectrum of the Dirac operator, then*

$$\lambda^2 \geq \frac{n}{4(n-1)}\mu.$$

Proof. We may assume $\mathrm{vol}(M, g) = 1$. If λ is in the essential spectrum of D, then 0 is in the essential spectrum of $D - \lambda$. Due to Lemma 1.2.11 there is a sequence $\phi_i \in C_c^\infty(M, S)$ such that $\|(D-\lambda)^2\phi_i\| \to 0$ and $\|(D-\lambda)\phi_i\| \to 0$ while $\|\phi_i\| = 1$. We may assume that $|\phi_i| \in C_c^\infty(M)$. That can always be achieved by a small pertubation.

Now let $\frac{1}{2} \leq \beta \leq 1$. Then $|\phi_i|^\beta \in H_1^2(M)$. Firstly, we will show that the sequence $\|d|\phi_i|^\beta\|$ is bounded:
By the Hölder inequality we have

$$\begin{aligned} 0 \leftarrow \|\phi_i\|^{2\beta-1}\|(D-\lambda)^2\phi_i\| &\geq \||\phi_i|^{2\beta-1}\|_{\{|\phi_i|\neq 0\}}\|(D-\lambda)^2\phi_i\| \\ &\geq \left| \int_{|\phi_i|\neq 0} |\phi_i|^{2\beta-2} < (D-\lambda)^2\phi_i, \phi_i > \mathrm{dvol}_g \right|. \end{aligned}$$

Using the Lichnerowicz formula (1.2) we obtain

$$\begin{aligned} \|(D-\lambda)^2\phi_i\| \geq &\left| \int_{|\phi_i|\neq 0} |\phi_i|^{2\beta-2} < \Delta^\lambda\phi_i, \phi_i > \mathrm{dvol}_g + \int \left(\frac{s}{4} - \frac{n-1}{n}\lambda^2\right) |\phi_i|^{2\beta}\mathrm{dvol}_g \right. \\ &\left. - 2\frac{n-1}{n}\int_{|\phi_i|\neq 0} |\phi_i|^{2\beta-2} < (D-\lambda)\phi_i, \lambda\phi_i > \mathrm{dvol}_g \right| \\ \geq &\left| \int_{|\phi_i|\neq 0} |\phi_i|^{2\beta-2}|\nabla^\lambda\phi_i|^2\mathrm{dvol}_g + 2(\beta-1)\int_{|\phi_i|\neq 0} |\phi_i|^{2\beta-3} < d|\phi_i|\cdot\phi_i, \nabla^\lambda\phi_i > \mathrm{dvol}_g \right. \\ &\left. + \int \left(\frac{s}{4} - \frac{n-1}{n}\lambda^2\right) |\phi_i|^{2\beta}\mathrm{dvol}_g - 2\frac{n-1}{n}\lambda\||\phi_i|^{2\beta-1}\|_{\{|\phi_i|\neq 0\}}\|(D-\lambda)\phi_i\| \right| \end{aligned}$$

With the Hölder inequality and the Kato inequality for the connection ∇^λ, see (1.13), we have

$$\begin{aligned}
0 \leftarrow &\|(D-\lambda)^2\phi_i\| \\
&\geq (2\beta-1)\int_{|\phi_i|\neq 0} |\phi_i|^{2\beta-2}|d|\phi_i|||\nabla^\lambda\phi_i|\mathrm{dvol}_g + \int\left(\frac{s}{4}-\frac{n-1}{n}\lambda^2\right)|\phi_i|^{2\beta}\mathrm{dvol}_g \\
&\quad - 2\frac{n-1}{n}\lambda\|\phi_i\|^{2\beta-1}\|(D-\lambda)\phi_i\| \\
&\geq (2\beta-1)\int_{|\phi_i|\neq 0} |\phi_i|^{2\beta-2}|d|\phi_i||^2\mathrm{dvol}_g + \int\left(\frac{s}{4}-\frac{n-1}{n}\lambda^2\right)|\phi_i|^{2\beta}\mathrm{dvol}_g \\
&\quad - 2\frac{n-1}{n}\lambda\|(D-\lambda)\phi_i\| \\
&\geq (2\beta-1)\frac{1}{\beta^2}\int_{|\phi_i|\neq 0} |d|\phi_i|^\beta|^2\mathrm{dvol}_g + \int\left(\frac{s}{4}-\frac{n-1}{n}\lambda^2\right)|\phi_i|^{2\beta}\mathrm{dvol}_g \\
&\quad - 2\frac{n-1}{n}\lambda\|(D-\lambda)\phi_i\|
\end{aligned}$$

