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MASTER THESIS

Orbit spaces and invariant theory

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Introduction

In order to describe fundamental particle interactions gauge theories are used in modern physics. These describe the forces the elementary particles exert on each other by the exchange of gauge bosons. The oldest gauge theory is electrodynamics, an Abelian theory with the gauge group $U(1)$. That electrodynamics can be seen in such a context was pointed out by Weyl [31]. In 1954, Yang and Mills introduced non-Abelian gauge theory as a model to understand the weak interaction [33]. They constructed a theory based on the $SU(2)$ symmetry group. Later, strong interaction was placed in the context of a gauge theory with symmetry group $SU(3)$.
Many developments have been made to integrate gauge fields into quantum field theory. The most common approach is the canonical quantization that led to quantum electrodynamics, quantum chromodynamics and the Standard Model that unifies the description of electromagnetism, weak and strong interactions.
Furthermore, the classical Yang-Mills theories have been enhanced to be formulated in bundle theory [10]. Classical gauge fields can be modelled mathematically by connections on a principal bundle whose structure group G, the gauge group, is given by a Lie group, often $SU(n)$. In the Hamiltonian formulation the configuration space is formed by all time-independent connections and its cotangent space is viewed as the phase space. Local gauge invariance results in the fact that different gauge field configurations are physically equivalent. This symmetry is represented by an action of the group of local gauge transformations, which lifts to an action on the phase space. Reduction of the phase space w.r.t. this symmetry gives the reduced phase space. Examinations of its mathematical structure are supposed to contribute to a better understanding of phenomena that are until now not satisfactorily explained by perturbation theory, e.g. the quark confinement. In general, the reduced space is not a manifold but decomposed into several manifolds: One that is open and dense in the reduced space, the principal stratum, and the others forming the singularities. The question of the physical meaning of the singularities arises [28].

The common approaches to quantization excel at good experimental predictions but have difficulties in mathematical rigour. An idea, followed by our research group, is to find a quantisation procedure based on the bundle theoretic formulation of classical Yang-Mills theory. First steps to overcome the difficulty of infinite dimensional configuration space are made by using Hamiltonian lattice gauge theory where time is still continuous. Singular Marsden-Weinstein reduction [9] is applicable to reduce the gauge symmetries. The reduced space can be described as a stratified symplectic space that can be quantized by using an extension of geometric quantization [16, 17].

Lattice gauge theory was introduced by Wilson [32] by viewing space-time as a four dimensional lattice. The lattice Λ is a pair (Λ^0, Λ^1) with Λ^0 being a set of sites and $\Lambda^1 \subseteq \Lambda^0 \times \Lambda^0$ a set of oriented links between the sites such that $(y,x) \in \Lambda^1$ for all $(x,y) \in \Lambda^1$. The gauge fields live on the links and the gauge transformations are placed on the sites. Analogously, a lattice can be used (by fixing temporal gauge) where only the spatial hypersurface at constant time is discretized and time is taken as continuous.
For working in lattice gauge theory notions like the time-independent field configurations have to be transferred to discrete space: The spatial hypersurface is substituted by a three dimensional lattice Λ. As in the continuous case the configuration space, denoted by $\mathcal{A}$, is the space of all connections, where a connection on the lattice Λ is a map

$$A : \Lambda^1 \to G$$

with $A(x,y) = A(y,x)^{-1}$ for all $(x,y) \in \Lambda^1$. A assigns to a path, consisting of several links, the product of the group elements of the links, multiplied in the corresponding order. $\mathcal{A}$ will be identified with the manifold $G^{|\Lambda^1|}$.
A gauge transformations is a map $\phi : \Lambda^0 \to G$ that enable a transformation of a connection $A \in \mathcal{A}$ into another one $A' \in \mathcal{A}$ by $A'(x,y) = \phi(x)A(x,y)\phi(y)^{-1}$. The set of all gauge transformations, denoted by $\mathcal{S}$, can be identified with $G^{|\Lambda^0|}$.
The phase space is the cotangent bundle $T^*\mathcal{A}$. In the following the identification $T^*\mathcal{A} \cong G^{|\Lambda^1|} \times \mathfrak{g}^{|\Lambda^1|}$, obtained by the right (or left) invariant vector fields on G, is used where $\mathfrak{g}$ is the Lie algebra of G.
On the lattice a maximal tree that connects all lattice points, passing through each lattice point once, can be defined [23]. It can be shown that a gauge can be chosen such that on the links belonging to the tree the gauge fields are fixed to be the unit element. Using such a maximal-tree gauge it is possible to proof that the isomorphy $(G^{|\Lambda^1|} \times \mathfrak{g}^{|\Lambda^1|})/G^{|\Lambda^0|} \cong (G^{|\tilde{\Lambda}^1|} \times \mathfrak{g}^{|\tilde{\Lambda}^1|})/G$ holds where $\tilde{\Lambda}$ consists of all off-tree links whose sites carry all the same group element.
As stated before, explicit knowledge of the reduced phase space $T^*\mathcal{A}/\mathcal{S}$ with its stratification is necessary for further examinations. Invariant theory is a way to analyze orbit spaces systematically [26, 29]. For this purpose, the orbit space of a manifold with symmetries is characterized as the image of certain G-invariant polynomials on the manifold.
Since symmetries occur in many physical contexts, e.g. in the analysis of phase spaces, in spontaneous symmetry breaking [1] and in polar structures of chiral biaxial liquid crystals (crystal field theory) [20], such actions were studied in depth by means of invariant theory and especially for the case of an action on Euclidean space, several algorithms were found to determine the image of invariants [27]. Since $G^{|\Lambda^1|} \times \mathfrak{g}^{|\Lambda^1|}$ is in general not a Euclidean space this thesis shall examine the orbit space structure for G acting on a G-invariant semialgebraic subset of $\mathbb{R}^n$ for certain

$n \in \mathbb{N}$. In [8], the case of $SU(3)$ acting diagonally on some copies of itself is analyzed.

The thesis is structured as follows: In the first chapter the orbit space and its stratification are introduced from a geometric point of view.
The approach of invariant theory is presented in chapter 2. At first, it is explained for an action of a compact Lie group G on $\mathbb{R}^n$ and in the second section this theory is generalized to the action of G on a semialgebraic subset of $\mathbb{R}^n$. For this purpose, the notion of a G-invariant description of a semialgebraic subset is established. With the help of this concept it will be possible to deduce the orbit space of G acting on the semialgebraic subset immediately from the orbit space of G acting on $\mathbb{R}^n$.
In chapter 3, the diagonal action of G on $(\mathbb{R}^n)^m$, where G acts on every copy of $\mathbb{R}^n$ separately, will be considered. The main question is how the bases, their relations and inequalities develop for an increasing number of copies and if there exists a number of copies m_0 from which on all orbit spaces for higher m are known. An idea of how to approach this question is given.
Chapter 4 states the algorithm of Abud and Sartori to determine the orbit space of a linear Lie group acting orthogonally on $\mathbb{R}^n$ and an alternative way to find relations is established. The applicability of this algorithm to the situation of a semialgebraic subset of $\mathbb{R}^n$ is established in chapter 5. There some remarks on computational difficulties are made, too.
The used statements of the Hamilton-Cayley polynomials and semialgebraic geometry are summarized in appendix A and B, respectively. In appendix C some further statements referring to examples given in the thesis are listed.

Chapter 1

Geometric description of the orbit space

1.1 The orbit space

Let M be a connected smooth manifold and let G be a compact Lie group that acts on M via

$$\Phi : G \times M \mapsto M; \qquad g \cdot m := \Phi(g, m). \tag{1.1}$$

such that Φ is a (left) action [5, p. 32]. For any $m \in M$ the set $\Omega_m = \{\, g \cdot m |\ g \in G\}$ is called the *orbit of* m. $G_m = \{\, g \in G |\ g \cdot m = m\}$ is the *stabilizer of* m. It contains all the elements of G which leave the point m fixed and forms a subgroup of G. Since for all $g \in G$ and $m \in M$ it is

$$G_{g \cdot m} = g\, G_m\, g^{-1}, \tag{1.2}$$

the stabilizers of points lying on the same orbit are conjugate in G. The conjugation forms an equivalence relation, where the equivalence class $[G_m] := \{g G_m g^{-1}\}_{g \in G}$, is called the *orbit type* of the orbit through m. All this gives rise to define the *orbit map*:

$$\pi : \ M \to M/G; \quad m \mapsto \Omega_m.$$

The points of M/G are the orbits of M. Equipped with the quotient topology induced by π, M/G is called the *orbit space* of M. M/G is Hausdorff if Φ is a proper map and even a manifold if Φ is additionally a free action [9, pp. 307, 310]. The properness of the action follows immediately from the compactness of G. But the actions that will be considered in the following are in general not free. Hence, it is necessary to give another description of the orbit space, that is until now only a topological space. This will be done below using the concept of *orbit type stratification*, but first an example for an orbit space is given.

Example 1.1.1. Let $SU(2)$ act on itself by conjugation. Each $m \in SU(2)$ is diagonalizable in such a way that the transformation matrix is in $SU(2)$. Hence, a matrix and its diagonal form are contained in the same orbit. Thus, due to (1.2) it suffices to consider diagonal matrices in $SU(2)$, i.e. matrices of the form $m = \operatorname{diag}(e^{\mathrm{i}\alpha}, e^{-\mathrm{i}\alpha})$

with $\alpha \in [0, 2\pi)$. Two matrices are on the same orbit if and only if their eigenvalues, i.e. $e^{\mathrm{i}\alpha}$ and $e^{-\mathrm{i}\alpha}$, are the same. Hence, in this case it is easy to describe the orbit space since the orbits can be parametrized by α, where α and $-\alpha$ correspond to the same orbit. Therefore, the orbit space is homeomorphic to the interval $[0, \pi]$.
Determining the orbit types, two cases are distinguished:

1. $m = \pm\mathrm{Id}$. Then, $G_m = SU(2)$ and $[G_m] = \{SU(2)\}$. This orbit type corresponds to the endpoints of the interval $[0, \pi]$.
2. $m \neq \pm\mathrm{Id}$. Calculation gives $G_m = \left\{ \begin{pmatrix} e^{\mathrm{i}\beta} & 0 \\ 0 & e^{-\mathrm{i}\beta} \end{pmatrix} \middle| \beta \in [0, 2\pi) \right\} \equiv U(1)$. All matrices with the orbit type $[G_m]$ are mapped under the orbit map to the open interval $(0, \pi)$.

The union of orbits having the same orbit type will be called *(orbit type) stratum of M* and denoted by Σ_m where m is any point on the stratum. The image under the orbit map $\hat{\Sigma}_m := \pi(\Sigma_m)$ is then called *(orbit type) stratum of M/G*. Σ_m is a submanifold of M [9, p. 304] and $\hat{\Sigma}_m$ is a manifold [29, p. 42]. The orbit types are partially ordered (with respect to inclusion).
The following result due to Montgomery, Samelson and Yang shows that there is a unique minimal orbit type.

Theorem 1.1.2. [29, p. 42] Let M and G be as before. There is a unique orbit type $[H]$ such that:
i) The stratum $(M/G)_{[H]}$, that is the stratum of M/G corresponding to the orbit type $[H]$, is connected, open and dense in M/G.
ii) $[H]$ is a minimal orbit type of M.
iii) The dimension of the submanifold corresponding to $[H]$ is determined by:

$$\dim(M/G)_{[H]} = \dim M - \dim G + \dim H. \tag{1.3}$$

Such an $[H]$ is called the *principal orbit type*, the corresponding stratum is the *principal stratum* and will be denoted by Σ_{p} in M and $\hat{\Sigma}_{\mathrm{p}}$ in M/G, respectively. All other, non-principal, strata will be referred to as *auxiliary strata*. The *dimension of the orbit space* is then the dimension of its principal stratum.

1.2 $SU(n)$ acting on itself or on its Lie algebra

In an analogous way as in example 1.1.1 the orbit types of $SU(n)$ acting by conjugation on itself or its Lie algebra can be determined in general.
Since matrices lying on the same orbit are similar and each element of $SU(n)$ and $\mathfrak{su}(n)$ is diagonalizable by a matrix out of $SU(n)$, the matrices can be assumed to be diagonal, i.e. of the form $A = \mathrm{diag}(e^{\mathrm{i}\alpha_1}, e^{\mathrm{i}\alpha_2}, \ldots, e^{\mathrm{i}\alpha_{n-1}}, e^{-\mathrm{i}(\alpha_1+\cdots+\alpha_{n-1})})$ if $A \in SU(n)$ and $A = \mathrm{diag}(\mathrm{i}x_1, \mathrm{i}x_2, \ldots, \mathrm{i}x_{n-1}, -\mathrm{i}(x_1 + \cdots + x_{n-1}))$ if $A \in \mathfrak{su}(n)$, respectively.
The stabilizer of A consists of all $G \in SU(n)$ commuting with A.

Lemma 1.2.1. Let $A \in SU(n)$ or $\mathfrak{su}(n)$ possess the eigenvalues $\lambda_1, \ldots, \lambda_k$ with multiplicities $\mu_1, \ldots, \mu_k$. Then, the orbit type of the orbit containing A is

$$[U(\mu_1) \times U(\mu_2) \times \cdots \times U(\mu_{k-1}) \times SU(\mu_k)].$$

The principal orbit type is $[\underbrace{U(1) \times \cdots \times U(1)}_{n-1}]$ and the dimension of the orbit space is $n-1$.

Proof: As stated above, A can be assumed to be of the form

$$A = \operatorname{diag}(\underbrace{\lambda_1, \ldots, \lambda_1}_{\mu_1}, \ldots, \underbrace{\lambda_k, \ldots, \lambda_k}_{\mu_k}).$$

Assume that $g = (g_{ij})$ lies in the stabilizer of $A = (a_{ij})$, only the a_{ii} are non-zero. Then g and A commute, which gives the condition $g_{ij}a_{jj} = a_{ii}g_{ij}$ for all $i,j \in \{1, \ldots, n\}$. If $a_{ii} \neq a_{jj}$, g_{ij} vanishes, otherwise it is arbitrary as far as g is still out of $SU(n)$. Therefore, g has to be of the following form

$$\begin{pmatrix} B_1 & & & & \\ & B_2 & & & \\ & & \ddots & & \\ & & & B_{k-1} & \\ & & & & (\prod_{i=1}^{k-1} \det B_i)^{-1} C \end{pmatrix} \text{ with } B_i \in U(\mu_i) \text{ and } C \in SU(\mu_k),$$

where the product of the determinant of the B_i inserted as a factor in front of C ensures that g has determinant 1. All elements out of the set

$$G^{(\mu_1,\ldots,\mu_k)}_{(\lambda_1,\ldots,\lambda_k)} := U(\mu_1) \times U(\mu_2) \times \cdots \times U(\mu_{k-1}) \times SU(\mu_k) \subseteq SU(n)$$

really commute with A. Hence, $G^{(\mu_1,\ldots,\mu_k)}_{(\lambda_1,\ldots,\lambda_k)}$ is already the stabilizer of A.

The principal orbit type is the minimal orbit type with respect to inclusion, that is, $[U(1)^{n-1} \times SU(1)]$. With $SU(1) = \{\mathrm{Id}\}$ the claim follows. Using equation (1.3), the dimension of the principal orbit type is given by $\dim \underbrace{U(1) \times \cdots \times U(1)}_{n-1} = n-1$.□

Each orbit of $SU(n)$ and $\mathfrak{su}(n)$, respectively, is in one-to-one correspondence to a set of eigenvalues λ_i with their multiplicities μ_i, i.e. $\{(\lambda_1, \mu_1), \ldots, (\lambda_k, \mu_k)\}$ such that $\prod \lambda_i^{\mu_i} = 1$ for $SU(n)$ or $\sum \mu_i \lambda_i = 0$ for $\mathfrak{su}(n)$. Thus, for these orbit spaces another description can be found:

$$SU(n)/SU(n) \cong \mathcal{S}_1^{n-1}/S_n \text{ and } \mathfrak{su}(n)/SU(n) \cong \mathrm{i}\mathbb{R}^{n-1}/S_n$$

where the permutation group S_n acts on $\mathcal{S}_1^{n-1}$, $\mathcal{S}_1$ being the one-dimensional unit circle, by the map:

$$(\sigma, (e^{\mathrm{i}\alpha_1}, \ldots, e^{\mathrm{i}\alpha_{n-1}}, e^{\mathrm{i}\alpha_n})) \mapsto (e^{\mathrm{i}\alpha_{\sigma(1)}}, \ldots, e^{\mathrm{i}\alpha_{\sigma(n-1)}}, e^{\mathrm{i}\alpha_{\sigma(n)}})$$

with $\alpha_n = -(\alpha_1 + \cdots + \alpha_{n-1})$, and on $\mathrm{i}\mathbb{R}^{n-1}$ by

$$(\sigma, (\mathrm{i}x_1, \ldots, \mathrm{i}x_{n-1}, \mathrm{i}x_n)) \mapsto (\mathrm{i}x_{\sigma(1)}, \ldots, \mathrm{i}x_{\sigma(n-1)}, \mathrm{i}x_{\sigma(n)}),$$

where $x_n = -(x_1 + \cdots + x_{n-1})$ holds.

Chapter 2

Approach by invariant theory

In the following, an approach will be introduced which allows to reformulate the problem of finding the orbit space in the language of invariants and semialgebraic varieties. This will be done for the special case that the manifold M equals $\mathbb{R}^n$ and then generalized to semialgebraic subsets of $\mathbb{R}^n$. But first some notions and ideas of invariant theory will be given.
All statements concerning semialgebraic geometry used in the following can be found in appendix B.

2.1 Group actions on $M = \mathbb{R}^n$

2.1.1 General theory

Let G be a reductive linear group and $V = \mathbb{R}^n$ be a representation space of G. A *polynomial* p on V is a function $p : V \to \mathbb{R}$ that is a polynomial function in the n coordinates of $V = \mathbb{R}^n$. The algebra of all polynomials on V will be denoted by $\mathrm{Pol}(V)$.

Definition 2.1.1. A polynomial p on V is *G-invariant* if $p\,(g\cdot x) = p\,(x)$ holds for each $g \in G$ and $x \in V$. A set of polynomials $P := \{p_1, \ldots, p_k\}$ is called *algebraically dependent* if there exists a polynomial function $f : \mathbb{R}^k \to \mathbb{R}$ such that its composition $f \circ (p_1, \ldots, p_k) = 0$ gives the zero map on V^k. Then, f is called a *relation* of P (w.r.t. an ordering of P). If there exists no relation of P, the set is called *algebraically independent*.

The G-invariant polynomials on V form a (graded) algebra, denoted by $\mathrm{Pol}_G(V)$. This algebra is finitely generated [15, p. 170]. For the finiteness it is important that the group is reductive. Since in this context only compact linear Lie groups are considered this is always fulfilled [24, p. 247]. Nagata gave a counterexample for an action of a non-reductive group where the algebra of G-invariant polynomials is not finitely generated [11, section 4.3].

A G-invariant polynomial is constant on an orbit. Further, the algebra of G-invariant polynomials separates the orbits [1, p. 368], i.e. for two different orbits there exists a G-invariant polynomial that takes different values on these orbits.

Definition 2.1.2. A set of generators $\{\theta_1, \ldots, \theta_q\}$ of an algebra with $q \in \mathbb{N}$ will be called *basis*. A basis is
i) *cofree* if it is algebraically independent, i.e. there exists no relation between its elements.
ii) *minimal* if no basis element can be expressed as a polynomial in the other basis elements.
iii) *homogeneous* if each basis element is a homogeneous polynomial.

Remark 2.1.3.
i) The constant functions are also elements of $\mathrm{Pol}_G(V)$. Since the constant polynomial with value 1 is always in the algebras considered it will be omitted when a set of generators is stated explicitly.
ii) The number of elements of a minimal basis is not unique.
iii) The number of elements of a cofree basis θ corresponds to the dimension of its image (cf. corollary B.8.i).
iv) In the literature, a basis is also referred to as *generating system* or *integrity basis*.

Lemma 2.1.4.
i) Let $p \in \mathrm{Pol}_G(V)$ be given and let $\sum_{i=1}^m p_i = p$ be its *homogeneous decomposition*, i.e $p_i \in \mathrm{Pol}(V)$ are homogeneous and $\deg\, p_i < \deg\, p_j$ for all $i < j$, where $\deg\, p$ denotes the degree of the polynomial p. Then, p_i is G-invariant for every i.
ii) There always exists a homogeneous basis of $\mathrm{Pol}_G(V)$.
iii) Let $\mathfrak{r}$ be a relation of a homogeneous basis and let $\mathfrak{r} = \sum_i \mathfrak{r}_i$ be its homogeneous decomposition. Then, all $\mathfrak{r}_i$ are relations.

Proof:
i) Let $x \in V$ and consider $p(tx)$ with parameter $t \in \mathbb{R}$. Then,

$$p(tx) = \sum_{i=1}^{m} t^{\deg\, p_i} p_i(x).$$

Due to the G-invariance of p, it follows

$$\sum_{i=1}^{m} t^{\deg\, p_i} p_i(g \cdot x) = \sum_{i=1}^{m} t^{\deg\, p_i} p_i(x)$$

for all $t \in \mathbb{R}$, $g \in G$ and $x \in V$. Hence, by comparison of coefficients of t all p_i are G-invariant.
ii) Take a finite basis B of $\mathrm{Pol}_G(V)$. A set $\mathcal{S}$ is constructed in such a way that $p_i \in \mathcal{S}$ if and only if p_i is a summand of the homogeneous decomposition of an element of B. Then, B is algebraically generated by $\mathcal{S}$ and, therefore, $\mathrm{Pol}_G(V)$ is a subset of the

algebra $\mathrm{alg}(\mathcal{S})$ generated by $\mathcal{S}$. Conversely, due to i) all generators and, hence, all elements of $\mathrm{alg}(\mathcal{S})$ are G-invariant. Therefore, $\mathcal{S}$ is a homogeneous basis of $\mathrm{Pol}_G(V)$.
iii) The claim follows by scaling $x \in V$ by a real parameter t. □

Let now $\theta_1, \ldots, \theta_q$ be such a (finite) set of homogeneous generators of $\mathrm{Pol}_G(\mathbb{R}^n)$. The map $\theta = (\theta_1, \ldots, \theta_q) : \mathbb{R}^n \to \mathbb{R}^q$ is called *Hilbert map.* Since this map is a polynomial function, the image set $S = \theta(\mathbb{R}^n)$ is semialgebraic (cf. corollary B.4.i) and exhibits a primary stratification (cf. definition B.15).
Now it is possible to combine the concept of orbit space with a result of invariant theory, the image under the Hilbert map, and even the both types of stratification turn out to be (nearly) equivalent.
Let $\bar{\theta}$ denote the induced map from $\mathbb{R}^n/G$ to $\mathbb{R}^q$ such that $\theta = \bar{\theta} \circ \pi$.

Theorem 2.1.5. [29, theorem (1.5)] Let $\theta, \bar{\theta}$ be as above and let G be a compact Lie group. Then it holds:
i) $\bar{\theta}$ maps the connected components of the orbit type strata of $\mathbb{R}^n/G$ one-to-one to the primary strata of $\theta(\mathbb{R}^n)$.
ii) $\bar{\theta}((\mathbb{R}^n/G)_{[H]})$ is semialgebraic for each orbit type $[H]$.

In the following, $\theta(\mathbb{R}^n)$ will be also referred to as orbit space and all other terms, like principal stratum, will be carried over, too.

Due to the correspondence to the primary strata it is now known how the orbit type strata are glued together: The boundary of a stratum is a union of lower dimensional strata.

Having a homogeneous basis, theorem 2.1.5 identifies the problem of the determination of the orbit space $\mathbb{R}^n/G$ with the computation of the semialgebraic set S, this means the determination of the relations and polynomial inequalities.

Remark 2.1.6. If G is non-compact, the correspondence of theorem 2.1.5 between orbit type strata and primary strata is in general not true. That's why in the context of non-compact groups the orbit type strata are often called orbit structures to not confuse both types of stratifications [5].

Example 2.1.7. A homogeneous basis of the invariants of the action

$$\phi : SU(2) \times \mathfrak{gl}(2, \mathbb{C}) \to \mathfrak{gl}(2, \mathbb{C})\,; \; (g, A) \mapsto gAg^{-1}$$

is given by

$$\theta_1 = \mathrm{Re}\ \mathrm{tr}A,\ \theta_2 = \mathrm{Im}\ \mathrm{tr}A,\ \theta_3 = \mathrm{Re}\ \mathrm{tr}A^2,\ \theta_4 = \mathrm{Im}\ \mathrm{tr}A^2,\ \theta_5 = \mathrm{tr}AA^*$$

(see subsection 2.1.2 below and lemma 4.6.2). If $\mathfrak{gl}(2, \mathbb{C})$ is naturally identified with $\mathbb{R}^8$, these polynomials, viewed as polynomials on $\mathbb{R}^8$, generate $\mathrm{Pol}_{SU(2)}(\mathbb{R}^8)$. Therefore, the orbit space S is a semialgebraic subset of $\mathbb{R}^5$. Its principal stratum has the orbit type $\{\pm\mathrm{Id}\}$ and, thus, its dimension is given by $\dim \mathfrak{gl}(2,\mathbb{C}) - \dim SU(2) + \dim\{\pm\mathrm{Id}\} = 8 - 3 = 5$. Hence, the basis is already cofree (see corollary B.8.i) and, therefore, minimal. The problem of finding the inequalities defining the orbit space remains (see subsection 4.6.2).

As it was already done in the last example, if M and G coincide, elements of the group will be denoted with small letters and elements of the manifold with capital ones. This convention makes clear which element acts and which is acted upon.

2.1.2 $SU(n)$ acting on $\mathfrak{gl}^m(n,\mathbb{C})$

In this section an example for an action and a generating set for its algebra of invariant polynomials are given. The actions considered are the diagonal adjoint actions of $U(n)$ and $SU(n)$ on $\mathfrak{gl}^m(n,\mathbb{C})$, given by:

$$(g,(A_1,\ldots,A_m)) \mapsto (gA_1g^{-1},\ldots,gA_mg^{-1}).$$

Diagonal actions will be considered in chapter 3. The diagonal adjoint action of $U(n)$ on $\mathfrak{gl}^m(n,\mathbb{C})$ was treated in [25]. There, a basis of the *-algebra of $U(n)$-invariant polynomials on $\mathfrak{gl}^m(n,\mathbb{C})$, denoted by $\mathrm{Pol}^*_{U(n)}(\mathbb{C}^n)$, was given:

$$B_n^m = \{\mathrm{tr}(W_{i_1}W_{i_2}\cdots W_{i_s}) \mid i_k \in \{1,\ldots,m\};\ W_j = A_j \text{ or } W_j = A_j^*\ ;\ s \leq 2^n - 1\} \quad (2.1)$$

where the traces are viewed as complex-valued functions on $\mathfrak{gl}^m(n,\mathbb{C})$. This result gives a set of real generators of $\mathrm{Pol}_{(S)U(n)}(\mathfrak{gl}^m(n,\mathbb{C}))$, too: In order to show this, a basis of $\mathrm{Pol}_{U(n)}(\mathfrak{gl}^m(n,\mathbb{C}) \cong \mathbb{R}^{2n^2})$ shall be found from a basis of $\mathrm{Pol}^*_{U(n)}(\mathfrak{gl}^m(n,\mathbb{C}) \cong \mathbb{C}^{n^2})$.

Lemma 2.1.8. Let B be a set of generators of the *-algebra $\mathrm{Pol}^*_G(\mathbb{C}^l)$ of G-invariant polynomials on $\mathbb{C}^l$. Then, the set $\mathrm{Re}B \cup \mathrm{Im}B$ generates $\mathrm{Pol}_G(\mathbb{R}^{2l})$.

Proof: If $\theta \in B$, $\mathrm{Re}\ \theta, \mathrm{Im}\ \theta \in \mathrm{Pol}_G(\mathbb{R}^{2l})$. It remains to show that every $\theta \in \mathrm{Pol}_G(\mathbb{R}^{2l})$ is generated by elements of $\mathrm{Re}\ B \cup \mathrm{Im}\ B$: With $\mathrm{Re}\ z = \frac{1}{2}(z+\bar z)$ and $\mathrm{Im}\ z = \frac{1}{2}(z-\bar z)$ the polynomial θ can also be interpreted as an element of $\mathrm{Pol}^*_G(\mathbb{C}^l)$, i.e. there exist $\theta_1,\ldots,\theta_q \in B$ such that θ is a complex polynomial in $(\theta_1,\ldots,\theta_q)$ and, hence, a complex polynomial in $(\mathrm{Re}\ \theta_1, \mathrm{Im}\ \theta_1, \ldots, \mathrm{Re}\ \theta_q, \mathrm{Im}\ \theta_q)$. But since $\theta \in \mathrm{Pol}_G(\mathbb{R}^{2l})$, it is already a real polynomial in $(\mathrm{Re}\ \theta_1, \mathrm{Im}\ \theta_1, \ldots, \mathrm{Re}\ \theta_q, \mathrm{Im}\ \theta_q)$. □

Lemma 2.1.9. $\mathrm{Pol}_{SU(n)}(\mathfrak{gl}^m(n,\mathbb{C})) = \mathrm{Pol}_{U(n)}(\mathfrak{gl}^m(n,\mathbb{C}))$.

Proof: Clearly, if a polynomial is invariant under the $U(n)$-action, it is also invariant under the action of the subgroup $SU(n)$. Therefore, it remains to show that every $SU(n)$-invariant polynomial is already $U(n)$-invariant:
Let an $SU(n)$-invariant polynomial $\theta : \mathfrak{gl}^m(n,\mathbb{C}) \to \mathbb{R}$ and an $h \in U(n)$ be given. Let λ be an n-th root of $\det(h)$. Then, $\lambda^{-1}h$ is still in $U(n)$ and $\det(\lambda^{-1}h) = 1$, i.e. $g = \lambda^{-1}h \in SU(n)$. Since λ is a scalar it holds

$$\theta(hAh^{-1}) = \theta(\lambda gAg^{-1}\lambda^{-1}) = \theta(gAg^{-1}) = \theta(A)$$

for all $A \in \mathfrak{gl}^m(n,\mathbb{C})$. Hence, θ is $U(n)$-invariant. $\square$

Thus, $\mathrm{Re}\, B_n^m \cup \mathrm{Im}\, B_n^m$ provides a homogeneous (real) basis for the action of $(S)U(n)$ on $\mathfrak{gl}^m(n,\mathbb{C})$.

2.2 Group actions on a semialgebraic subset of $\mathbb{R}^n$

2.2.1 Restriction to the semialgebraic subset

Starting with the setting and notions of the last section, now similar statements shall be found for a G-invariant semialgebraic subset X of $\mathbb{R}^n$, i.e. the semialgebraic set X is closed under the action of G. Let G be again a compact linear Lie group. Clearly, the orbit space X/G is included in $\mathbb{R}^n/G$. The basic idea is to use the information found for $\mathbb{R}^n/G$ to describe X/G. In general, $\mathbb{R}^n/G$ is much higher dimensional but the advantage will be that $\mathbb{R}^n$ is a vector space (see chapter 4).

Remark 2.2.1. If X is not a manifold, the notions listed in section 1.1 still make sense when they are interpreted as to be induced by the restriction from $\mathbb{R}^n$ to X. Thus, the orbit space X/G is the image $\pi(X)$ and is still composed of orbit type strata. But these strata need not to be manifolds if X is none.

Lemma 2.2.2. The restriction

$$\mathrm{Pol}_G(X) := \{p\restriction_X \mid p \in \mathrm{Pol}_G(\mathbb{R}^n)\}$$

of $\mathrm{Pol}_G(\mathbb{R}^n)$ is again an algebra with unit and the restricted basis $\{\theta_1\restriction_X, \dots, \theta_q\restriction_X\}$, $\{\theta_1, \dots, \theta_q\}$ being a basis of $\mathrm{Pol}_G(\mathbb{R}^n)$, is a basis of $\mathrm{Pol}_G(X)$.

Proof: Since the multiplication of a polynomial by a real number and the multiplication and addition of polynomials, respectively, commute with the restriction of a polynomial to the subset X, $\mathrm{Pol}_G(X)$ is an algebra with the unit element $1\restriction_X$.
Let $q \in \mathrm{Pol}_G(X)$. By definition, there exists $p \in \mathrm{Pol}_G(\mathbb{R}^n)$ such that $q = p\restriction_X$. But p can be expressed as a polynomial in the θ_i. Restriction to X shows that q is a polynomial in the $\theta_i\restriction_X$. Hence, $\{\theta_1\restriction_X, \dots, \theta_q\restriction_X\}$ is a basis. $\square$

$\mathrm{Pol}_G(X)$ will be referred to as the *algebra of G-invariant polynomials on X.*

Since θ is a polynomial function and X is semialgebraic, $\theta(X)$ is also semialgebraic (see corollary B.4.ii) and therefore, also exhibits a primary stratification (see definition B.15). As in the case of $X = \mathbb{R}^n$, the connected components of the orbit type strata and corresponding primary strata are identical:

Lemma 2.2.3. Let θ, $\bar\theta$, G and X be as above. Then, it holds:
i) $\bar\theta$ maps the connected components of the orbit type strata of X/G one-to-one to the primary strata of $S = \theta(X)$. Each stratum of $\theta(X)$ is contained in the corresponding stratum (the stratum with the same orbit type) of $\theta(\mathbb{R}^n)$.
ii) $\bar\theta((X/G)_{[H]})$ is semialgebraic for each orbit type $[H]$.

Proof:
i) from (2.1.5.i)
ii) $\bar{\theta}((X/G)_{[H]}) = \bar{\theta}((\mathbb{R}^n/G)_{[H]}) \cap \theta(X)$ is an intersection of two semialgebraic sets (see theorem 2.1.5.ii) and hence, semialgebraic itself. □

The idea for finding a description of the orbit space $\theta(X)$ (as subset of $\mathbb{R}^q$) is to use at first the relations and inequalities determining $\theta(\mathbb{R}^n)$ to identify which points of $\mathbb{R}^q$ have to be taken into consideration at all. Then, the points belonging to $\theta(X)$ have to be determined:

Theorem 2.2.4. Let X and θ be as above. Then it holds:
i) There exists a *G-invariant description* of the G-invariant semialgebraic set X, i.e. G-invariant polynomials $\{p_i\}_{1\leq i\leq k_p}$, $\{q_j\}_{1\leq j\leq k_q}$ on $\mathbb{R}^n$ with $k_p,\ k_q \in \mathbb{N}$ such that

$$X = \{x \in \mathbb{R}^n \mid p_i(x) = 0,\ q_j(x) \geq 0 \text{ for all } i, j\}.$$

ii) Let a G-invariant description of X as proposed in i) be given. Using the above notations, let $\tilde{p}_i$ and $\tilde{q}_j$ be defined by $p_i = \widetilde{p_i} \circ \theta$ and $q_j = \widetilde{q_j} \circ \theta$. Then, it holds:

$$\theta(X) = \theta(\mathbb{R}^n) \cap \{x \in \mathbb{R}^q \mid \tilde{p}_i = 0,\ \tilde{q}_j \geq 0 \text{ for all } i, j\}.$$

Proof:
i) Since $\theta(X)$ is semialgebraic there exist relations $\tilde{p}_i$ and inequalities $\tilde{q}_j$ defining $\theta(X)$. Then $p_i = \tilde{p}_i \circ \theta$, $q_j = \tilde{q}_j \circ \theta$ are polynomials on $\mathbb{R}^n$. Let

$$Y := \{x \in \mathbb{R}^n \mid p_i(x) = 0,\ q_j(x) \geq 0 \text{ for all } i, j\} \subseteq \mathbb{R}^n.$$

Clearly, $X \subseteq Y$. Thus, the inverse inclusion remains to show:
Take any $y \in Y$. Then, $\theta(y)$ fulfills all $\tilde{p}_i$ and $\tilde{q}_j$ and therefore, $\theta(y) \in \theta(X)$. But since θ separates the orbits in $\mathbb{R}^n$, y has to be contained in the orbit of an element of X. X is G-invariant and thus, only contains full orbits, and hence, $y \in X$. This completes the proof.
ii) Let $\theta(\mathbb{R}^n)$ be given by the relations $\{r_i\}_{1\leq i\leq k_r}$, $k_r \in \mathbb{N}$ and the inequalities $\{s_j\}_{1\leq j\leq k_s}$, $k_s \in \mathbb{N}$. Due to their G-invariance, the p_i and q_j really define the corresponding $\widetilde{p_i}$ and $\widetilde{q_j}$. Hence, it remains to show that

$$\theta(X) = Y := \{y \in \mathbb{R}^q \mid r_i(y) = 0,\ \widetilde{p_i}(y) = 0,\ s_i(y) \geq 0,\ \widetilde{q_i}(y) \geq 0\}.$$

Clearly, $\theta(X) \subseteq Y$. Take any $y \in Y$. Due to $Y \subseteq \theta(\mathbb{R}^n)$ there exists $z \in \mathbb{R}^n$ with $y = \theta(z)$. Then, z already fulfills p_i and q_j for all i, j, i.e. $z \in X$ and $y = \theta(z) \in \theta(X)$. Hence, $Y = \theta(X)$. □

Remark 2.2.5.
i) The proof of i) already used the description of $\theta(X)$. In such a case the characterization of $\theta(\mathbb{R}^n)$ would not be needed anymore to give $\theta(X)$. But the G-invariant description of X can also be found if the orbit space is not known, as it can be seen

in the following example.
ii) In proposition 5.2.1 a G-invariant description for $SU(n)$ acting on the algebraic set $SU(n)$ and $\mathfrak{su}(n)$, respectively, is found that only consists of equations. Thus, the question arises whether for a G-invariant algebraic set X this description can be always realized by equations only.