Since s is bounded from below, $\int s|\phi_i|^{2\beta}\mathrm{dvol}_g \geq \inf s\ \|\phi_i\|_{2\beta}^{2\beta} \geq \min\{\inf s, 0\}$ is also bounded from below. Thus, with $\|(D-\lambda)\phi_i\| \to 0$ we obtain that $\|d|\phi_i|^\beta\|$ is also bounded.

Next we fix $\alpha = \frac{n-2}{n-1}$ and obtain

$$\begin{aligned}
\frac{\mu}{4} - \frac{n-1}{n}\lambda^2 &\leq \left(\frac{\mu}{4}-\frac{n-1}{n}\lambda^2\right)\||\phi_i|^\alpha\|^2 \\
&\leq \frac{1}{4}\int |\phi_i|^\alpha L|\phi_i|^\alpha\mathrm{dvol}_g - \frac{n-1}{n}\lambda^2\||\phi_i|^\alpha\|^2 \\
&= \int |\phi_i|^{2\frac{n-2}{n-1}-2}\Bigg(\frac{n}{n-1}|d|\phi_i||^2 + \frac{1}{2}d^*d|\phi_i|^2 \\
&\quad + \left(\frac{s}{4}-\frac{n-1}{n}\lambda^2\right)|\phi_i|^2\Bigg)\mathrm{dvol}_g
\end{aligned}$$

where we used the definition of μ as infimum of the spectrum of $L = 4\frac{n-1}{n-2}\Delta + s$. The third line is obtained from

$$|\phi_i|^\alpha d^*d|\phi_i|^\alpha = \frac{\alpha}{2}|\phi_i|^{2\alpha-2}d^*d|\phi_i|^2 - \alpha(\alpha-2)|\phi_i|^{2\alpha-2}|d|\phi_i||^2.$$

Next, using

$$\frac{1}{2}d^*d < \phi_i, \phi_i > = < \nabla^*\nabla\phi_i, \phi_i > - |\nabla\phi_i|^2 = < D^2\phi_i, \phi_i > - \frac{s}{4}|\phi_i|^2 - |\nabla\phi_i|^2$$

and

$$|\nabla^\lambda\phi_i|^2 = |\nabla\phi_i|^2 - 2\mathrm{Re}\frac{\lambda}{n} < (D-\lambda)\phi_i, \phi_i > - \frac{\lambda^2}{n}|\phi_i|^2,$$