Example 2.2.6. Let ϕ be the adjoint action of $SU(2)$ on $\mathfrak{gl}(2,\mathbb{C})$. A minimal basis was given in example 2.1.7. The group $SU(2)$ itself is an invariant subset under ϕ. Therefore,

$$\theta_1\restriction_{SU(2)} = \mathrm{tr}A,\ \theta_2\restriction_{SU(2)} = 0,\ \theta_3\restriction_{SU(2)} = \mathrm{tr}^2 A - 2,$$
$$\theta_4\restriction_{SU(2)} = 0,\ \theta_5\restriction_{SU(2)} = \mathrm{tr}AA^* = 2$$

form a basis of invariants of $\phi\restriction_{SU(2)}$.
Even if the basis of ϕ is minimal, as in the case given above, the restricted one is in general not. Some of the restricted invariants are algebraically dependent, like $\theta_1\restriction_{SU(2)}$ and $\theta_3\restriction_{SU(2)}$ or even trivial, like $\theta_2\restriction_{SU(2)}$. Thus, a minimal basis of $\mathrm{Pol}_{SU(2)}(SU(2))$ is given by just one invariant: $\mathrm{tr}A$.
The new relations $\theta_3\restriction_{SU(2)} = {\theta_1}^2\restriction_{SU(2)} - 2$ and $\theta_5\restriction_{SU(2)} = 2$ occur. Here, these two equations already give the $SU(2)$-invariant description looked for in theorem 2.2.4.i, i.e.

$$SU(2) = \{A \in \mathfrak{gl}(2,\mathbb{C}) \mid\ \mathrm{tr}AA^* = 2 \text{ and } \frac{1}{2}(\mathrm{tr}^2 A - \mathrm{tr}A^2) = \det A = 1\}.$$

This identity will be shown in proposition 5.2.1. There a generalization to $SU(n)$ is given, too.

2.2.2 Invariant theoretic treatment of the action of $SU(n)$ on $SU(n)$ or on $\mathfrak{su}(n)$

In this subsection minimal bases for the actions of $SU(n)$ on itself and on its Lie algebra will be given. Further, for small n the orbit space is stated explicitly.

Lemma 2.2.7.
i-a) Let $n = 2l$ with $l \in \mathbb{N}$. Then, $\mathrm{Pol}_{SU(n)}(SU(n))$ is generated by the cofree basis consisting of Re $\mathrm{tr}A^i$ with $i \in \{1,2,\dots,l\}$ and Im $\mathrm{tr}A^j$ with $j \in \{1,2,\dots,l-1\}$.
i-b) Let $n = 2l+1$ with $l \in \mathbb{N}$. Then, $\mathrm{Pol}_{SU(n)}(SU(n))$ is generated by the cofree basis: Re $\mathrm{tr}A^j$ and Im $\mathrm{tr}A^j$ with $j \in \{1,2,\dots,l\}$.
ii) $\mathrm{Pol}_{SU(n)}(\mathfrak{su}(n))$ is generated by the cofree basis: $\mathrm{tr}B^j$ (for j even) and $\mathrm{i}\cdot\mathrm{tr}B^j$ (for j odd) with $j \in \{2,\dots,n\}$.

Proof: The dimension of all the considered orbit spaces is $n-1$ (see lemma 1.2.1). Thus, if a basis of $\mathrm{Pol}_{SU(n)}(SU(n))$ and $\mathrm{Pol}_{SU(n)}(\mathfrak{su}(n))$, respectively, only consists of $n-1$ elements, it is already cofree (see corollary B.8.i). Hence, it suffices to show that the given sets are bases.
Due to (2.1) and lemma 2.1.9 the set

$$B_n^1 = \{\mathrm{tr}(W_1\cdots W_s) \mid W_i = A \text{ or } W_i = A^*, s \le 2^n - 1\},$$

Dimension	Orbit type strata	Orbit type
2	$0 < -\theta_1^3 < 6\theta_2^2$	$[U(1) \times U(1)]$
1	$0 < -\theta_1^3 = 6\theta_2^2$	$[U(2)]$
0	$\theta_1 = \theta_2 = 0$	$[SU(3)]$

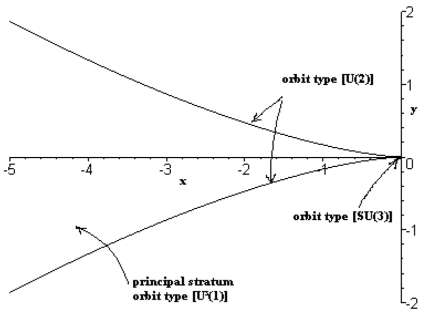

Figure 2.1: The orbit space $\mathfrak{su}(3)/SU(3)$

where $A \in \mathfrak{gl}(n, \mathbb{C})$, generates the *-algebra $\mathrm{Pol}^*_{SU(n)}(\mathfrak{gl}(n, \mathbb{C}))$.
i) The restriction of B^1_n to $SU(n)$ generates $\mathrm{Pol}^*_{SU(n)}(SU(n))$ (see lemma 2.2.2). Let $j, i \in \mathbb{N}$ with $j \leq i$ and $A \in SU(n)$. Then, every element of $B^1_n\lceil_{SU(n)}$ consisting of j times A^* and i times A equals $\mathrm{tr}A^{i-j}$. Due to the identity $\mathrm{tr}A^{*i} = \overline{\mathrm{tr}A^i}$ the set $\{\mathrm{tr}A^i |\ i \leq 2^n - 1\}$ already generates $\mathrm{Pol}^*_{SU(n)}(SU(n))$ (generated means here as in section 2.1.2 generated as a *-algebra, i.e a generator p also generates $\overline{p}$). Inserting the Hamilton-Cayley polynomial (see appendix A) yields

$$\mathrm{tr}A^{n+i} = \mathrm{tr}(A^n A^i) = \mathrm{tr}(p_1 A^{n+i-1} + \cdots + p_n A^i) = p_1 \mathrm{tr}A^{n+i-1} + \cdots + p_n \mathrm{tr}A^i,$$

where every p_i is a polynomial of $\mathrm{tr}A^j$ with $j \leq i$, see corollary A.2. By induction over i it follows that $\{\mathrm{tr}A^i |\ i < n\}$ generates $\mathrm{Pol}^*_{SU(n)}(SU(n))$. With $p_i = (-1)^n \overline{p_{n-i}}$, $i = 1, \ldots, n$, see example A.3, one even gets a polynomial expression for every $\mathrm{tr}A^j$, $j > l$, in terms of $\{\mathrm{tr}A^i, \overline{\mathrm{tr}A^i} |\ i \leq l\}$ with $l = [\frac{n}{2}]$. Due to lemma 2.1.8, $\mathrm{Pol}_{SU(n)}(SU(n))$ is then generated by $\{\mathrm{Re}\ \mathrm{tr}A^i, \mathrm{Im}\ \mathrm{tr}A^i |\ i \leq l\}$.
i-a) Let now $n = 2l$. Due to example A.3, the coefficient p_l of the Hamilton-Cayley polynomial for matrices of $SU(n)$ is real. Together with corollary A.2 this gives:

$$p_l = \frac{1}{l}\mathrm{tr}A^l + \ \text{polynomial}(\mathrm{tr}A^j,\ j < l) = \frac{1}{l}\overline{\mathrm{tr}A^l} + \ \text{polynomial}(\overline{\mathrm{tr}A^j},\ j < l) = \overline{p_l}.$$

Therefore, $2\mathrm{Im}(\mathrm{tr}A^l) = \mathrm{tr}A^l - \overline{\mathrm{tr}A^l} = \text{polynomial}(\mathrm{tr}A^j, \overline{\mathrm{tr}A^l},\ j < l)$. Now, the generating set of $\mathrm{Pol}_{SU(n)}(SU(n))$ is reduced to $\{\mathrm{Re}\ \mathrm{tr}A^j |\ j \leq l\} \cup \{\mathrm{Im}\ \mathrm{tr}A^j |\ j < l\}$.
i-b) For $n = 2l + 1$ the claim is already shown.
ii) $\mathrm{Pol}^*_{SU(n)}(\mathfrak{su}(n))$ is generated by $B^1_n\lceil_{\mathfrak{su}(n)}$ (cf. lemma 2.2.2). Let $j, i \in \mathbb{N}$ with $j \leq i$ and $A \in \mathfrak{su}(n)$. Since $A^* = -A$, $\{\mathrm{tr}A^i,\ i \leq 2^n - 1\}$ already generates $\mathrm{Pol}^*_{SU(n)}(\mathfrak{su}(n))$. Using the same argumentation as in i) it is even sufficient to consider $i \leq n$. Since each $A \in \mathfrak{su}(n)$ can be diagonalized by a matrix out of $SU(n)$, A is assumed to be diagonal, i.e. having the form $\mathrm{diag}(\mathrm{i}x_1, \mathrm{i}x_2, \ldots, \mathrm{i}x_{n-1}, -\mathrm{i}(x_1 + \cdots + x_{n-1}))$, $x_i \in \mathbb{R}$. Then calculation gives immediately that every $\mathrm{tr}A^i$ is real for even i and imaginary for odd i. With $\mathrm{tr}A = 0$ and because of lemma 2.1.8 $\mathrm{Pol}_{SU(n)}(\mathfrak{su}(n))$ is generated by $\mathrm{tr}A^i$ (i even) and $\mathrm{i} \cdot \mathrm{tr}A^i$ (i odd) with $i \in \{2, \ldots, n\}$. $\square$

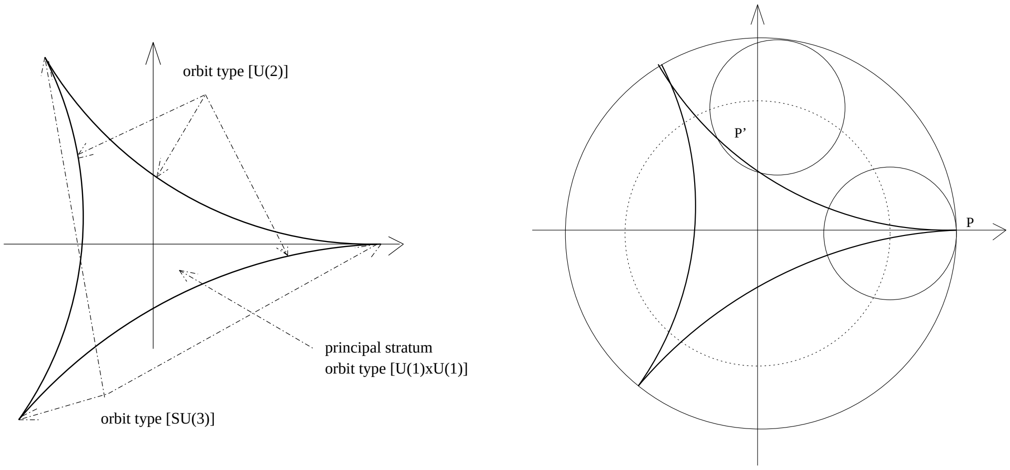

Figure 2.2: On the left: The orbit space $SU(3)/SU(3)$, on the right: Construction of the hypocycloid

Example 2.2.8.

i) Let $B \in \mathfrak{su}(n)$. As mentioned above,

$$B = \mathrm{diag}(\mathrm{i}x_1, \mathrm{i}x_2, \ldots, \mathrm{i}x_{n-1}, -\mathrm{i}(x_1 + \cdots + x_{n-1})), \quad x_i \in \mathbb{R},$$

can be assumed.

i-a) For the case $n = 2$, $\theta_1 = \mathrm{tr}B^2 = -2x_1^2$ already generates $\mathrm{Pol}_{SU(2)}(\mathfrak{su}(2))$. θ_1 can take values in all non-positive real numbers. Thus, the orbit space is given by

$$\mathfrak{su}(2)/SU(2) \cong \{x \in \mathbb{R} \mid x \leq 0\}.$$

i-b) For $n = 3$, $\mathrm{Pol}_{SU(3)}(\mathfrak{su}(3))$ is generated by

$$\begin{aligned}\theta_1 &= \mathrm{tr}B^2 = -2(x_1^2 + x_1x_2 + x_2^2),\\ \theta_2 &= \mathrm{i} \cdot \mathrm{tr}B^3 = x_1^3 + x_2^3 - (x_1 + x_2)^3.\end{aligned}$$

Firstly, the auxiliary strata are calculated. If B is in such a stratum, at least two of its eigenvalues have to coincide (see lemma 1.2.1). W.l.o.g. $x_1 = x_2$ is assumed. Then, $\theta_1 = -6x_1^2$ and $\theta_2 = -6x_1^3$. This gives a description of the auxiliary strata with an orbit type $[U(2)]$ and $[SU(3)]$, respectively, (see figure 2.1). The orbit type $[SU(3)]$ then corresponds to three equal eigenvalues: $x_1 = x_2 = -(x_1 + x_2)$, i.e. $x_1 = 0$. Since these strata form the boundary of the principal stratum, a characterization of the whole orbit space, that is given in the table of figure 2.1, is obtained easily.

ii) Let $A \in SU(n)$. Then, A can be assumed to be diagonal and parametrized by $\alpha_i \in [0, 2\pi)$, $i = 1, \ldots, n-1$:

$$A = \mathrm{diag}(e^{\mathrm{i}\alpha_1}, \ldots, e^{\mathrm{i}\alpha_{n-1}}, e^{-\mathrm{i}(\alpha_1 + \ldots + \alpha_{n-1})})$$

ii-a) For $n = 2$, $\theta_1 = \mathrm{tr}A$ generates $\mathrm{Pol}_{SU(2)}(SU(2))$. The parametrization yields $\theta_1 = 2\cos\alpha_1$. Since $\cos\alpha_1$ can take all values in the interval $[-1, 1]$ the orbit space is given by (cf. example 1.1.1):

$$SU(2)/SU(2) = \{\theta_1 \in \mathbb{R} |\ \theta_1^2 \leq 4\}.$$

ii-b) In the case $n = 3$, $\mathrm{Pol}_{SU(3)}(SU(3))$ is generated by

$$\begin{aligned} \theta_1 &= \mathrm{Retr}A = \cos\alpha_1 + \cos\alpha_2 + \cos(\alpha_1 + \alpha_2), \\ \theta_2 &= \mathrm{Imtr}A = \sin\alpha_1 + \sin\alpha_2 - \sin(\alpha_1 + \alpha_2). \end{aligned}$$

Firstly, a description of the auxiliary strata will be established. If A belongs to an orbit of the auxiliary strata at least two of its eigenvalues have to coincide, w.l.o.g. $\alpha_1 = \alpha_2$. Then $\theta_1 = 2\cos\alpha_1 + \cos 2\alpha_1$ and $\theta_2 = 2\sin\alpha_1 - \sin 2\alpha_1$ give the parametric description of a hypocycloid, a curve that is produced by a fixed point P on the circumference of a small circle of radius b that rolls around the inside of a large circle of radius $a > b$ [6, p. 82]. Here, $a = 1$ and $b = 3$, see figure 2.2.

Chapter 3

Diagonal actions

Let $\Psi : G \times \mathbb{R}^z \to \mathbb{R}^z$ be an action of the compact linear group G on $\mathbb{R}^z$. Then the action

$$\Psi^m : G \times (\mathbb{R}^z)^m \to (\mathbb{R}^z)^m;\ (g, (A_1, \dots, A_m)) \mapsto (\Psi(g, A_1), \dots, \Psi(g, A_m))$$

is called *diagonal action* of Ψ.

Let $V := \mathbb{R}^z$. The notion generalizes to the action of G on a G-invariant subset X of V (or on a manifold).
Due to theorem 2.2.4 a description of X^m/G can be found by a characterization of V^m/G and a G-invariant description of X^m, which is immediately given by a G-invariant description of X itself. Thus, it suffices to consider diagonal actions on V. Nevertheless, the examples will often work with X^m/G, just for the sake of reducing complexity and manageability.

Every invariant of Ψ^m gives a set of invariants of Ψ^{m+1} just by renaming the indices of A_i. The same is true for the relations and inequalities. This chapter shall shed light on the question which conclusions can be drawn for the determination of V^{m+1}/G from the knowledge of V^m/G.

3.1 Preliminaries

Let the stabilizer of Ψ at $x \in V$ be denoted by G_x. Then, for the stabilizers of Ψ^m it holds

$$G_{(x_1,\dots,x_m)} = \cap_{i=1}^m G_{x_i}$$

for every $(x_1, \dots, x_m) \in V^m$. Thus, the orbit types can be calculated by the intersection of stabilizers.

Example 3.1.1. (orbit types of $SU(2)$ acting on $SU^m(2)$)
Let $SU(2)$ act on two copies, $SU^2(2)$, and let $(A_1, A_2) \in SU^2(2)$. For calculating the stabilizer $G_{(A_1,A_2)}$ three cases are distinguished (see example 1.1.1):

1. $G_{A_1}=G_{A_2}=SU(2)$ implies $G_{(A_1,A_2)}=SU(2)$.
2. $G_{A_1}=SU(2)$ and $G_{A_2}=\{\ g\,\mathrm{diag}(e^{\mathrm{i}\beta},e^{-\mathrm{i}\beta})g^{-1}\ \}_{\beta\in[0,2\pi)}$ for fixed $g\in SU(2)$ or vice versa. Then, $G_{(A_1,A_2)}=G_{A_2}$.
3. $G_{A_1}=\{\mathrm{diag}(e^{\mathrm{i}\beta},e^{-\mathrm{i}\beta})\}_{\beta\in[0,2\pi)}$ and $G_{A_2}=\{\ g\,\mathrm{diag}(e^{\mathrm{i}\beta},e^{-\mathrm{i}\beta})g^{-1}\ \}_{\beta\in[0,2\pi)}$ for fixed $g\in SU(2)$. One can assume G_{A_1} to have such a form, since for determining orbit types it suffices to consider only one element of an orbit and in an orbit there always exists an element whose first component is a diagonal matrix.

Since the intersection is non-empty (the identity element Id is always contained) there should be some β and γ such that

$$\begin{pmatrix} e^{\mathrm{i}\beta} & 0 \\ 0 & e^{-\mathrm{i}\beta} \end{pmatrix} = g\begin{pmatrix} e^{\mathrm{i}\gamma} & 0 \\ 0 & e^{-\mathrm{i}\gamma} \end{pmatrix} g^{-1}.$$

This can only be fulfilled if the eigenvalues of both matrices are the same, i.e. $\beta=\pm\gamma$. If $e^{\mathrm{i}\beta}$ can only take the values ± 1, the intersection is $\{\mathrm{Id},-\mathrm{Id}\}$. Thus, assume $e^{\mathrm{i}\beta}\neq\pm 1$. Then, computation gives that g has to have the form

$$\begin{pmatrix} e^{\mathrm{i}\alpha} & 0 \\ 0 & e^{-\mathrm{i}\alpha} \end{pmatrix} \text{or} \begin{pmatrix} 0 & e^{\mathrm{i}\alpha} \\ -e^{-\mathrm{i}\alpha} & 0 \end{pmatrix}, \text{ for fixed } \alpha\in[0,2\pi).$$

Hence, g commutes with each $\mathrm{diag}(e^{\mathrm{i}\delta},e^{-\mathrm{i}\delta})$, $\delta\in[0,2\pi)$, both stabilizers are identical and $G_{(A_1,A_2)}=G_{A_1}$.

So for two copies of $SU(2)$ there exist only 3 different orbit types, namely the ones corresponding to $\{\mathrm{Id},-\mathrm{Id}\}$, $\{\mathrm{diag}(e^{\mathrm{i}\alpha},e^{-\mathrm{i}\alpha})|\ \alpha\in[0,2\pi)\}$ and $SU(2)$. The intersections of the stabilizers determined above show that also for a higher number of copies of $SU(2)$ the orbit types will remain the same. Thus, each orbit space $SU^k(2)/SU(2)$ with $k\geq 2$ consists of 3 orbit type strata. The principal orbit type is $[\{\mathrm{Id},-\mathrm{Id}\}]$ and due to equation (1.3) the orbit space has the dimension $3k-3$. The argumentation showed that each tuple of matrices representing an element of the auxiliary strata can be diagonalized simultaneously and the converse is also true.

Further all conclusions found in lemma 2.1.4 for the special case of $m=1$ can be generalized to diagonal actions:

Lemma 3.1.2.
i) Let $p\in\mathrm{Pol}_G(V^n)$ and $\sum_{i=(i_1,\ldots,i_n)}p_i=p$ its *multihomogeneous decomposition*, i.e. the degree of every monomial of p_i w.r.t. the j-th copy of V is i_j. Then, p_i is G-invariant for every multi-index i.
ii) There always exists a *multihomogeneous basis* of $\mathrm{Pol}_G(V^n)$, i.e. the multihomogeneous decomposition of each basis element consists of only one summand.
iii) Let $\mathfrak{r}$ be a relation between the elements of a multihomogeneous basis and $\mathfrak{r}=\sum_i\mathfrak{r}_i$ its multihomogeneous decomposition, where i is a multi-index. Then, all $\mathfrak{r}_i$ are relations. The multi-index i is called the *multihomogeneous degree* of $\mathfrak{r}_i$.

Proof: The proof can be achieved analogously as for lemma 2.1.4, but the elements of each copy have to be scaled separately, i.e.: $(t_1x_1,\ldots,t_nx_n)\in V^n$ with scaling parameters $t_1,\ldots,t_n\in\mathbb{R}$. □

There exists a natural embedding of $\mathrm{Pol}(V^{m-1})$ into $\mathrm{Pol}(V^m)$. In the following $\mathrm{Pol}(V^{m-1})$ will be viewed as a subset of $\mathrm{Pol}(V^m)$.

Definition 3.1.3. The *height* of a polynomial $f \in \mathrm{Pol}(V^n)$ is defined to be the number of coordinates x_i such that f is non-constant w.r.t. this i-th coordinate (= the i-th copy) of V^n, where $(x_1, \dots, x_n) \in V^n$.
Two polynomials $f \in \mathrm{Pol}(V^n)$, $g \in \mathrm{Pol}(V^m)$ (w.l.o.g. $n \geq m$) are *of the same type* if they have the same height and if g is viewed as polynomial out of $\mathrm{Pol}(V^n)$, there exists a permutation σ of the coordinates of V^n such that $f = g \circ \sigma$.

Being of the same type forms an equivalence relation. The space of all equivalence classes is denoted by $\mathfrak{T}$ and an element is called a *polynomial type.* Since two polynomials of the same type have the same height, the notion of the height of a polynomial is carried over to the notion of the height of a polynomial type.

Example 3.1.4. $\mathfrak{t} = [x_1^2 + x_2] \in \mathfrak{T}$ is a polynomial type of height 2 and the polynomials $x_1^2 + x_2, x_1^2 + x_4, x_3^2 + x_2$ are representatives of $\mathfrak{t}$.

From now on, only diagonal actions with the property

($*$) There exist a number $m_0 \in \mathbb{N}$ and multihomogeneous bases B_m of $\mathrm{Pol}_G(V^m)$ such that the elements of B_m are of height less than or equal to m_0 for all $m \geq m_0$.

will be considered. As seen in section 2.1.2 an example of such an action is $SU(n)$ acting on $V^m = \mathfrak{gl}^m(n, \mathbb{C})$. Having a set of bases with property ($*$) it is always possible to find a common description of B_m for all m being sufficiently large:

Lemma 3.1.5. Let Ψ^m be the diagonal action to $\Psi : G \times V \to V$ and let B_m be a family of multihomogeneous bases of $\mathrm{Pol}_G(V^m)$ with the property ($*$). Then

$$\tilde{B}_m := \{f \in \mathrm{Pol}(V^m) |\text{ there exists } g \in B_{m_0} \text{ that is of the same type as } f\}$$

is a basis for all $m \geq m_0$. $\tilde{B}_m$ is even minimal if B_{m_0} is a minimal basis.

Proof: To show that $\tilde{B}_m$ is a basis of $\mathrm{Pol}_G(V^m)$ it suffices to prove that every element can be generated by $\tilde{B}_m$: Let $f \in \mathrm{Pol}_G(V^m)$. Then f can be expressed as a polynomial in the elements $f_1, \dots, f_q \in B_m$. Each f_i depends on at most m_0 coordinates, say $(x_{i_1}, \dots, x_{i_{m_0}})$. Mapping $(x_{i_1}, \dots, x_{i_{m_0}}) \mapsto (x_1, \dots, x_{m_0})$, f_i is mapped to $g_i \in \mathrm{Pol}_G(V^{m_0})$ where g_i is a polynomial in elements of B_{m_0}. Reversing the map, f_i is a polynomial in elements of $\tilde{B}_m$. Hence, f is also a polynomial in elements of $\tilde{B}_m$.

Let now B_{m_0} be a minimal basis of $\mathrm{Pol}_G(V^{m_0})$. Assume $\tilde{B}_m$ for $m > m_0$ is not minimal. Then, there exists $p \in \tilde{B}_m$ that can be expressed by a polynomial in $p_1, \dots, p_q \in \tilde{B}_m \setminus \{p\}$ that is multihomogeneous w.r.t. $(x_1, \dots, x_m)$ (see lemma 3.1.2). Since p is at most of height m_0 it can be assumed to depend at most on the coordinates $(x_{i_1}, \dots, x_{i_{m_0}})$. Mapping again $(x_{i_1}, \dots, x_{i_{m_0}}) \mapsto (x_1, \dots, x_{m_0})$ shows that then, there exists $p' \in B_{m_0}$ that can be expressed by a polynomial in $p'_1, \dots, p'_q \in B_{m_0} \setminus \{p'\}$. That is a contradiction to the minimality of B_{m_0}. $\square$

Remark 3.1.6. Let $X \subseteq V$ be a G-invariant semialgebraic subset and let $\tilde{B}_m$ be the basis for G acting on V^m obtained in lemma 3.1.5 for large enough m. Then, due to lemma 2.2.2 the restriction $\tilde{B}_m \restriction_{X^m}$ forms a basis of $\mathrm{Pol}_G(X^m)$ but as seen in example 2.2.6 minimality is in general not preserved.

Example 3.1.7. Let $SU(2)$ act on $\widetilde{SU}^m(2)$ by diagonal conjugation where

$$\widetilde{SU}^m(2) := \{(A_1, \ldots, A_k) \in SU^m(2) |\ [A_i, A_j] = 0 \text{ for all } i, j\}.$$

This action is not an example of a diagonal action in the sense introduced above since $\widetilde{SU}^m(2) \neq \widetilde{SU}(2) \times \widetilde{SU}(2) \times \cdots \times \widetilde{SU}(2)$ with $\widetilde{SU}(2) = SU(2)$. However, there also exist bases having the property $(*)$ and it is an easy example for getting a feeling of the development of the orbit space while the number of copies increases. At the same time, it gives a full description of the auxiliary strata of $SU(2)$ acting on $SU^m(2)$ since due to example 3.1.1 all matrices of an element $SU^m(2)$ commute if and only if it belongs to the auxiliary strata.

A basis of $\mathrm{Pol}_{SU(2)}(\widetilde{SU}^m(2))$ is given by the set (cf. subsection 5.2.2):

$$\{\mathrm{tr}A_i, \mathrm{tr}A_jA_k, \mathrm{tr}A_lA_pA_q\},$$

where $1 \leq i \leq m,\ 1 \leq j < k \leq m,\ 1 \leq l < p < q \leq m$.
Since the multiplication of the A_i is commutative, $\mathrm{tr}A_iA_jA_k$ can be expressed using the fundamental trace identity (cf. equation (A.2)):

$$2\mathrm{tr}A_iA_jA_k = \mathrm{tr}A_i\mathrm{tr}A_jA_k + \mathrm{tr}A_j\mathrm{tr}A_kA_i + \mathrm{tr}A_k\mathrm{tr}A_iA_j - \mathrm{tr}A_i\mathrm{tr}A_j\mathrm{tr}A_k. \tag{3.1}$$

Hence, $\mathrm{tr}A_i, \mathrm{tr}A_jA_k$ already give a basis of $\mathrm{Pol}_{SU(2)}(\widetilde{SU}^m(2))$.
Further, since all A_i commute, they can be diagonalized simultaneously, i.e. each orbit contains an m-tuple where all entries are diagonal. Thus, the matrices can be assumed to be of the following form:

$$A_i = \begin{pmatrix} e^{\mathrm{i}\alpha_i} & 0 \\ 0 & e^{-\mathrm{i}\alpha_i} \end{pmatrix}, \text{ with } \alpha_i \in [0, 2\pi).$$

Then the invariants are given by: $\mathrm{tr}A_i = 2\cos\alpha_i$ and $\mathrm{tr}A_iA_j = 2\cos(\alpha_i + \alpha_j)$. Obviously, the invariants $\mathrm{tr}A_i$ are independent from each other. But for each the inequality

$$\mathrm{tr}^2 A_i \leq 4 \tag{3.2}$$

has to hold. If $\mathrm{tr}A_i$ is chosen, fulfilling the last inequality, α_i is automatically determined up to sign. Hence, then $\mathrm{tr}A_iA_j$ can only take two values depending on the relative sign between α_i and α_j. This includes the degenerated case where $\alpha_i = \alpha_j$ and further $\mathrm{tr}A_iA_j$ is determined uniquely. On the level of invariants, this observation is expressed by the relation

$$(2\mathrm{tr}A_iA_j - \mathrm{tr}A_i\mathrm{tr}A_j)^2 = (4 - \mathrm{tr}^2 A_i)(4 - \mathrm{tr}^2 A_j).$$

Dim	Orbit space strata
m	$\mathrm{tr}A_i^2 \leq 4$ (not all inequalities may saturate simultaneously) $(2\mathrm{tr}A_iA_j - \mathrm{tr}A_i\mathrm{tr}A_j)^2 = (4 - \mathrm{tr}^2 A_i)(4 - \mathrm{tr}^2 A_j)$ $(2\mathrm{tr}A_iA_j - \mathrm{tr}A_i\mathrm{tr}A_j)(2\mathrm{tr}A_iA_k - \mathrm{tr}A_i\mathrm{tr}A_k)$ $= (\mathrm{tr}^2 A_i - 4)(2\mathrm{tr}A_jA_k - \mathrm{tr}A_j\mathrm{tr}A_k)$ for all i, j, k
0	$\mathrm{tr}A_i^2 = 4, 2\mathrm{tr}A_iA_j = \mathrm{tr}A_i\mathrm{tr}A_j$

Table 3.1: The auxiliary strata of $SU^m(2)/SU(2)$

The choice of one of these two values determines the relative sign between α_i and α_j. But then the choice of the relative sign between α_i and α_j and the one between α_j and α_k fully determines the relative sign between α_i and α_k. Thus, the choices of $\mathrm{tr}A_iA_j$ are not independent. Something like a consistency relation is needed:

$$(2\mathrm{tr}A_iA_j - \mathrm{tr}A_i\mathrm{tr}A_j)(2\mathrm{tr}A_iA_k - \mathrm{tr}A_i\mathrm{tr}A_k) = (\mathrm{tr}^2 A_i - 4)(2\mathrm{tr}A_jA_k - \mathrm{tr}A_j\mathrm{tr}A_k).$$

With these relations and inequalities the orbit space is fully described:
Summarizing, $\mathrm{tr}A_i$ can be chosen freely (inside the interval $[-2, 2]$ given by the inequality (3.2)) and afterwards all other invariants are determined up to a discrete set. Hence, $\{\mathrm{tr}A_i\}$ is a maximal set of algebraically independent basis elements and the dimension of the orbit space is m (cf. lemma B.7). The result is summarized in table 3.1.

In an analogous way, a characterization of the auxiliary strata of $SU(2)$ acting on $SU^k(2) \times \mathfrak{su}^m(2)$ is established in appendix C.1.

In the last example the orbit space $\widetilde{SU}^m(2)/SU(2)$ is already known for general m when the case $m = 3$ is calculated. Then all relations and inequalities needed to describe the orbit space for general m can be extracted by renaming the coordinates, that are here the matrices A_i.
In the following sections it will be examined how the orbit space develops for an increasing number of copies. Clearly, all relations and inequalities that hold for V^{m_0}/G give, by renaming the coordinates, relations and inequalities for every V^m/G with $m \geq m_0$. However, in general new relations and inequalities will occur. The new occurring relations shall be analysed by distinguishing between the different types of relations (see remark B.9).
At first it will be concentrated in section 3.2 on the dimension-reducing relations. In section 3.2 it will be shown that, from a certain number of copies on, all relations of this type are known.

Then, some ideas for the second type of relation and the inequalities will be given; but there are still many open questions.

3.2 Dimension-reducing relations

The main statement of this section (lemma 3.2.3) is that roughly speaking, there are only finitely many dimension-reducing relations (see remark B.9).
The following will be examined for arbitrary polynomial types that do not need to be bound to a basis of an action. Firstly, the considerations will be restricted to only one polynomial type, but this will fully generalize to several polynomial types.

Let $\mathfrak{t} \in \mathfrak{T}$ be a polynomial type of height m and

$$S_{\mathfrak{t}}^n := \{p \in \mathrm{Pol}(V^n) |\ p \text{ is a representative of } \mathfrak{t}\}.$$

It holds $S_{\mathfrak{t}}^{n-1} \subseteq S_{\mathfrak{t}}^n$. For every n let a maximal set $\tilde{S}_{\mathfrak{t}}^n \subseteq S_{\mathfrak{t}}^n$ of algebraically independent elements be chosen such that $\tilde{S}_{\mathfrak{t}}^{n-1} \subseteq \tilde{S}_{\mathfrak{t}}^n$ (always possible due to lemma B.7). Further, let $s_{\mathfrak{t}}^n := |\tilde{S}_{\mathfrak{t}}^n|$ be the number of elements of $\tilde{S}_{\mathfrak{t}}^n$ and $\Delta_{\mathfrak{t}}^n := \tilde{S}_{\mathfrak{t}}^n \backslash \tilde{S}_{\mathfrak{t}}^{n-1}$ and let $z_{\mathfrak{t}}^n$ denote the number of coordinates with the following property: There exists an element out of $\Delta_{\mathfrak{t}}^n$ that is non-constant w.r.t. this coordinate.

Lemma 3.2.1. Let $(\tilde{S}_{\mathfrak{t}}^n)_{n \in \mathbb{N}}$ as introduced above be given. Then it holds:
i) There exists $n_0 \in \mathbb{N}$ such that $|\Delta_{\mathfrak{t}}^n| = s_{\mathfrak{t}}^n - s_{\mathfrak{t}}^{n-1} \leq m \dim V$ for all $n \geq n_0$.
ii) There exists $n_0 \in \mathbb{N}$ such that $z_{\mathfrak{t}}^n \leq m^2 \dim V$ for all $n \geq n_0$.

Proof: Due to example B.6.iii the dimension of $S_{\mathfrak{t}}^n(V^n)$ is bounded by a linear function of n. That will turn out to be the main reason for the claim.
i) Let c_i denote the number of elements p of $\tilde{S}_{\mathfrak{t}}^n$ such that p is constant w.r.t. the i-th coordinate. Since p is of height m, it only depends explicitly on m coordinates. Therefore, every p is counted for $n - m$ different c_i. Hence,

$$\sum_{i=1}^{n} c_i = (n - m) s_{\mathfrak{t}}^n.$$

Further, $c_i \leq s_{\mathfrak{t}}^{n-1}$, otherwise an algebraically independent subset of $S_{\mathfrak{t}}^{n-1}$ with more than $s_{\mathfrak{t}}^{n-1}$ elements can be found by renaming the coordinates. It follows

$$(n - m) s_{\mathfrak{t}}^n \leq n s_{\mathfrak{t}}^{n-1}$$

and

$$s_{\mathfrak{t}}^n - s_{\mathfrak{t}}^{n-1} \leq \frac{n}{n-m} s_{\mathfrak{t}}^{n-1} - s_{\mathfrak{t}}^{n-1} = \frac{m}{n-m} s_{\mathfrak{t}}^{n-1}.$$

Further, it holds $s_{\mathfrak{t}}^n = |\tilde{S}_{\mathfrak{t}}^n| = \dim S_{\mathfrak{t}}^n(V^n) \leq \dim V^n = n \dim V$ (cf. example B.6.iii and lemma B.7) and therewith:

$$|\Delta_{\mathfrak{t}}^n| = s_{\mathfrak{t}}^n - s_{\mathfrak{t}}^{n-1} \leq \frac{n-1}{n-m} m \dim V.$$

Since the left side of the inequality is always an integer and since the right side tends to $m \dim V$ for n going to infinity the claim holds for large enough n_0.

ii) Every polynomial of type $\mathfrak{t}$ is non-constant w.r.t. m coordinates. Therefore, $z_\mathfrak{t}^n \leq m|\Delta_\mathfrak{t}^n|$. Choosing n large enough such that i) is applicable the claim follows.$\square$

The important statement of this lemma will be that $z_\mathfrak{t}^n$ is bounded by above and the bound is independent of n.