we have

$$\begin{aligned}
\frac{\mu}{4}-\frac{n-1}{n}\lambda^2 \le & \int_{|\phi_i|\neq 0}|\phi_i|^{2\frac{n-2}{n-1}-2}\left(\frac{n}{n-1}|d|\phi_i||^2-|\nabla^\lambda\phi_i|^2\right)\mathrm{dvol}_g\\
&+\int_{|\phi_i|\neq 0}|\phi_i|^{2\frac{n-2}{n-1}-2}<(D^2-\lambda^2)\phi_i,\phi_i>\mathrm{dvol}_g\\
&-\int_{|\phi_i|\neq 0}2|\phi_i|^{2\frac{n-2}{n-1}-2}\mathrm{Re}\frac{\lambda}{n}<(D-\lambda)\phi_i,\phi_i>\mathrm{dvol}_g\\
\le & \int_{|\phi_i|\neq 0}|\phi_i|^{2\frac{n-2}{n-1}-2}\left(\frac{n}{n-1}|d|\phi_i||^2-|\nabla^\lambda\phi_i|^2\right)\mathrm{dvol}_g\\
&+\int_{|\phi_i|\neq 0}|\phi_i|^{2\frac{n-2}{n-1}-2}<(D-\lambda)^2\phi_i,\phi_i>\mathrm{dvol}_g\\
&+\int_{|\phi_i|\neq 0}2\left(1-\frac{1}{n}\right)\lambda|\phi_i|^{2\frac{n-2}{n-1}-2}\mathrm{Re}<(D-\lambda)\phi_i,\phi_i>\mathrm{dvol}_g.
\end{aligned}$$

The last two summands vanish in the limit since

$$\left|\int_{|\phi_i|\neq 0}|\phi_i|^{2\frac{n-2}{n-1}-2}<(D-\lambda)^2\phi_i,\phi_i>\mathrm{dvol}_g\right|\le\|(D-\lambda)^2\phi_i\|\ \||\phi_i|^{\frac{n-3}{n-1}}\|\to 0$$

and

$$\left|\int_{|\phi_i|\neq 0}|\phi_i|^{2\frac{n-2}{n-1}-2}\mathrm{Re}<(D-\lambda)\phi_i,\phi_i>\mathrm{dvol}_g\right|\le\|(D-\lambda)\phi_i\|\ \||\phi_i|^{\frac{n-3}{n-1}}\|\to 0.$$

For the other summand we use the Kato-type inequality of Lemma 1.7.1

$$|d|\psi||\le|(D-\lambda)\psi|+k|\nabla^\lambda\psi|$$

which holds outside the zero set of ψ. Due to Example 1.7.2 we have $k=\sqrt{\frac{n-1}{n}}$. Thus, for $n\ge 5$ we can estimate

$$\begin{aligned}
&\int_{|\phi_i|\neq 0}|\phi_i|^{2\frac{n-2}{n-1}-2}\left(\frac{n}{n-1}|d|\phi_i||^2-|\nabla^\lambda\phi_i|^2\right)\mathrm{dvol}_g\\
&\quad=\int_{|\phi_i|\neq 0}|\phi_i|^{2\frac{n-2}{n-1}-2}(k^{-1}|d|\phi_i||-|\nabla^\lambda\phi_i|)(k^{-1}|d|\phi_i||+|\nabla^\lambda\phi_i|)\mathrm{dvol}_g\\
&\quad\le k^{-1}\int_{\{|d|\phi_i||\ge k|\nabla^\lambda\phi_i|\}\cap\{|\phi_i|\neq 0\}}|\phi_i|^{2\frac{n-2}{n-1}-2}|(D-\lambda)\phi_i|(k^{-1}|d|\phi_i||+|\nabla^\lambda\phi_i|)\mathrm{dvol}_g\\
&\quad\le 2k^{-2}\int_{\{|d|\phi_i||\ge k|\nabla^\lambda\phi_i|\}\cap\{|\phi_i|\neq 0\}}|\phi_i|^{2\frac{n-2}{n-1}-2}|(D-\lambda)\phi_i||d|\phi_i||\mathrm{dvol}_g\\
&\quad\le 2k^{-2}\int\left(2\frac{n-2}{n-1}-1\right)^{-1}|(D-\lambda)\phi_i||d|\phi_i|^{2\frac{n-2}{n-1}-1}|\mathrm{dvol}_g\\
&\quad\le 2k^{-2}\frac{n-1}{n-3}\|(D-\lambda)\phi_i\|\ \|d|\phi_i|^{\frac{n-3}{n-1}}\|.
\end{aligned}$$