Let $R_\mathfrak{t}^n := \{(i_1, \ldots, i_m) \in \mathbb{N}^m |\ 1 \leq i_j \leq n \text{ for all } j \text{ and } i_j \neq i_k \text{ for all } j \neq k\}$ be the set of all ordered subsets of $\{1, \ldots, n\}$ with cardinality m. For a fixed representative $f(x_1, \ldots, x_m)$ of $\mathfrak{t}$ the surjective map $\phi : R_\mathfrak{t}^n \to S_\mathfrak{t}^n$ is defined such that $(i_1, \ldots, i_m) \mapsto f(x_{i_1}, \ldots, x_{i_m})$.
Two sets $(i_1, \ldots, i_m), (j_1, \ldots, j_m)$ will be compared by the lexicographic ordering: $(i_1, \ldots, i_m) < (j_1, \ldots, j_m)$ if and only if $l \in \mathbb{N},\ 0 \leq l < m$, exists with $i_k = j_k$ for all $k \leq l$ and $i_{l+1} < j_{l+1}$. This relation is a total order on $R_\mathfrak{t}^n$.

$\Delta_\mathfrak{t}^n = \tilde{S}_\mathfrak{t}^n \setminus \tilde{S}_\mathfrak{t}^{n-1}$ can only contain polynomials which are non-constant w.r.t. the n-th coordinate, otherwise $\tilde{S}_\mathfrak{t}^{n-1}$ would not be maximal algebraically independent. Let $\tilde{R}_\mathfrak{t}^n := \{(i_1, \ldots, i_m) \in R_\mathfrak{t}^n |$ there exists j such that $i_j = n\}$ be defined. Then, $\phi(\tilde{R}_\mathfrak{t}^n) = S_\mathfrak{t}^n \setminus S_\mathfrak{t}^{n-1}$. As a subset of $R_\mathfrak{t}^n$, $\tilde{R}_\mathfrak{t}^n$ is ordered. Then, corresponding to this order, the elements of $\tilde{R}_\mathfrak{t}^n$ are numbered such that $r_1 < r_2 < r_3 < \ldots$ and r_i will be referred to as the i-th element of $\tilde{R}_\mathfrak{t}^n$.

From now on, each $\tilde{S}_\mathfrak{t}^n$ will be chosen in the following way:
Start with $S_0 = \tilde{S}_\mathfrak{t}^{n-1}$. If the union of S_0 with $\phi(r_1)$ is algebraically independent, $S_1 = S_0 \cup \{\phi(r_1)\}$, otherwise $S_1 = S_0$. Then, the same procedure will be done with $\phi(r_2)$, $\phi(r_3)$ and so on. Since $\tilde{R}_\mathfrak{t}^n$ is finite, the procedure terminates. Due to construction and the surjectivity of ϕ the resulting set S is maximal algebraically independent. Then, $\tilde{S}_\mathfrak{t}^n := S$.

Lemma 3.2.2. Using the family of sets $(\tilde{S}_\mathfrak{t}^n)_{n \in \mathbb{N}}$ constructed above there exists $n_0 \in \mathbb{N}$ such that $s_\mathfrak{t}^{n+1} - s_\mathfrak{t}^n$ is constant for all $n \geq n_0$.
Let $\mathfrak{R}_{n_0}$ be a set of relations of $S_\mathfrak{t}^{n_0}$ such that the first two algebraic sets in the inclusions $S_\mathfrak{t}^{n_0}(V^{n_0}) \subseteq \mathcal{Z}(\mathfrak{R}_{n_0}) \subseteq \mathbb{R}^{s_\mathfrak{t}^{n_0}}$ have the same dimension, where $\mathcal{Z}(\mathfrak{R}_{n_0})$ denotes the algebraic set determined by the relations of $\mathfrak{R}_{n_0}$ (see definition B.1). Then, the set

$$\tilde{R}_n := \{\mathfrak{r} \text{ is a relation of } S_\mathfrak{t}^n |\ \text{there exists } \mathfrak{r}' \in \mathfrak{R}_{n_0} \text{ that has the same type as } \mathfrak{r}\}$$

gives all dimension-reducing relations for every $n \geq n_0$, i.e. the dimension of $\mathcal{Z}(\tilde{R}_n) \subseteq \mathbb{R}^{|S_\mathfrak{t}^n|}$ equals $\dim S_\mathfrak{t}^n(V^n) = |\tilde{S}_\mathfrak{t}^n|$.

Proof: Due to the construction of $\tilde{S}_\mathfrak{t}^n$ there exist finitely many indices $i_1, \ldots, i_p$ with $p = s_\mathfrak{t}^n - s_\mathfrak{t}^{n-1}$ such that $\tilde{S}_\mathfrak{t}^n = \tilde{S}_\mathfrak{t}^{n-1} \cup \{\phi(r_{i_1}), \ldots, \phi(r_{i_p})\}$ with $r_{i_j} \in \tilde{R}_\mathfrak{t}^n$ holds. Let n

be chosen sufficiently large such that lemma 3.2.1 and $m^2 \dim V < n$ holds. Then, there exists $k \in \mathbb{N}$ with $1 \leq k \leq n$ such that

$$r_{i_1}, \ldots, r_{i_p} \in \tilde{R}^n_{\mathfrak{t},k} := \{(j_1, \ldots, j_m) \in \tilde{R}^n_{\mathfrak{t}} |\ j_i \neq k \text{ for all } i\}.$$

The map $\rho^n_k : R^n_{\mathfrak{t},k} \to R^{n-1}_{\mathfrak{t}}$,

$$(i_1, \ldots, i_m) \mapsto (j_1, \ldots, j_m) \text{ such that } j_l = \begin{cases} i_l & : \ i_l < k \\ i_l - 1 & : \ i_l > k \end{cases} \quad \text{for all } l$$

is bijective and preserves the order. Furthermore, there exists a unique bijective map $\tilde{\rho}^n_k : \phi(R^n_{\mathfrak{t},k}) \to S^{n-1}_{\mathfrak{t}}$ such that the diagram

$$\begin{tikzcd} R^n_{\mathfrak{t},k} \arrow[r, "\rho^n_k"] \arrow[d, "\phi"] & R^{n-1}_{\mathfrak{t}} \arrow[d, "\phi"] \\ S^n_{\mathfrak{t}} \supset \phi(R^n_{\mathfrak{t},k}) \arrow[r, "\tilde{\rho}^n_k"] & S^{n-1}_{\mathfrak{t}} \end{tikzcd}$$

is commutative. The restriction $\rho^n_k \restriction_{\tilde{R}^n_{\mathfrak{t},k}} : \tilde{R}^n_{\mathfrak{t},k} \to \tilde{R}^{n-1}_{\mathfrak{t}}$ is bijective, too.

Let $\tilde{S}^{n-1}_{\mathfrak{t}} \setminus \tilde{S}^{n-2}_{\mathfrak{t}} = \{f_1, \ldots, f_q\}$ with $q = s^{n-1}_{\mathfrak{t}} - s^{n-2}_{\mathfrak{t}}$. Assume $q < p$: Then, there exists j with $1 \leq j \leq q$ and $\phi(\rho^n_k(r_{i_j})) \notin \{f_1, \ldots, f_q\}$. Due to the construction of the f_i and since ρ^n_k preserves the order there exists a relation $\mathfrak{r}$ in the elements of $\tilde{S}^{n-2}_{\mathfrak{t}}$ and $\phi(\rho^n_k(r_l))$ with $l \leq i_j$. Since $\tilde{\rho}^n_k$ lifts to a bijection between the set of polynomials on $\phi(R^n_{\mathfrak{t},k})$ and the set of polynomials on $S^{n-1}_{\mathfrak{t}}$ the relation $\mathfrak{r}$ gives a relation between elements of $(\tilde{\rho}^n_k)^{-1}(\tilde{S}^{n-2}_{\mathfrak{t}}) \subseteq \tilde{S}^{n-1}_{\mathfrak{t}}$ and $(\tilde{\rho}^n_k)^{-1} \circ \phi \circ \rho^n_k(r_l) = \phi(r_l)$ with $l \leq i_j$. But, due to construction $\phi(r_l)$ is then not in $\tilde{S}^n_{\mathfrak{t}}$. This is a contradiction.
Thus, $q \geq p$ and $s^n_{\mathfrak{t}} - s^{n-1}_{\mathfrak{t}} \geq s^{n+1}_{\mathfrak{t}} - s^n_{\mathfrak{t}} \geq s^{n+2}_{\mathfrak{t}} - s^{n+1}_{\mathfrak{t}} \geq \cdots$. Since $s^n_{\mathfrak{t}} - s^{n-1}_{\mathfrak{t}}$ is bounded from below by 0 this chain of inequalities becomes stationary. Thus, there exists $n_0, c \in \mathbb{N}$ such that for all $n \geq n_0$ the difference $s^n_{\mathfrak{t}} - s^{n-1}_{\mathfrak{t}} = c$.
The rest of the claim concerning the relations follows immediately from the construction of the maximal sets of algebraically independent invariants $\tilde{S}^n_{\mathfrak{t}}$ and the considerations above. □

The lemma above shows that there exists a number of copies, thenceforwards no new types of dimension-reducing relations will occur. Further, then there exists a common description of the maximal sets of algebraically independent invariants.

As announced, this result can be generalized to several polynomial types:
Let $\mathfrak{t} := \{\mathfrak{t}_1, \ldots, \mathfrak{t}_q\}$ be a set of finitely many polynomial types. Define $S^n_{\mathfrak{t}} := \cup_{i=1}^q S^n_{\mathfrak{t}_i}$. Then, $\tilde{S}^n_{\mathfrak{t}}$ is constructed out of all $\tilde{S}^n_{\mathfrak{t}_i}$ in the following way: $S = \tilde{S}^n_{\mathfrak{t}_1}$. Then, if the first element p of $\tilde{S}^n_{\mathfrak{t}_2}$ and the elements of S (the $\tilde{S}^n_{\mathfrak{t}_i}$ are considered to be ordered in the way they were obtained in their construction) are algebraically independent, S is

set to $S \cup \{p\}$ otherwise it remains again unchanged. This will be done further with the second, third,... element of $\tilde{S}^n_{\mathfrak{t}_2}$, then with the first,... element of $\tilde{S}^n_{\mathfrak{t}_3}$ and so on. The resulting set S is then defined to be $\tilde{S}^n_{\mathfrak{t}}$, that is again a maximal algebraically independent subset of $S^n_{\mathfrak{t}}$. A fully analogous proof as for one polynomial type (lemma 3.2.2) gives:

Lemma 3.2.3. Let $\mathfrak{t} := \{\mathfrak{t}_1, \ldots, \mathfrak{t}_q\}$ and $(\tilde{S}^n_{\mathfrak{t}})_{n\in\mathbb{N}}$ be given as above. Then, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ the number of elements of $\tilde{S}^n_{\mathfrak{t}} \setminus \tilde{S}^{n-1}_{\mathfrak{t}}$ being of type $\mathfrak{t}_i$ is a constant c_i for each i and the second part of lemma 3.2.2 concerning the relations holds.

Remark 3.2.4.
i) From the above lemmata the following can be stated: Let m_0 be the number of copies such that lemma 3.1.5 holds and let the dimension of $S^n_{\mathfrak{t}}(V^n) \subseteq \mathbb{R}^{s^n_{\mathfrak{t}}}$ equal $cn + d$ for constants $c, d \in \mathbb{N}$. Then, the knowledge of maximal sets of algebraically independent invariants E_k and E_{k+1} for k and $k+1$ copies, respectively, with $k \geq m_0$ and $|E_{k+1} \setminus E_k| = c$ implies that the sets $E_{l+1} := E_l \cup \{q|\ q$ is obtained from $q' \in E_l \setminus E_{l-1}$ by mapping the $(l-1)$th coordinate to the lth$\}$ are maximal sets of algebraically independent elements of $S^{l+1}_{\mathfrak{t}}$ for each $l > k$.
ii) All the statements found in this section also hold when V is replaced by a subset $X \subseteq V$. Then the proofs are obtained by just replacing V by X. This is possible since the vector space structure of V is not used in the proofs.

Example 3.2.5. Consider the example of $SU(2)$ acting diagonally on $SU^m(2)$ by conjugation. A basis will be given in lemma 4.6.2. Using restriction to $SU(2)$ this basis can be reduced to

$$\mathrm{tr}A_i, \mathrm{tr}A_jA_k, \mathrm{tr}A_lA_pA_q, \text{ with } 1 \leq i \leq m, 1 \leq j < k \leq m, 1 \leq l < p < q \leq m.$$

The idea is to find a maximal set S^m of algebraically independent basis elements:
Claim: $S^m = \{\mathrm{tr}A_i,\ \mathrm{tr}A_1A_j, \mathrm{tr}A_2A_k\}_{m\geq i\geq 1;\ m\geq j,k\geq 2}$.

Proof: For $m = 1$ and $m = 2$ the set S^m is already a cofree basis. Since S^m consists of $m + (m-1) + (m-2) = 3m - 3$ elements for $m \geq 3$, that corresponds to the dimension of the orbit space (cf. example 3.1.1), it is maximal if all elements are algebraically independent. If it is algebraically independent, relations of including the following invariants are expected, say for getting the "right" dimension:
1. relation in $\mathrm{tr}A_1A_2A_3, \mathrm{tr}A_1, \mathrm{tr}A_2, \mathrm{tr}A_3, \mathrm{tr}A_1A_2, \mathrm{tr}A_1A_3$ and $\mathrm{tr}A_2A_3$
2. relation in $\mathrm{tr}A_1, \mathrm{tr}A_2, \mathrm{tr}A_3, \mathrm{tr}A_4, \mathrm{tr}A_1A_2, \mathrm{tr}A_1A_3, \mathrm{tr}A_1A_4, \mathrm{tr}A_2A_3, \mathrm{tr}A_2A_4$ and $\mathrm{tr}A_3A_4$

A relation of the first type was found:

$$\begin{aligned}
&\mathrm{tr}^2\ A_iA_jA_k \\
&+\mathrm{tr}A_iA_jA_k(\mathrm{tr}A_i\mathrm{tr}A_j\mathrm{tr}A_k - \mathrm{tr}A_iA_j\mathrm{tr}A_k - \mathrm{tr}A_iA_k\mathrm{tr}A_j - \mathrm{tr}A_jA_k\mathrm{tr}A_i) \\
&+\mathrm{tr}^2\ A_iA_j + \mathrm{tr}^2\ A_iA_k + \mathrm{tr}^2\ A_jA_k + \mathrm{tr}^2\ A_i + \mathrm{tr}^2\ A_j + \mathrm{tr}^2\ A_k - \mathrm{tr}A_i\mathrm{tr}A_j\mathrm{tr}A_iA_j \\
&-\mathrm{tr}A_i\mathrm{tr}A_k\mathrm{tr}A_iA_k - \mathrm{tr}A_j\mathrm{tr}A_k\mathrm{tr}A_jA_k - \mathrm{tr}A_i\mathrm{tr}A_j\mathrm{tr}A_k = 4. \qquad (3.3)
\end{aligned}$$

This relation was established, using a computer, by the restriction of the relations determining $\mathfrak{gl}^3(2,\mathbb{C})/SU(2)$ found by a multihomogeneous ansatz function of degree $(2,2,2)$ (see subsection 4.4).
Even now, when the second relation is not found yet, it can be shown that the guess of a maximal set of algebraically independent generators was right: This will be proved at first for $m = 3, 4$.
The relation found for $\mathrm{tr}A_1A_2A_3$ shows that the guess is right for S^3. The set $T_1 := S^3 \cup \{\mathrm{tr}A_4\}$ is still algebraically independent. Further $T_1 \cup \{\mathrm{tr}A_1A_4\}$ is also algebraically independent since T_1 only determines the eigenvalues of A_4.
It remains to show that $T_1 \cup \{\mathrm{tr}A_1A_4, \mathrm{tr}A_2A_4\}$ is still algebraically independent. Let the opposite be assumed: Then, by mapping A_2 to A_3 the same holds for the set $T_1 \cup \{\mathrm{tr}A_1A_4, \mathrm{tr}A_3A_4\}$. Since $T_1 \cup \{\mathrm{tr}A_1A_4, \mathrm{tr}A_2A_4\}$ is algebraically dependent but $T_1 \cup \{\mathrm{tr}A_1A_4\}$ not, $T_1 \cup \{\mathrm{tr}A_1A_4,\ \mathrm{tr}A_1A_2A_4\}$ is also algebraically dependent (due to relation (3.3) and corollary B.8). Analogous argumentation leads to the algebraic dependence of the sets $T_1 \cup \{\mathrm{tr}A_1A_4,\ \mathrm{tr}A_1A_3A_4\}$ and $T_1 \cup \{\mathrm{tr}A_1A_4,\ \mathrm{tr}A_2A_3A_4\}$. But this implies that $T_1 \cup \{\mathrm{tr}A_1A_4\}$ is already maximal algebraically independent. This is a contradiction to the dimension of the orbit space and lemma B.7.
Due to remark 3.2.4.ii the set S^m is also maximal algebraically independent for all $m \geq 5$. □

The knowledge of the second expected relation would give together with the relation found for $\mathrm{tr}A_iA_jA_k$ a set of relations for S^m such that each new occurring relation does not reduce the dimension.

3.3 Further relations

In the last section of this chapter some ideas for treating the other types of relations and the inequalities are given. At the end open questions and assumptions are listed.