For $n \geq 5$ we have $1 \geq \frac{n-3}{n-1} \geq \frac{1}{2}$ and, thus, $\|d|\phi_i|^{\frac{n-3}{n-1}}\|$ is bounded. Together with $\|(D-\lambda)\phi_i\| \to 0$ we obtain the following: For all $\epsilon > 0$ there is an i_0 such that for all $i \geq i_0$ we have

$$\int_{|\phi_i|\neq 0} |\phi_i|^{2\frac{n-2}{n-1}-2}\left(\frac{n}{n-1}|d|\phi_i||^2 - |\nabla^\lambda \phi_i|^2\right) \mathrm{dvol}_g \leq \epsilon.$$

Hence, we have $\frac{\mu}{4} \leq \frac{n-1}{n}\lambda^2$. □

4.3 The conformal inequality

Now we want to use the results of the previous section to prove the conformal Hijazi inequality for conformally parabolic manifolds. Since we proved the Hijazi inequality only for complete manifolds, we first need to express the λ^+_{min}-invariant using only complete metrics:

Lemma 4.3.1. *Let (M, g, σ) be an open complete Riemannian spin manifold of unit volume. Then* $\lambda^+_{min} = \inf\{\lambda^+_1(\overline{g}) \mid \mathrm{vol}(\overline{g}) = 1,\ f \equiv 1 \text{ near infinity}\}$ *where "near infinity" refers to the existence of a compact subset $U \subset M$ such that $f \equiv 1$ on $M \setminus U$.*

Proof. Let $g_i = f_i^2 g$ be a sequence of conformal metrics of unit volume with $\lambda^+_1(g_i) \to \lambda^+_{min}$ for $i \to \infty$. Thus, there is a sequence $\phi_i \in C^\infty_c(M, S)$ such that

$$F(\phi_i, g_i) := \frac{\| D_{g_i}\phi_i \|^2_{g_i}}{(D_{g_i}\phi_i, \phi_i)_{g_i}} \to \lambda^+_{min}$$

Now choose the conformal factor h_i such that h_i is equal to f_i on the support of ϕ_i, $h_i = 1$ near infinity and $\int_M h_i^n \mathrm{dvol}_g = 1$. Then, $F(\phi_i, h_i^2 g) = F(\phi_i, g_i) \to \lambda^+_{min}$, and the metrics $h_i^2 g$ are complete, since g is complete, and they have unit volume. □

In particular, we have:

Corollary 4.3.2. *Let (M, g, σ) be a conformally parabolic Riemannian spin manifold. Then there exists a sequence of complete conformal metrics g_i of unit volume such that $\lambda^+_1(g_i) \to \lambda^+_{min}(g)$ and $g_i \equiv g_1$ near infinity, i.e.*

$$\lambda^+_{min}(M, g, \sigma) = \inf\{\lambda^+_1(M, \overline{g}, \sigma) \mid \overline{g} \equiv g_1 \text{ near infinity},\ \mathrm{vol}(M, \overline{g}) = 1\}.$$

With this Corollary the conformal inequality for conformally parabolic manifolds follows from Theorem 4.0.5:

Proof of Theorem 4.0.6. For $Q < 0$ the inequality is trivially satisfied. Thus, we restrict ourselves to the case $Q \geq 0$:
We may assume that g is itself a complete metric of finite volume satisfying $0 \notin \sigma_{ess}(D_g)$. Due to Corollary 4.3.2 there exists a sequence g_i of complete metrics of unit volume with $g_i \equiv g$ near infinity and $\lambda^+_1(g_i) \to \lambda^+_{min}$.