Assume now a sequence of maximal sets of algebraically independent invariants $(S^m)_{m\in\mathbb{N}}$ as the one described in the last section and the corresponding set of relations $\mathfrak{R}_{n_0}$ are given:

$$\text{orbit space } = S_{\mathfrak{t}}^n(V^n) \subseteq \mathcal{Z}(\tilde{\mathfrak{R}}_n) \subseteq \mathbb{R}^{|S_{\mathfrak{t}}^n|},$$

where $\tilde{\mathfrak{R}}_n$ follows from $\mathfrak{R}_{n_0}$ as in lemma 3.2.2. Due to lemma 3.2.3 this implies the knowledge of all dimension-reducing relations. Thus, only relations of the second type (cf. remark B.9) will be now taken into account (Relations of the third type may be still unknown, but they are not necessary to give a semialgebraic description of the orbit space). Let $\mathfrak{v}$ be such a type. Consider now the projection of the orbit space onto the coordinates belonging to the maximal set of algebraically independent invariants:

$$\mathrm{pr}: \ S^n_{\mathfrak{t}}(V^n) \subseteq \mathbb{R}^{|S^n_{\mathfrak{t}}|} \to S^n(V^n) \subseteq \mathbb{R}^{|S^n|}.$$

As the projection of the connected orbit space, $S^n(V^n)$ is connected, too. Since S^n is algebraically independent, it is described as a semialgebraic variety just by a set of inequalities $\mathfrak{U}^n$ (cf. corollary B.8.i). For the following, such a sequence $(\mathfrak{U}^n)_{n\in\mathbb{N}}$ is assumed to be known. Thus, the considerations can be restricted to new relations $\mathfrak{v}$ of type 2.2 (cf. remark B.9).
Since the set S^n is algebraically independent every point of $S^n(V^n)$ has only finitely many preimages in the connected orbit space $S^n_{\mathfrak{t}}(V^n)$.
In the following a basis of the action Ψ^m with the following property is assumed:

($**$) There exists a sequence of maximal sets $(S^m)_{m\in\mathbb{N}}$ of algebraically independent invariants of Ψ^m such that each set S^m separates the orbit types of Ψ^m.

Example 3.3.1. The set S^m for the diagonal actions of $SU(2)$ on $SU^m(2)$ of example 3.2.5 separates the orbit types:
As seen in example 1.1.1 there exist only three different orbit types. Assume now there exists a number of copies $m \in \mathbb{N}$ such that S^m does not separate the orbit types. Then, there exist two orbits $\mathcal{O}_1$ going through $(A_1, \ldots, A_m)$ and $\mathcal{O}_2$ going through $(B_1, \ldots, B_m)$ having different orbit types but $S^m(A_1, \ldots, A_m) = S^m(B_1, \ldots, B_m)$. Two cases can be distinguished:

1. $\mathcal{O}_1$ has the orbit type $[SU(2)]$ and $\mathcal{O}_2$ not: Then, $\mathrm{tr}^2 B_i = \mathrm{tr}^2 A_i = 4$ for all i. This implies $B_i = \pm\mathrm{Id}$ and thus, has also orbit type $[SU(2)]$. That is a contradiction.
2. $\mathcal{O}_1$ has the orbit type $[\{\mathrm{diag}(e^{\mathrm{i}\beta}, e^{-\mathrm{i}\beta})\}_\beta]$ and $\mathcal{O}_2$ $[\{\mathrm{Id}, -\mathrm{Id}\}]$. Then, $[A_i, A_j] = 0$ for all i, j. Since S^2 is even a basis, it separates the orbits of Ψ^2. Hence, (A_i, A_j) and (B_i, B_j) are in the same orbit of Ψ^2. This means, $[B_i, B_j] = 0$ for all i, j, which is in contradiction to the orbit type assumed for $(B_1, \ldots, B_m)$.

An idea to use the property ($**$) that it reduces the "number of components" a new relation of type 2.2 cuts off: Considering the transition from k copies to $k+1$ and looking for a new relation of type 2.2 (it will be assumed that this is only one relation since this can be always achieved by squaring some relations and add them) the property ($**$) allows to view the new relation as cutting of only whole components living over $S^n(V^n)$. An assumption is that by counting arguments one can show that there is an end of this procedure, i.e. there exists a k_0 such that no such relation occur by the transition from k_0 to $k_0 + 1$.

The following questions remain:

1. Does there exist a common description of the sequence of inequalities $(\mathfrak{U}^n)_{n\in\mathbb{N}}$?

2. For which groups does there exist a maximal set of algebraically invariants that separates the orbit types?

3. The main question: Does there exist a common description for the orbit spaces of an arbitrary number of copies as in the case when only dimension-reducing relations are considered (see lemma 3.2.3)?

Chapter 4

Orthogonal actions on $\mathbb{R}^n$

In this section a method given by Abud and Sartori in [1] for determining the orbit space for an orthogonal action of a compact linear Lie group G on $\mathbb{R}^n$ will be introduced. For applying the algorithm a minimal homogeneous basis is assumed to be already established. Then fulfilling this assumption all relations and inequalities can be extracted - in principle. After introducing the algorithm some remarks on the actual computation and for the providing of the minimal homogeneous basis are given.

4.1 The algorithm of Abud and Sartori

Starting with a homogeneous minimal basis $\theta = (\theta_1, \ldots, \theta_q) \in \mathrm{Pol}_G(\mathbb{R}^n)^q$, the gradients $\partial\theta_i : \mathbb{R}^n \to \mathbb{R}^n$ are determined for all $i \in \{1, \ldots, q\}$. Then one defines the matrix P by

$$P := \partial\theta^T \partial\theta, \quad P_{ij} = \langle \partial\theta_i, \partial\theta_j \rangle,$$

where $\langle\, .\,, .\rangle$ denotes the Euclidean scalar product in $\mathbb{R}^n$. The action is assumed to be orthogonal, i.e. for all $x, y \in \mathbb{R}^n$ and $g \in G$ one has $\langle x, y\rangle = \langle g \cdot x, g \cdot y\rangle$. Then the following holds:

Lemma 4.1.1. [1, pp. 337-338]
i) $P = \partial\theta^T \partial\theta$ is real, symmetric and positive semidefinite.
ii) Each entry of P is a G-invariant polynomial.
iii) rk $P(\phi)$ = rk $\partial\theta_i(\phi)$ = the dimension of the stratum $\hat{\Sigma}(\phi)$ of $\mathbb{R}^n/G$ containing the image of $\phi \in \mathbb{R}^n$. In particular, if the rank is maximal, the basis is cofree.

From ii), the existence of a matrix $\hat{P}$ such that $P = \hat{P} \circ \theta$ follows and, due to iii), it holds

$$\mathrm{rk}\ \hat{P}(\theta(\phi)) = \mathrm{rk}\ P(\phi) = \dim\ \hat{\Sigma}(\phi).$$

Now, all the information needed to describe the strata in $\mathbb{R}^n$ is established:
The set

$$V^{(k)} = \{\phi \in \mathbb{R}^n \,|\, \mathrm{rk}\ P(\phi) \leq k\}$$

contains all the strata of $\mathbb{R}^n$ whose images in $\mathbb{R}^n/G$ have dimension less than or equal to k and, analogously,

$$W^{(k)} = \{\phi \in \mathbb{R}^n \mid \mathrm{rk}\, P(\phi) = k\}$$

contains all strata whose images have dimension k. It holds:

$$V^{(k+1)} \setminus V^{(k)} = W^{(k+1)} \text{ and } \overline{W^{(k)}} = V^{(k)}.$$

However, the final description of the strata in $\mathbb{R}^n/G$ is still missing. At the level of the orbit space, θ is seen as a vector in $\mathbb{R}^q$ and the information that it is an image point of a vector of polynomials is lost. Defining the two sets

$$\begin{aligned} &\hat{\Lambda}^{(k)} = \{\theta \in \mathbb{R}^q \mid \hat{P}(\theta) \geq 0,\ \mathrm{rk}\, \hat{P}(\theta) \leq k\} \\ \text{and} \quad &\hat{\Gamma}^{(k)} = \{\theta \in \mathbb{R}^q \mid \hat{P}(\theta) \geq 0,\ \mathrm{rk}\, \hat{P}(\theta) = k\}, \end{aligned}$$

where $\hat{P}(\theta) \geq 0$ means that the matrix is positive semidefinite, one immediately gets

$$\hat{V}^{(k)} := \pi(V^{(k)}) \subseteq \hat{\Lambda}^{(k)} \text{ and } \hat{W}^{(k)} := \pi(W^{(k)}) \subseteq \hat{\Gamma}^{(k)},$$

where π denotes the projection onto the orbit space, the orbit map.
At the transition from P to $\hat{P}$ the information of being positive semidefinite is lost, that's why this condition is reinserted in the definition of $\hat{\Lambda}^{(k)}$ and $\hat{\Gamma}^{(k)}$.
In general, equality does not hold (see example 4.1.2.ii). In particular, the principal stratum $\pi(W^{(\mu)}) =: \hat{\Sigma}_p$, where μ is the dimension of the principal stratum, is connected, but $\hat{\Gamma}^{(\mu)}$ in general not. However, a connected component of $\hat{\Lambda}^{(\mu)}$ does always coincide with $\hat{\Sigma}_p$:
Let ϕ_p be a principal point of $\mathbb{R}^n$, i.e. a point in the principal stratum, and denote by $\hat{\Gamma}_p^{(\mu)}$ the connected component of $\hat{\Gamma}^{(\mu)}$ containing $\theta(\phi_p)$. Due to theorem 1.1.2.i, $\hat{\Gamma}_p^{(\mu)}$ then contains the whole principal stratum. Since the boundary of $\hat{\Sigma}_p$ does not intersect $\hat{\Gamma}_p^{(\mu)}$, $\hat{\Gamma}_p^{(\mu)} = \hat{\Sigma}_p$ follows. All other strata are then given by the intersections

$$\overline{\hat{\Sigma}_p} \cap \hat{\Gamma}^{(k)} = \text{ union of all strata of dimension } k,$$

since the union of all auxiliary strata form the boundary of the principal stratum.

Example 4.1.2.
i) Let $SO(2)$ act on $x \in \mathbb{R}^2$ by matrix multiplication that corresponds to a rotation of x by a certain angle. The orbits are concentric circles (including the degenerated case of the origin) and each orbit corresponds to a fixed radius. The invariant polynomials $\mathrm{Pol}_{SO(2)}(\mathbb{R}^2)$ are generated by $\theta_1 = x_1^2 + x_2^2$, hence $\partial\theta_1 = (2x_1, 2x_2)$. The algorithm is performed in the following table:

	$\mathbb{R}^2$	orbit space $\mathbb{R}^2/SO(2) \subseteq \mathbb{R}$
	$P = \langle \partial\theta_1,\ \partial\theta_1\rangle = 4x_1^2 + 4x_2^2$	$\hat{P} = 4\theta_1$
strata	$W^{(1)} = \{x\in\mathbb{R}^2 \mid 4x_1^2+4x_2^2 \neq 0\} = \mathbb{R}^2 - \{0\}$	$\widehat{\Gamma}^{(1)} = \{\theta_1 \in \mathbb{R} \mid \theta_1 \geq 0,\ \theta_1 \neq 0\} = \mathbb{R}_{>0}$
	$W^{(0)} = \{0\}$	$\widehat{\Gamma}^{(0)} = \{0\}$

Since $\hat{\Gamma}^{(1)}$ is connected it already coincides with the principal stratum, i.e. the orbit space $\mathbb{R}^2/SO(2)$ is given by the semialgebraic set $\{x\in\mathbb{R}^2 \mid x \geq 0\}$.
ii) [1, p. 344 example 3] The following is an example where the set $\hat{\Gamma}^{(\mu)}$ is not connected: The group $\mathbb{Z}_4 \times SO(3)$ acts on $\mathbb{R}^6 \cong \mathbb{C}^3$ by:

$$(\mathbb{Z}_4 \times SO(3)) \times \mathbb{C}^3 \ni ((k,A),z) \to \mathrm{i}^k Az \in \mathbb{C}^3 \text{ with } z = x + \mathrm{i}y,\ x,y\in\mathbb{R}^3.$$

A minimal basis of $\mathrm{Pol}_{\mathbb{Z}\times SO(3)}(\mathbb{R}^6)$ is given by:

$$\theta_1 = \| z \|^2 = \| x \|^2 + \| y \|^2,\ \theta_2 = (\mathrm{Re}\ z^2)^2 = (\| x \|^2 - \| y \|)^2$$
$$\theta_3 = (\mathrm{Im}\ z^2)^2 = 4\langle x,y\rangle^2,\ \theta_4 = (\mathrm{Re}\ z^2)(\mathrm{Im}\ z^2) = 2\langle x,y\rangle(\| x \|^2 - \| y \|^2).$$

The basis is not cofree since the relation $\theta_4^2 = \theta_2\theta_3$ is fulfilled. The $\hat{P}$-matrix of this action reads:

$$\frac{1}{4}\hat{P} = \begin{pmatrix} \theta_1 & 2\theta_2 & 2\theta_3 & 2\theta_4 \\ & 4\theta_1\theta_2 & 0 & 2\theta_1\theta_4 \\ & & 4\theta_1\theta_3 & 2\theta_1\theta_4 \\ & & & \theta_1(\theta_2+\theta_3) \end{pmatrix},$$

where all the omitted entries are determined by the symmetry of the matrix. The relation $\theta_4^2 = \theta_2\theta_3$ can also be found by calculating the determinant of $\hat{P}$:

$$\det \frac{1}{4}\hat{P} = 16\,\theta_1{}^2\,(\theta_2+\theta_3)\left(\theta_3 - \theta_1{}^2 + \theta_2\right)\left(-\theta_2\theta_3 + \theta_4{}^2\right) = 0.$$

Hence, the rank of $\hat{P}$ is at most 3. The set $\hat{\Gamma}^{(3)}$ contains all points $\theta\in\mathbb{R}^4$ such that $\mathrm{rk}\ \hat{P}(\theta) = 3$ and $\hat{P}(\theta)$ is positive semidefinite. After extracting all these conditions $\hat{\Gamma}^{(3)}$ is found to be the disjoint and disconnected union of

$$\begin{aligned} &\mathcal{C}_1 &&:= \{\theta_2\theta_3 = \theta_4^2,\ \theta_1^2 > \theta_2+\theta_3 > 0,\ \theta_1 > 0\} \\ \text{and}\ &\mathcal{C}_2 &&:= \{\theta_2+\theta_3 = \theta_1^2 > 0,\ \theta_2\theta_3 > \theta_4^2\}. \end{aligned}$$

Using any principal point or the relation already known, it is found that the principal stratum has to be a subset of $\mathcal{C}_1$. Since $\mathcal{C}_1$ is connected, it is the principal stratum.

In general, one cannot immediately read off $\hat{P}$ from P. Therefore, an algorithm is necessary to find for a given G-invariant polynomial p a polynomial $\tilde{p}$ with $p = \tilde{p} \circ \theta$, θ being the Hilbert map. That will also solve the problem of finding a minimal homogeneous basis if an arbitrary basis is given.
Furthermore, it has to be figured out how to extract the inequalities and equalities from the rank and positive semidefiniteness condition, how to choose the "right" connected component and which new inequalities or equalities this choice implies. This will be done in the next sections.

4.2 Minimization of a given basis of $\mathrm{Pol}_G(\mathbb{R}^n)$

Let a homogeneous basis $\theta = (\theta_1, \ldots, \theta_q)$ of $\mathrm{Pol}_G(\mathbb{R}^n)$ be given. Its homogeneity and the vector space structure of $\mathbb{R}^n$ allow to give an algorithm to minimize θ.
Firstly, it will be stated how to find for a given G-invariant polynomial p on $\mathbb{R}^n$ a polynomial $\tilde{p}$ on $\mathbb{R}^q$ such that $p = \tilde{p} \circ \theta$ (that is also necessary to find the matrix $\hat{P}$).

Lemma 4.2.1. Let p be a homogeneous G-invariant polynomial on $\mathbb{R}^n$ of degree d. Then there exists $\tilde{p}$ fulfilling $p = \tilde{p} \circ \theta$ that can be written as a sum of monomials of the following form: $wy_1^{i_1} \cdots y_q^{i_q}$ with $d = \sum_{j=1}^q i_j \deg\, \theta_j$ and w being a real constant.

Proof: Since p is G-invariant, it is generated by θ. That means that there exists $\tilde{p} \in \mathrm{Pol}(\mathbb{R}^q)$ with $p = \tilde{p} \circ \theta$. Let $\tilde{p}_i = wy_1^{i_1} \cdots y_q^{i_q}$, $(y_1, \ldots, y_q) \in \mathbb{R}^q$, be a monomial of $\tilde{p}$. Define $\deg_\theta \tilde{p}_i := \sum_{j=1}^q i_j \deg\, \theta_j$ and decompose $\tilde{p}$ into polynomials r_k where $\tilde{p}_j$ is a summand of $\mathfrak{r}_k$ if and only if $\deg_\theta \tilde{p}_j = k$ holds:

$$p = \sum_k r_k \circ \theta.$$

Scaling $x \in \mathbb{R}^n$ by a real parameter t gives

$$t^{\deg\, p}(p - r_{\deg\, p} \circ \theta) - \sum_{k \neq \deg\, p} t^k r_k \circ \theta = 0.$$

Since t is arbitrary $p = r_{\deg\, p} \circ \theta$. □

For finding such a polynomial in θ an ansatz function is generated. This means, all tuples $(i_1, \ldots, i_q)$ with $\deg\, p = \sum_{j=1}^q i_j \deg\, \theta_j$ are listed. The corresponding monomials are added each with a coefficient w_k, k is an index of the monomial. Then, p as well as the ansatz function is a polynomial in $x_1, \ldots, x_n$, the coordinates of x, and comparison of coefficients determines the coefficients. In general, if there exists a relation between θ of degree j, this determination of the coefficients is not unique, but ends up with

$$p = \mathrm{polynomial}(\theta) + \sum_j w_j P_j$$

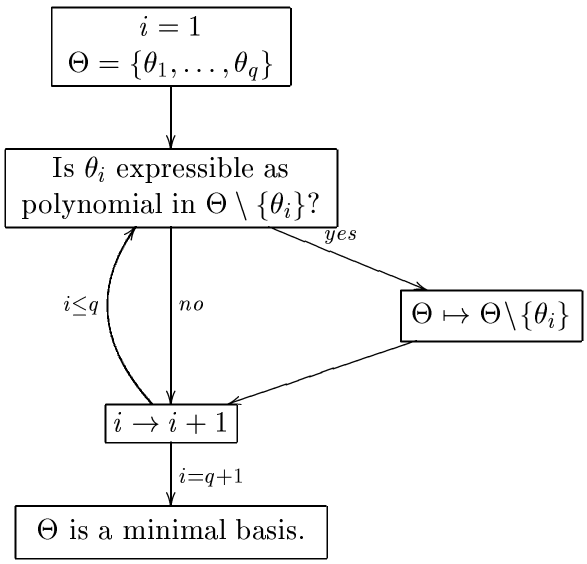

Figure 4.1: Minimization of a basis Θ

where for every j the polynomial P_j is a relation in θ, maybe even the zero polynomial. Setting all variables w_j to 0, a polynomial of the form asserted in lemma 4.2.1 is found.
For an arbitrary G-invariant polynomial p this procedure is done for every summand of its homogeneous decomposition (due to lemma 2.1.4 these summands are G-invariant).