We first consider the case that there is an infinite subsequence g_{i_j} such that $\lambda_1^+(g_{i_j})$ is an eigenvalue of $D_{g_{i_j}}$. Then we can apply Theorem 4.0.5 and equality (4.1) and obtain

$$\lambda_1^+(M, g_{i_j}, \sigma)^2 \geq \frac{n}{4(n-1)}\mu(M, g_{i_j}) \geq \frac{n}{4(n-1)}Q(M, g).$$

Thus, for $j \to \infty$ we obtain the conformal Hijazi inequality.
Now we consider the remaining case – only finitely many $\lambda_1^+(g_i)$ are eigenvalues. Thus, from Lemma 4.1.2 we know that then there is an infinite subsequence g_{i_j} such that $\lambda_1^+(g_{i_j}) \in \sigma_{ess}(D_{g_{i_j}})$. But if for two metrics g_i and g_k we have $\sigma_{ess}(D_{g_i}) \ni \lambda_1^+(g_i) \geq \lambda_1^+(g_k) \in \sigma_{ess}(D_{g_k})$, then $\lambda_1^+(g_i)$ already equals $\lambda_1^+(g_k)$ since $g_k \equiv g_i$ near infinity and the essential spectrum only depends on the manifold at infinity, see Theorem 1.2.6. Hence, there has to exist a constant subsequence $\lambda_{min}^+ = \lambda_1^+(g_{i_j}) \in \sigma_{ess}(D_{g_{i_j}}) = \sigma_{ess}(D_g)$. Lemma 4.1.1.iv then gives $\lambda_{min}^+ = 0$ and, thus, $0 \in \sigma_{ess}(D_g)$. This is a contradiction to the assumption. $\square$

Now, the question remains whether for any complete metric of finite volume with $0 \in \sigma_{ess}(D)$ the Yamabe constant is negative. Then we could omit the condition on the essential spectrum in Theorem 4.0.6. But for now we only know this for some classes of manifolds:

Remark 4.3.3.
i) Let (M, g, σ) be complete, of finite volume and with non-positive scalar curvature. Then, let η_i be a smooth function that is compactly supported on $B_{2i}(p)$ for a fixed $p \in M$, $\eta_i = 1$ on $B_i(p)$ and in between such $|\nabla \eta_i| \leq \frac{2}{i}$. This gives

$$\begin{aligned}
\mu &\leq \|\eta_i\|^{-2} \int_M \eta_i L \eta_i \mathrm{dvol}_g \\
&\leq \|\eta_i\|^{-2} \left(4\frac{n-1}{n-2}\frac{4}{i^2}\mathrm{vol}(M, g) + \int_M s|\eta_i|^2 \mathrm{dvol}_g\right) \\
&\leq 4\frac{n-1}{n-2}\frac{4}{i^2}\mathrm{vol}(M, g)\mathrm{vol}(B_i(p), g)^{-1}
\end{aligned}$$

Hence, $Q \leq 0$.
ii) If the scalar curvature of the complete manifold (M, g, σ) is uniformly positive, i.e. there is a constant c with $s \geq c > 0$, then $Q \geq 0$ but $0 \notin \sigma_{ess}(D)$ due to the Lichnerowicz formula.
iii) Let (M, g, σ) be complete and of finite volume. Furthermore, let $0 \in \sigma_{ess}(D)$ and let the scalar curvature be nonnegative outside a compact set $U \subset M$.
Thus, there exists a sequence $\phi_i \in C_c^\infty(M \setminus U, S)$ with $\|D\phi_i\| \to 0$ and $\|\phi_i\| = 1$. With the Lichnerowicz formula we obtain

$$\begin{aligned}
0 \leftarrow \|D\phi_i\|^2 &= \|\nabla\phi_i\|^2 + \int_{M\setminus U} \frac{s}{4}|\phi_i|^2 \mathrm{dvol}_g = \|d|\phi_i|\|^2 + \int_{M\setminus U} \frac{s}{4}|\phi_i|^2 \mathrm{dvol}_g \\
&= \frac{n-2}{4(n-1)} \int |\phi_i| L |\phi_i| \mathrm{dvol}_g + \frac{1}{n-1}\int_{M\setminus U} \frac{s}{4}|\phi_i|^2 \mathrm{dvol}_g \geq \mu\frac{n-2}{4(n-1)}
\end{aligned}$$

and, thus, $Q \leq 0$.