If one wants to test whether e.g. θ_1 can be expressed by a polynomial in $\theta_2, \dots, \theta_q$, the procedure is essentially the same. First, an ansatz function is generated and comparison of coefficients gives a system of equations in the variables w_i which has a solution if and only if θ_1 is a polynomial in $\theta_2, \dots, \theta_q$. This gives rise to a minimization procedure (see figure 4.1).

Remark 4.2.2. With the help of lemma 3.1.2, both lemma 4.2.1 and the minimization algorithm can be extended straightforward to diagonal actions. Instead of homogeneous polynomials, multihomogeneous ansatz functions are used. For this purpose, the considerations above generalize if the degrees are studied for each copy separately.

4.3 Positive semidefiniteness

Before giving some equivalent definitions of positive semidefiniteness, some notions regarding quadratic forms are introduced.
Let $A = (a_{ij})$ be a symmetric $n \times n$ matrix. Then, $q = q(x) = x^T A x = \sum_{i,j} a_{ij} x_i x_j$ is the corresponding quadratic form. Two quadratic forms given by A and A' are

equivalent if and only if there exists a non-singular matrix B with $A' = B^T AB$. Using such transformations, every quadratic form of rank r (= the rank of the corresponding symmetric matrix) can be transformed to a form ([2])

$$q'(x) = h_1 x_1^2 + \ldots + h_r x_r^2,\ h_i \neq 0, \tag{4.1}$$

that is then called the *reduced form*.

Definition 4.3.1. [14, pp. 279, 281] A real symmetric $n \times n$ matrix A is called *positive semidefinite* if one of the following equivalent statements holds:
i) $x^T Ax \geq 0$ for all $x \in \mathbb{R}^n$.
ii) All eigenvalues of A are non-negative.
iii) Each of the principal minors has non-negative determinant.
iv) The corresponding reduced quadratic form has only non-negative coefficients.

Here, condition iv) will be used to figure out the inequalities that have to be fulfilled such that a given matrix is positive semidefinite. But for applying this condition the reduced quadratic form has to be found. There exist several algorithms for reducing a given quadratic form. Next Lagrange's reduction will be stated and worked with (Another method is e.g. Kronecker's reduction which can also be found in [2]).

Lagrange's reduction. [2, p. 132] This algorithm is essentially based on repeating quadratic completion: Let the non-reduced quadratic form $q = \sum_{i,j} a_{ij}x_ix_j$ be given.
1. Choose, if possible, a pair (i, j) such that $a_{ii}, a_{ij} \neq 0$. If such a pair does not exist, choose a pair (i, j) such that $a_{ij} \neq 0$ (such a pair has to exist as otherwise q would already be a reduced quadratic form) and substitute x_i by $x_i + x_j$. Then, in the resulting form, again denoted by q, the pair (i, j) fulfills $a_{ii}, a_{ij} \neq 0$.
2. Write the quadratic form and the resulting one after the substitution, respectively, as

$$q = \underbrace{a_{ii}}_{\neq 0} x_i^2 + \underbrace{(\sum_{i \neq j} a_{ij}x_j)}_{=r} x_i + \text{rest(not depending on } x_i \text{ anymore)}$$

and substitute x_i by $x_i - \frac{r}{2a_{ii}}$. The obtained quadratic form contains x_i only as squared term.
The steps 1 and 2 are repeated until the quadratic form is a reduced one. All substitutions used during the algorithm are non-singular transformations, hence all new obtained forms are equivalent to the original q and the resulting reduced form is indeed the reduced form of the original q. Furthermore, the algorithm terminates.

Example 4.3.2. (application of Lagrange's reduction)

The quadratic form

$$\begin{aligned} q = {} & 2x_1^2 + 4\theta_1 x_1 x_3 + 4\theta_2 x_1 x_4 + 4\theta_1 x_1 x_5 + 2x_2^2 - 4\theta_2 x_2 x_3 + 4\theta_1 x_2 x_4 \\ & + 4\theta_2 x_2 x_5 + 4\theta_5 x_3^2 + 8\theta_3 x_3 x_5 + 4\theta_5 x_4^2 + 8\theta_4 x_4 x_5 + 4\theta_5 x_5^2 \end{aligned}$$

shall be reduced. Substitute $x_1 \to x_1 - (\theta_1 x_3 + \theta_2 x_4 + \theta_1 x_5)$; for the sake of sim–

$$
\begin{tikzcd}
 & q \arrow[d] \arrow[dr] & \\
 & A>0 \arrow[dl] \arrow[d] & A=B=C=0 \\
4A^2-B^2-C^2>0 & 4A^2=B^2+C^2 & \\
\mathrm{rk}\ q=5 & \mathrm{rk}\ q=4 & \mathrm{rk}\ q=2
\end{tikzcd}
$$

Figure 4.2: Scheme of the conditions for the positive semidefiniteness of the quadratic form in example 4.3.2 and the corresponding ranks

plicity, all resulting forms will also be referred to as q.

$$
\begin{aligned}
q = {} & 2x_1^2 + 2x_2^2 - 4\theta_2 x_2 x_3 + 4\theta_1 x_2 x_4 + 4\theta_2 x_2 x_5 + (4\theta_5 - 2\theta_1^2)x_3^2 - 4\theta_1\theta_2 x_3 x_4 \\
& + (8\theta_3 - 4\theta_1^2)x_3 x_5 + (4\theta_5 - 2\theta_2^2)x_4^2 + (8\theta_4 - 2\theta_2\theta_1)x_4 x_5 + (4\theta_5 - 2\theta_1^2)x_5^2.
\end{aligned}
$$

Substitution of $x_2 \to x_2 - (-\theta_2 x_3 + \theta_1 x_4 + \theta_2 x_5)$ gives

$$
\begin{aligned}
q = {} & 2x_1^2 + 2x_2^2 + \underbrace{(4\theta_5 - 2\theta_1^2 - 2\theta_2^2)}_{A} x_3^2 + \underbrace{(8\theta_3 - 4\theta_1^2 + 4\theta_2^2)}_{B} x_3 x_5 \\
& + (4\theta_5 - 2\theta_2^2 - 2\theta_1^2)x_4^2 + \underbrace{(8\theta_4 - 8\theta_2\theta_1)}_{C} x_4 x_5 + (4\theta_5 - 2\theta_1^2 - 2\theta_2^2)x_5^2.
\end{aligned}
$$

q is positive semidefinite iff $q - 2x_1^2 - 2x_2^2$ is. Then, q is positive semidefinite if one of the following cases apply:

1. $A = 4\theta_5 - 2\theta_1^2 - 2\theta_2^2 > 0$ and $q = A^2x_3^2 + ABx_3x_5 + A^2x_4^2 + ACx_4x_5 + A^2x_5^2$ is positive semidefinite. Quadratic completion gives:

$$
q = (Ax_3 + \frac{1}{2}Bx_5)^2 + (Ax_4 + \frac{1}{2}Cx_5)^2 + (A^2 - \frac{1}{4}B^2 - \frac{1}{4}C^2)x_5^2
$$

In this case, $A^2 - \frac{1}{4}B^2 - \frac{1}{4}C^2 = 4(\theta_5 - \theta_3 - \theta_2^2)(\theta_5 + \theta_3 - \theta_1^2) - 4(\theta_4 - \theta_1\theta_2)^2 \geq 0$ has to be fulfilled.

2. $A = 0$ and $q = Bx_3x_5 + Cx_4x_5$ is positive semidefinite.
After substitution of $x_3 \to x_3 + x_5$ and $x_4 \to x_4 + x_5$ the resulting form reads

$$
q = B(x_3 + \frac{1}{2}x_5)^2 + C(x_4 + \frac{1}{2}x_5)^2 - \frac{1}{4}(B + C)x_5^2.
$$

This one can only be positive semidefinite if $B = C = 0$.

All these conditions are summarized in table 4.2.

4.4 Calculation of relations

A matrix is of rank k if every $(k+1) \times (k+1)$ minor has zero determinant and there exists a $k \times k$ minor with non-zero determinant. Hence, finding relations means calculating several determinants. Since the complexity of the computation of the determinant of an $n \times n$ matrix grows exponentially with n, it becomes very fast a huge computational effort to calculate the relations.

In lemma 3.1.2 it was shown that there always exists a multihomogeneous basis of $\mathrm{Pol}_G(V^n)$ and then, the relations can also be assumed to be multihomogeneous. If further a relation is expected between a certain set $\mathcal{S}$ of basis elements, e.g. with the help of a maximal set of algebraically independent basis elements already found, there is still the possibility to generate an ansatz function for $\mathcal{S}$. The disadvantage is now that the degree of the relation is not known. But when starting with the lowest possible degree one can create a multihomogeneous ansatz function of that degree and test whether this already gives a relation. If the outcome is negative, the same is done with a next higher degree and so on.

Example 4.4.1. Consider the action of $SU(2)$ on $\mathfrak{su}^3(2)$ by diagonal conjugation. Since $\mathfrak{su}(2)$ is a vector space the strategy of finding a relation will be the same. The only reason why it is chosen here as an example is the lower computational effort.

A (multi)homogeneous basis is given by (see lemma 5.2.3):

$$B := \{\mathrm{tr}B_1^2,\ \mathrm{tr}B_2^2,\ \mathrm{tr}B_3^2,\ \mathrm{tr}B_1B_2,\ \mathrm{tr}B_1B_3,\ \mathrm{tr}B_2B_3,\ \mathrm{tr}B_1B_2B_3\}.$$

Since the orbit space has dimension 6 (see lemma 5.2.3) a relation is expected. Using a parametrization of $\mathfrak{su}(2)$

$$B_i = \begin{pmatrix} \mathrm{i}x_{i3} & x_{i1} + \mathrm{i}x_{i2} \\ -x_{i1} + \mathrm{i}x_{i2} & -\mathrm{i}x_{i3} \end{pmatrix}, \quad x_{i1}, x_{i2}, x_{i3} \in \mathbb{R}$$

the invariants read:

$$\begin{aligned} \mathrm{tr}B_iB_j &= -2(x_{i1}x_{j1} + x_{i2}x_{j2} + x_{i3}x_{j3}), \quad i,j \in \{1,2,3\}, \\ \mathrm{tr}B_1B_2B_3 &= 2(x_{12}x_{21}x_{33} + x_{13}x_{22}x_{31} + x_{23}x_{11}x_{32} \\ &\quad - x_{23}x_{12}x_{31} - x_{33}x_{11}x_{22} - x_{13}x_{21}x_{32}). \end{aligned}$$

The multihomogeneous degrees of the invariants are $(1,1,0)$ (up to symmetry), where the polynomial is meant to be homogeneous of degree 'i-th element of this vector' w.r.t. to the i-th copy (see lemma 3.1.2), and $(1,1,1)$, respectively. Because of lemma 3.1.2.iii a multihomogeneous relation is looked for, that should further contain all three matrices since there is no relation for $SU(2)$ acting on $\mathfrak{su}^2(2)$ (see section C.3). Thus, the relation has minimal total degree 3. Total degree 3 can only correspond to a multihomogeneous degree of $(1,1,1)$ and results in an ansatz function where only

$\mathrm{tr}B_1B_2B_3$ occurs. Total degree 4 and multihomogeneous $(1,1,2)$, respectively, gives the ansatz function

$$w_1\mathrm{tr}B_1B_2\mathrm{tr}B_3^2 + w_2\mathrm{tr}B_1B_3\mathrm{tr}B_2B_3 = 0$$

with the only solution $w_1 = w_2 = 0$. The ansatz function of total degree 5, i.e. $(1,2,2,)$, factors into $\mathrm{tr}B_1B_2B_3$ times a polynomial of degree 2. Hence, the first degree which really stands a chance to succeed is 6 and $(2,2,2)$, respectively, and it gives in fact a relation:

$$\begin{aligned}\mathrm{tr}^2B_1B_2B_3 = {} & \mathrm{tr}B_1^2\mathrm{tr}^2B_2B_3 + \mathrm{tr}B_2^2\mathrm{tr}^2B_1B_3 + \mathrm{tr}B_3^2\mathrm{tr}^2B_1B_2 \\ & - \mathrm{tr}B_1^2\mathrm{tr}B_2^2\mathrm{tr}B_3^2 - 2\mathrm{tr}B_1B_2\mathrm{tr}B_1B_3\mathrm{tr}B_2B_3.\end{aligned}$$

4.5 Connected components

The last step of the algorithm of Abud and Sartori is to decide whether the found set $\hat{\Gamma}^{(\mu)}$ is connected and if not to find the right connected component.
This results in the problem of decomposing a semialgebraic set into its connected components. The connected components are again semialgebraic and the number of connected components is finite [4, p. 35].

The problem of the (efficient) computation of the decomposition of a semialgebraic set was studied in the last thirty years and is still an area of active research. Several algorithms that are essentially based on Collin's cylindrical decomposition (CAD, cf. for example [4]) were found [22]. The CAD is a solution to the key problem of computing at least one point in each connected component of a semialgebraic set. Although the CAD-algorithm is doubly exponential in the number of variables, there also exist algorithms that are able to decompose a semialgebraic set in subexponential time [7].
Further, there are also many results for an upper bound of the number of connected components, the zeroth Betti number, [21] that are also used to estimate the complexity of algorithms.
In most of the examples calculated here, the sets established are connected and it was straightforward to verify this property (cf. subsection 4.6.2).

4.6 Examples

4.6.1 $SU(n)$ acting on $\mathfrak{gl}^m(n,\mathbb{C})$

The algorithm introduced in section 4.1 is applicable to examples of the form $(S)U(n)$ is acting on $\mathfrak{gl}(n,\mathbb{C})$ by diagonal conjugation. For that purpose, it has to be shown that the action is diagonal, i.e. when identifying $\mathfrak{gl}(n,\mathbb{C})$ with $\mathbb{R}^{2n^2}$ the standard scalar product is invariant.

Lemma 4.6.1. Let $\iota : \mathfrak{gl}(n,\mathbb{C}) \to \mathbb{R}^{2n^2}$ be the canonical identification and let $\psi_g : \mathfrak{gl}(n,\mathbb{C}) \to \mathfrak{gl}(n,\mathbb{C})$, $\psi_g(A) := gAg^{-1}$ be the action of an element $g \in U(n)$ on $\mathfrak{gl}(n,\mathbb{C})$. Then,

$$\Phi_g := \iota \circ \psi_g \circ \iota^{-1} \in O(2n^2).$$

Proof: Let $g = (g_{ij})$ and $A = (a_{ij}) \in \mathfrak{gl}(n,\mathbb{C})$ be given. Then, $g^{-1} = (\overline{g_{ji}})$ and $\psi_g(A) = (\sum_{\alpha,\beta} g_{i\alpha}\overline{g_{j\beta}}a_{\alpha\beta})$. W.l.o.g. ι is assumed to be given by:

$$\iota : (a_{ij}) \mapsto (\mathrm{Re}\ a_{11}, \mathrm{Re}\ a_{12}, \ldots, \mathrm{Re}\ a_{nn}, \mathrm{Im}\ a_{11}, \ldots, \mathrm{Im}\ a_{nn}).$$

Then, Φ_g corresponds to the matrix:

$$\left(\begin{array}{ccc|ccc}
\mathrm{Re}\ g_{11}\overline{g_{11}} & \mathrm{Re}\ g_{11}\overline{g_{12}} & \cdots & -\mathrm{Im}\ g_{11}\overline{g_{11}} & -\mathrm{Im}\ g_{11}\overline{g_{12}} & \cdots \\
\mathrm{Re}\ g_{11}\overline{g_{21}} & \ddots & & -\mathrm{Im}\ g_{11}\overline{g_{21}} & \ddots & \\
\vdots & & & \vdots & & \\
\mathrm{Re}\ g_{n1}\overline{g_{n1}} & & & -\mathrm{Im}\ g_{n1}\overline{g_{n1}} & & \\
\hline
\mathrm{Im}\ g_{11}\overline{g_{11}} & \mathrm{Im}\ g_{11}\overline{g_{12}} & \cdots & \mathrm{Re}\ g_{11}\overline{g_{11}} & \mathrm{Re}\ g_{11}\overline{g_{12}} & \cdots \\
\mathrm{Im}\ g_{11}\overline{g_{21}} & \ddots & & \mathrm{Re}\ g_{11}\overline{g_{21}} & \ddots & \\
\vdots & & & \vdots & & \\
\mathrm{Im}\ g_{n1}\overline{g_{n1}} & & & \mathrm{Re}\ g_{n1}\overline{g_{n1}} & &
\end{array}\right).$$

In order to show that $\Phi_g \in O(2n^2)$, it suffices to verify that each column has magnitude 1 and the scalar product between different columns is zero:
The magnitude of a column is given by

$$\sum_{i,j}(\mathrm{Re}^2\ g_{i\alpha}\overline{g_{j\beta}} + \mathrm{Im}^2\ g_{i\alpha}\overline{g_{j\beta}}) = \sum_{i,j}|g_{i\alpha}\overline{g_{j\beta}}|^2 = (\sum_i |g_{i\alpha}|^2)(\sum_j |\overline{g_{j\beta}}|^2) = 1.$$