Theorem 4.2.2 allows to formulate such a result for another class of Riemannian spin manifolds:

Corollary 4.3.4. *Let (M, g, σ) be a complete Riemannian spin manifold of finite volume and of dimension $n \geq 5$. If, additionally, the scalar curvature is bounded from below, then the conformal Hijazi inequality holds:*

$$\lambda^+_{min}(M, g, \sigma)^2 \geq \frac{n}{4(n-1)} Q(M, g).$$

Proof. The case where $0 \notin \sigma_{ess}(D_g)$ the inequality is already proven in Theorem 4.0.6. So we assume now that $0 \in \sigma_{ess}(D)$. Then Theorem 4.2.2 implies $Q \leq 0$. □

Spin conformal compactifications give another class of manifolds for which the conformal Hijazi inequality hold. Such manifolds do not necessary have to be conformally parabolic, one example is the hyperbolic space.

Corollary 4.3.5. *Let (M, g, σ) be a Riemannian spin manifold that is spin conformally equivalent to $(N \setminus \Omega, h, \chi)$ where (N, h, χ) is a closed Riemannian spin manifold and Ω is a closed and bounded subset that is contained in a countable union of m-dimensional submanifolds $(m \leq n-2)$ which has locally finite m-dimensional Hausdorff measure. Then the conformal Hijazi inequality is valid.*

Proof. The claim follows from the conformal Hijazi inequality on closed manifolds, Lemma 3.1.2 and Remark 3.1.4.ii which imply $\lambda^+_{min}(M, g, \sigma) = \lambda^+_{min}(N, h, \chi)$ and $Q(M, g) = Q(N, h)$. □

Example 4.3.6. We consider the Riemannian manifold $(\mathbb{H}^{n-1} \times S^1, g_{\mathbb{H}} + dt^2)$ that is conformally compactifiable to S^n by two points. Hence, we have $Q(\mathbb{H}^{n-1} \times S^1, g_{\mathbb{H}} + dt^2) = Q(S^n)$, see Remark 3.1.4.ii.
Furthermore, $\mathbb{H}^{n-1} \times S^1$ admits two spin structures. The one that is induced from the spin structure of S^n is σ_{nt}, the spin structure whose restriction to S^1 is non-trivial. Thus, Lemma 3.1.2 implies $\lambda^+_{min}(\mathbb{H}^{n-1} \times S^1, g_{\mathbb{H}} + dt^2, \sigma_{nt}) = \lambda^+_{min}(S^n, g_{st}, \chi_{st})$.
Next, we want to examine the case if the manifold is equipped with the other spin structure σ_{tr}:
We know that $(S^n \setminus \{p_1,\ p_2\}, g_{st})$ is conformally parabolic, since $(\mathbb{R}^n, g_E)$ is, and $(\mathbb{R}^n, g_E)$ and $(S^n \setminus \{p\}, g_{st})$ are conformally equivalent. Thus, due to Example 1.6.4 in the conformal class of $(\mathbb{H}^{n-1} \times S^1, g_{\mathbb{H}} + dt^2)$ there is a complete metric of finite volume and whose scalar curvature is bounded from below. Hence, with Corollary 4.3.4 we now know that at least for $n \geq 5$ we have also for the trivial spin structure that $\lambda^+_{min}(\mathbb{H}^{n-1} \times S^1, g_{\mathbb{H}} + dt^2, \sigma_{tr}) = \lambda^+_{min}(S^n, g_{st}, \chi_{st})$.

4.4 Proof of Theorem 4.0.7

From the Lichnerowicz formula (1.2) we get

$$((D-\lambda)^2\psi, \psi) = (\Delta^\lambda \psi, \psi) + \int_M \left(\frac{s}{4} - \frac{n-1}{n}\lambda^2\right) |\psi|^2 \mathrm{dvol}_g - 2\frac{n-1}{n}(D\psi - \lambda\psi, \lambda\psi).$$

With $((D-\lambda)^2\psi,\psi)=((D-\lambda)\psi,D\psi)-((D-\lambda)\psi,\lambda\psi)$ for compactly supported ψ we obtain

$$((D-\lambda)\phi_i,D\phi_i)=\|\nabla^\lambda\phi_i\|^2+\int_M\left(\frac{s}{4}-\frac{n-1}{n}\lambda^2\right)|\phi_i|^2\mathrm{dvol}_g-\frac{n-2}{n}(D\phi_i-\lambda\phi_i,\lambda\phi_i).$$

Since ϕ_i is a normalized minimizing sequence, we have for $\lambda=\lambda_1^+(M,g,\sigma)$

$$0\leftarrow\frac{\|D\phi_i\|^2-\lambda(D\phi_i,\phi_i)}{(D\phi_i,\phi_i)}=\frac{(D\phi_i,(D-\lambda)\phi_i)}{(D\phi_i,\phi_i)}.$$

With

$$\lambda\leftarrow\frac{\|D\phi_i\|^2}{(D\phi_i,\phi_i)}\geq(D\phi_i,\phi_i)$$

we see that the denominator $(D\phi_i,\phi_i)$ cannot diverge since otherwise λ could not be finite. Thus, the numerator $(D\phi_i,(D-\lambda)\phi_i)$ converges to 0. Furthermore, $(D\phi_i,\phi_i)\to\leq\lambda$ that shall denote the following: For all $\epsilon>0$ exists i_0 such that for all $i\geq i_0$ we have $(D\phi_i,\phi_i)\leq\lambda+\epsilon$. Thus, we have $((D-\lambda)\phi_i,\phi_i)\to\leq 0$.

Summarizing we have:

$$\underbrace{((D-\lambda)\phi_i,D\phi_i)}_{\to 0}\geq\int_M\left(\frac{s}{4}-\frac{n-1}{n}\lambda^2\right)|\phi_i|^2\mathrm{dvol}_g-\underbrace{\frac{n-2}{n}(D\phi_i-\lambda\phi_i,\lambda\phi_i)}_{\to\leq 0}.$$

Hence, for all $\epsilon>0$ there is an i_0 such that for all $i\geq i_0$ we have

$$4\frac{n-1}{n}\lambda^2+\epsilon\geq\sup_{i\geq i_0}\int s|\phi_i|^2\mathrm{dvol}_g$$

and, thus, for $\epsilon\to 0$

$$4\frac{n-1}{n}\lambda^2\geq\limsup_{i\to\infty}\int s|\phi_i|^2\mathrm{dvol}_g$$

which implies the claimed inequalities.

For the equality of

$$\lambda_1^+(g)^2=\frac{n}{4(n-1)}\inf_M s$$

it is necessary and sufficient that $\nabla^\lambda\phi_i\to 0$ and if $n>2$, $((D-\lambda)\phi_i,\phi_i)\to 0$ for $i\to\infty$. This implies $(D-\lambda)\phi_i\to 0$.

Let now (M,g,σ) be complete. Then $\lambda:=\lambda_1^+$ is either a positive eigenvalue or in the non-negative part of the essential spectrum, cf. Lemma 4.1.2

Firstly, let $\lambda>0$ be an eigenvalue with normalized eigenspinor ϕ. We set $\phi_i=\eta_i\phi$

where $\eta_i \in C_c^\infty(M,S)$ is a cut-off function such that supp $\eta_i \subset B_{2i}(p)$ and $|\nabla\eta_i| \leq \frac{4}{i}$. Inserting ϕ_i in the Lichnerowicz formula above gives

$$0 \leftarrow \|\nabla^\lambda(\eta_i\phi)\| \geq \|\eta_i\nabla^\lambda\phi\| - \|\nabla\eta_i\cdot\phi\| \geq \|\eta_i\nabla^\lambda\phi\| - \frac{4}{i}\|\phi\|.$$

The limit $i\to\infty$ gives $\nabla^\lambda\phi = 0$. Hence, by Theorem 1.1.4 (M,g,σ) is Einstein with positive scalar curvature and, hence, due to the Theorem of Bonnet-Myers M is already closed.
If otherwise $\lambda \in \sigma_{ess}(D)$ and (M,g) has additionally finite volume, then Lemma 4.1.1.iv shows $\lambda^+_{min}(M,g,\sigma) = 0$. The conformal Hijazi inequality then implies $Q(M,g)\leq 0$. Thus, $\inf s \leq 0$. To obtain equality we necessarily have $\inf s = 0$ and, thus, $\lambda = 0$. □

Remark 4.4.1.
i) If $\lambda_1^+ \in \sigma_{ess}(D)$, the spinors ϕ_i can be chosen such that supp $\phi_i \subset M\setminus B_i(p)$ for fixed $p\in M$. Thus, we get the following improvement:

$$\frac{4(n-1)}{n}\lambda_1^+(g)^2 \geq \limsup_{i\to\infty}\inf_{M\setminus B_i(p)} s.$$

ii) For the special case of a complete Riemannian spin manifold of finite volume whose Dirac operator has pure point spectrum one can obtain the same result by using the test function in the proof of Theorem 4.0.5 and without using a conformal change.
iii) Examples for the case of equality of the Friedrich inequality can be obtained from closed manifolds that also fulfill the equality: Let (N,h,χ) be such a closed manifold. Then due to Lemma 3.1.2 for each bounded, closed and m-C^∞-rectifiable subset $\Omega\subset M$ of locally finite m-dimensional Hausdorff density (with $m\leq n-2$) the manifold $(N\setminus\Omega,h,\chi)$ will also give equality. But, only incomplete manifolds arise in these examples.

4.5 Different λ_{min}-invariants

In the introduction we stressed out that sometimes other definitions of the λ^+_{min}-invariant occur in the literature. Firstly, one can define

$$\lambda^-_{min} = \inf\left\{\frac{\|D\phi\|_q^2}{|(D\phi,\phi)|}\ \Big|\ (D\phi,\phi)<0,\ \phi\in C_c^\infty(M,S)\right\}$$

and

$$\lambda^\pm_{min} = \min\{\lambda^+_{min},\lambda^-_{min}\}.$$

In this case nothing new happens. All results and definitions carry over straightforwardly, e.g., for complete manifolds one has $\lambda_1^- = -\sup\sigma(D)\cup\{-\infty,0\}$ and $\lambda_1^\pm = \inf|\sigma(D)|\cup\{0,\infty\}$.

The invariants of these definitions do not see the kernel of the Dirac operator. In contrast to this, in [7] for closed manifolds the following definition was considered:

$$\lambda_{min} = \inf_{\overline{g} \in g} \lambda_1(\overline{g}) \mathrm{vol}(M, g)^{\frac{1}{n}}$$

where λ_1 is the first eigenvalue of the square of the Dirac operator. The invariant λ_{min} is zero if and only if the Dirac operator has a nontrivial kernel [38].
For our purpose, this is exactly the disadvantage of this definition. We are interested in spin conformal compactification. For that we rely on the positivity of λ^+_{min} on closed manifolds to obtain the obstruction 3.0.1. That's why we restricted ourselves to λ^+_{min}.

Nevertheless, we could also extend the notion of λ_{min} to open manifolds by defining

$$\lambda_1 = \inf \left\{ \frac{\|D\phi\|^2}{\|\phi\|^2} \;\Big|\; \phi \in C^\infty_c(M, S) \right\}$$

and

$$\lambda_{min}(M, g, \sigma) = \inf_{\overline{g} \in [g], \mathrm{vol}(M, \overline{g}) = 1} \lambda_1.$$

Then, we can also obtain estimates for λ_{min} and λ_1 as we did for λ^+_{min}.

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