The scalar product of a column out of the left half of Φ_g with a column of the right half vanishes. It remains to calculate the scalar product of two different columns in the same half $((\alpha,\beta) \neq (\gamma,\delta))$:

$$\begin{aligned}
\sum_{i,j}(\mathrm{Re}\ g_{i\alpha}\overline{g_{j\beta}} \cdot \mathrm{Re}\ g_{i\gamma}\overline{g_{j\delta}} + \mathrm{Im}\ g_{i\alpha}\overline{g_{j\beta}} \cdot \mathrm{Im}\ g_{i\gamma}\overline{g_{j\delta}}) &= \sum_{i,j}\mathrm{Re}\ (g_{i\alpha}\overline{g_{j\beta}} \cdot \overline{g_{i\gamma}\overline{g_{j\delta}}}) \\
&= \sum_{i,j}(\mathrm{Re}\ g_{i\alpha}\overline{g_{i\gamma}} \cdot \mathrm{Re}\ \overline{g_{j\beta}}g_{j\delta} - \mathrm{Im}\ g_{i\alpha}\overline{g_{i\gamma}} \cdot \mathrm{Im}\ \overline{g_{j\beta}}g_{j\delta}) \\
&= \sum_{i,j}(g_{i\alpha}\overline{g_{i\gamma}} \cdot \overline{g_{j\beta}}g_{j\delta} - \mathrm{i}(\mathrm{Re}\ g_{i\alpha}\overline{g_{i\gamma}}\mathrm{Im}\ \overline{g_{j\beta}}g_{j\delta} + \mathrm{Re}\ \overline{g_{j\beta}}g_{j\delta}\mathrm{Im}\ g_{i\alpha}\overline{g_{i\gamma}})) \\
&= \underbrace{(\sum_i g_{i\alpha}\overline{g_{i\gamma}})(\sum_j \overline{g_{j\beta}}g_{j\delta})}_{=0} - \mathrm{i}\underbrace{\sum_{i,j}(\mathrm{Re}\ g_{i\alpha}\overline{g_{i\gamma}}\mathrm{Im}\ \overline{g_{j\beta}}g_{j\delta} + \mathrm{Re}\ \overline{g_{j\beta}}g_{j\delta}\mathrm{Im}\ g_{i\alpha}\overline{g_{i\gamma}})}_{real}
\end{aligned}$$

Since the scalar product is real it has to vanish. $\square$

4.6.2 The special case of $n = 2$

Now the special case of $SU(2)$ acting on $\mathfrak{gl}^m(2,\mathbb{C})$ is considered. At first, a minimal basis of invariants is given:

Lemma 4.6.2. Let $SU(2)$ act on $\mathfrak{gl}^m(2,\mathbb{C})$ by diagonal conjugation. The invariants

$$\begin{aligned}
&\mathrm{Re}\ \mathrm{tr}A_i, \mathrm{Im}\ \mathrm{tr}A_i, \mathrm{Re}\ \mathrm{tr}A_i^2, \mathrm{Im}\ \mathrm{tr}^2 A_i, \mathrm{tr}A_iA_i^*, \qquad 1 \le i \le m,\\
&\mathrm{Re}\ \mathrm{tr}A_iA_j, \mathrm{Im}\ \mathrm{tr}A_iA_j, \mathrm{Re}\ \mathrm{tr}A_iA_j^*, \mathrm{Im}\ \mathrm{tr}A_iA_j^*,\\
&\quad \mathrm{Re}\ \mathrm{tr}A_iA_jA_j^*, \mathrm{Im}\ \mathrm{tr}A_iA_jA_j^*, \mathrm{Re}\ \mathrm{tr}A_jA_iA_i^*, \mathrm{Im}\ \mathrm{tr}A_jA_iA_i^*, \qquad 1 \le i < j \le m,\\
&\mathrm{Re}\ \mathrm{tr}A_iA_jA_k^*, \mathrm{Re}\ \mathrm{tr}A_iA_j^*A_k, \mathrm{Re}\ \mathrm{tr}A_i^*A_jA_k,\\
&\quad \mathrm{Im}\ \mathrm{tr}A_iA_jA_k^*, \mathrm{Im}\ \mathrm{tr}A_iA_j^*A_k, \mathrm{Im}\ \mathrm{tr}A_i^*A_jA_k, \qquad 1 \le i < j < k \le m
\end{aligned}$$

form a minimal homogeneous basis $\mathcal{B}_m$, consisting of $|\mathcal{B}_m| := 5m + 8\binom{m}{2} + 6\binom{m}{3}$ elements. The dimension of the orbit space $\mathfrak{gl}^m(2)/SU(2)$ is $8m-3$.

Proof: For $m \le 3$ it can be easily verified, when using a computer, by starting with the basis given in section 2.1.2 and using the minimization procedure stated in section 4.2, that the above-mentioned set really gives a basis of $\mathrm{Pol}_{SU(2)}(\mathfrak{gl}^3(2,\mathbb{C}))$. Due to lemma 3.1.5 the set given above is a minimal basis for all m. The dimension of the orbit space follows from equation (1.3) and the principal orbit type is $[\{\mathrm{Id}, -\mathrm{Id}\}]$ (see example 3.1.1). $\square$

From the above lemmata it can be seen that each basis element contains at most three different matrices. Hence, an entry of P is a polynomial in at most six different matrices. Thus, if the matrix $\hat{P}$ is calculated for $m = 6$, $\hat{P}$ is known for every m.

The case $\mathbf{m = 1}$.
Let $SU(2)$ act on $\mathfrak{gl}(2,\mathbb{C})$ and let $A = (a_{ij} + \mathrm{i}b_{ij}) \in \mathfrak{gl}(2,\mathbb{C})$, $a_{ij}, b_{ij} \in \mathbb{R}$. Due to lemma 4.6.2 the invariants

$$\begin{aligned}
\theta_1 &= \mathrm{Re}\ \mathrm{tr}A &&= a_{11} + a_{22},\\
\theta_2 &= \mathrm{Im}\ \mathrm{tr}A &&= b_{11} + b_{22},\\
\theta_3 &= \mathrm{Re}\ \mathrm{tr}A^2 &&= a_{11}^2 - b_{11}^2 + 2a_{12}a_{21} - 2b_{12}b_{21} + a_{22}^2 - b_{22}^2,\\
\theta_4 &= \mathrm{Im}\ \mathrm{tr}A^2 &&= 2a_{11}b_{11} + 2a_{12}b_{21} + 2a_{21}b_{12} + 2a_{22}b_{22},\\
\theta_5 &= \mathrm{tr}AA^T &&= a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2 + b_{11}^2 + b_{12}^2 + b_{21}^2 + b_{22}^2
\end{aligned}$$

form a minimal homogeneous basis for $\mathrm{Pol}_{SU(2)}(\mathbb{R}^8)$. According to equation (1.3) the dimension of the orbit space is 5. Hence, the basis is cofree (cf. corollary B.8.i) and the ideal of relations is trivial. Now, the method of Abud and Sartori shall be

Dim	Orbit space strata	Orbit type
5	$2\,\theta_5 > {\theta_1}^2 + {\theta_2}^2$ $(\theta_4 - \theta_2\theta_1)^2 < \left(\theta_5 - {\theta_2}^2 - \theta_3\right)\left(\theta_5 - {\theta_1}^2 + \theta_3\right)$	$[\{\mathrm{Id}, -\mathrm{Id}\}]$
4	$(\theta_4 - \theta_2\theta_1)^2 = \left(\theta_5 - {\theta_2}^2 - \theta_3\right)\left(\theta_5 - {\theta_1}^2 + \theta_3\right)$ $2\,\theta_5 > {\theta_1}^2 + {\theta_2}^2$	$[\{\mathrm{diag}(e^{\mathrm{i}\alpha}, e^{-\mathrm{i}\alpha})\}_{\alpha\in[0,2\pi)}]$
2	$2\,\theta_5 = {\theta_1}^2 + {\theta_2}^2$ $2\,\theta_3 = {\theta_1}^2 - {\theta_2}^2$ $\theta_4 = \theta_1\theta_2$	$[SU(2)]$

Table 4.1: The orbit space $\mathfrak{gl}(2,\mathbb{C})/SU(2)$

applied. Then, the matrix $\hat{P}$ is given by:

$$\hat{P} = \begin{pmatrix} 2 & 0 & 2\,\theta_1 & 2\,\theta_2 & 2\,\theta_1 \\ 0 & 2 & -2\,\theta_2 & 2\,\theta_1 & 2\,\theta_2 \\ 2\,\theta_1 & -2\,\theta_2 & 4\,\theta_5 & 0 & 4\,\theta_3 \\ 2\,\theta_2 & 2\,\theta_1 & 0 & 4\,\theta_5 & 4\,\theta_4 \\ 2\,\theta_1 & 2\,\theta_2 & 4\,\theta_3 & 4\,\theta_4 & 4\,\theta_5 \end{pmatrix}. \tag{4.2}$$

The positive semidefiniteness gives the inequalities calculated in example 4.3.2; the quadratic form reduced there is the one corresponding to $\hat{P}$. Hence, until now the candidates for the strata listed in table 4.1 were calculated. It remains to show that the candidate for $\hat{\Lambda}^{(5)}$ is really connected. Then, it is the principal stratum and in fact, the table gives the description of the orbit space $\mathfrak{gl}(2,\mathbb{C})/SU(2)$.

Claim: The semialgebraic set

$$S := \{\theta \in \mathbb{R}^5 \,|\, 2\theta_5 > \theta_1^2 + \theta_2^2,\ (\theta_4 - \theta_2\theta_1)^2 < (\theta_5 - \theta_2^2 - \theta_3)(\theta_5 - \theta_1^2 + \theta_3)\}$$

is connected.

Proof: S is connected if and only if the set

$$S' := \{(\theta_1, \theta_2, \theta_5, \theta_6, \theta_7) \in \mathbb{R}^5 \,|\ \theta_6 > \theta_1^2,\ \theta_7 > \theta_2^2,\ (\theta_7 - \theta_2^2)(\theta_6 - \theta_1^2) > (\theta_4 - \theta_2\theta_1)^2\},$$

the image of S under the substitution $\theta_6 := \theta_5 + \theta_3$ and $\theta_7 := \theta_5 - \theta_3$, is connected. It remains to show that S' is in fact connected:
For that purpose, the sets

$$\begin{aligned}
S_1 &= \{(\theta_2, \theta_7) \in \mathbb{R}^2 |\ \theta_7 > \theta_2^2, \} \\
S_2 &= \{(\theta_1, \theta_6) \in \mathbb{R}^2 |\ \theta_6 > \theta_1^2, \} \\
S_3 &= S_1 \times S_2, \\
S_4 &= \{(\theta_1, \theta_2, \theta_4, \theta_6, \theta_7) \in \mathbb{R}^5 |\ (\theta_7 - \theta_2^2)(\theta_6 - \theta_1^2) > (\theta_4 - \theta_2\theta_1)^2\}
\end{aligned}$$

are defined. Then, $S' = (S_3 \times \mathbb{R}) \cap S_4$. Clearly, S_1, S_2 and hence, S_3 and $S_3 \times \mathbb{R}$ are connected. For all $(\theta_1, \theta_2, \theta_6, \theta_7) \in S_3$ the inequality determining S_4 defines a non-empty, open and connected interval as the set of possible values of θ_4. Hence, $S' = (S_3 \times \mathbb{R}) \cap S_4$ is connected. $\square$

Chapter 5

The algorithm of Abud and Sartori applied to a semialgebraic subset

Let G be a compact linear Lie group acting orthogonally on $\mathbb{R}^n$. Then one can determine the corresponding orbit space (cf. chapter 4). Further, let X be a semialgebraic subset of $\mathbb{R}^n$. A G-invariant description of X together with the semialgebraic characterization of $\mathbb{R}^n/G$ gives X/G (cf. theorem 2.2.4).
But in general the orbit space $\mathbb{R}^n/G$ is much higher dimensional than X/G. Thus, for the purpose of calculation it is better to work solely with X, if possible. Possibilities for this shall be discussed in the following.
Then the G-invariant description of $SU(n)$ and $\mathfrak{su}(n)$ will be established to consider afterwards examples of the form: $X = SU^k(n) \times \mathfrak{su}^l(n)$.
At the end, some remarks on the difficulties of computations are stated.

5.1 The algorithm

Let $\theta_1, \ldots, \theta_q$ be a minimal homogeneous basis of $\mathrm{Pol}_G(\mathbb{R}^n)$ and X a G-invariant semialgebraic subset of $\mathbb{R}^n$. Further, a G-invariant description of X(see theorem 2.2.4) given by relations p_i and inequalities q_j is assumed to be known and the matrix $\hat{P}$ ($P = \partial\theta^T \partial\theta = \hat{P} \circ \theta$, cf. lemma 4.1.1) shall be calculated.
Due to theorem 2.2.4, it holds $\theta(X) = \theta(\mathbb{R}^n) \cap \{x \in \mathbb{R}^q |\ \tilde{p_i} = 0, \tilde{q_j} \geq 0 \text{ for all } i,j\}$ with $p_i = \tilde{p_i} \circ \theta$ and $q_j = \tilde{q_j} \circ \theta$. Since one is only interested in orbits belonging to X it suffices to consider $\hat{P}\restriction_X := \hat{P}\restriction_{p_i=0,\ q_j \geq 0}$ and to calculate the rank and positive semidefiniteness conditions for $\hat{P}\restriction_X$.

If the G-invariant description of X is given as polynomials of the form: one invariant is expressed as a polynomial of the others (as it will be the case of $SU(n)$ acting on $SU(n)$ and $\mathfrak{su}(n)$, respectively, see proposition 5.2.1 and remark 5.2.2), such a restriction is always possible by just inserting these polynomials in $\hat{P}$. In other cases, if the G-invariant description contains inequalities or polynomials that are not

resolvable to an invariant, it is not clear how to insert the information of "belonging" to X into $\hat{P}$ for restricting it to X. But nevertheless $\hat{P}$ shall be restricted as much as possible to reduce computational effort, for example by polynomials having the form stated above or by other equations in the invariants known for X.
So assume it is possible to reduce $\hat{P}$ to a set Y with $X \subseteq Y \subseteq \mathbb{R}^n$. Evaluating the rank and positive semidefiniteness of $\hat{P}\restriction_Y$ relations r_k and inequalities s_l are obtained. Then the orbit space is given by

$$\theta(X) = \{x \in \mathbb{R}^q \mid \tilde{p_i} = 0, r_k = 0, \tilde{q_j} \geq 0, s_l \geq 0 \text{ for all } i, j, k, l\}.$$

Further, let $\tilde{\theta} = \{\theta_1\restriction_X, \ldots, \theta_l\restriction_X\}$ for $l \leq q$ be a minimal basis of $\mathrm{Pol}_G(X)$. The projection of $\theta(X)$ onto the first l coordinates then already gives the orbit space.

Example 5.1.1. Although $SU(2)/SU(2)$ was already given in example 1.1.1 it shall be established now by using the restriction of $\mathfrak{gl}(2,\mathbb{C})$ to $SU(2)$:
The restriction of the $\hat{P}$-matrix for $SU(2)$ acting on $\mathfrak{gl}(2,\mathbb{C})$ (see equation (4.2)) to $SU(2)$ (cf. example 2.2.6) gives

$$\hat{P}\restriction_{SU(2)} = \begin{pmatrix} 2 & 0 & 2\,\theta_1 & 0 & 2\,\theta_1 \\ 0 & 2 & 0 & 2\,\theta_1 & 0 \\ 2\,\theta_1 & 0 & 8 & 0 & 4\,(\theta_1^2-2) \\ 0 & 2\,\theta_1 & 0 & 8 & 0 \\ 2\,\theta_1 & 0 & 4\,(\theta_1^2-2) & 0 & 8 \end{pmatrix}.$$

The determinant of $\hat{P}$ is zero and the matrix is positive semidefinite if and only if $\mathrm{tr}^2 A \leq 4$. For the rank, there are then two possibilities:

1. $\mathrm{rk}\hat{P} = 2$ if $\mathrm{tr}^2 A = 4$.
2. $\mathrm{rk}\hat{P} = 4$ if $\mathrm{tr}^2 A < 4$.

After restriction, the rank of $\hat{P}$ on a stratum no longer coincides with the dimension of the stratum.

5.2 Examples

5.2.1 $SU(n)$ acting on $SU^m(n) \times \mathfrak{su}^k(n)$

Let $SU(n)$ act on $\mathfrak{gl}^{m+k}(n,\mathbb{C})$ by diagonal conjugation and let an $SU(n)$-invariant subset $X = SU^m(n) \times \mathfrak{su}^k(n) \subseteq \mathfrak{gl}^{m+k}(n,\mathbb{C})$ be given. For calculating the orbit space $X/SU(n)$, an $SU(n)$-invariant description of X is needed (see theorem 2.2.4):

Proposition 5.2.1.
i) $SU(n) = \{A \in \mathfrak{gl}(n,\mathbb{C}) \mid \det A = 1 \text{ and } \mathrm{tr}AA^* = n\}$
ii) $\mathfrak{su}(n) = \{B \in \mathfrak{gl}(n,\mathbb{C}) \mid \mathrm{tr}B = 0 \text{ and } \mathrm{tr}BB^* = -\mathrm{tr}B^2\}$

Proof: Since clearly each matrix $A \in SU(n)$ and $B \in \mathfrak{su}(n)$, respectively, fulfills the characterisations on the left sides of the equations it suffices to show the other inclusion. In the following matrices fulfilling these left sides will be assumed:
i) Let v_i be the i-th column vector of A. Then

$$\mathrm{tr} AA^* = \sum_i^n \| v_i \|^2 = n.$$

Using the arithmetic-geometric mean inequality (AGM), it follows:

$$1 = (\frac{1}{n}\mathrm{tr} AA^*)^n \geq \prod_{i=1}^n \| v_i \|^2 \ . \tag{5.1}$$

Let $A = Q{\cdot}R$ be the first part of the Iwasawa decomposition of A, i.e. $Q \in U(n)$ and R is an upper triangular matrix. Then, the i-th column vector of $R = (r_{ij})$, where $r_{ij} = 0$ for all $i > j$, has the same magnitude as v_i. Therefore,

$$\prod_{i=1}^n \| v_i \|^2 = \prod_{j=1}^n \sum_{i=1}^j |r_{ij}|^2 \geq \prod_{i=1}^n |r_{ii}|^2 = |\det R|^2. \tag{5.2}$$

For the last estimate several non-negative summands are neglected. As $\det A = 1$ and $Q \in U(n)$ holds, $|\det R| = 1$. Combining (5.1) and (5.2), the two inequalities have to saturate. But in the AGM, equality holds if and only if $\| v_i \|^2 = 1$ for all i. Saturation of the second inequality means that all neglected summands should vanish. For all $i, j \in \{1, \dots, n\}$ with $i < j$ there is one vanishing summand of the form $|r_{ij}|^2 \cdot \prod_{i=1, i\neq j}^n |r_{ii}|^2$. Each $|r_{ii}| \neq 0$, since the determinant of R has magnitude 1. Hence, $|r_{ij}| = 0$ for all $i \neq j$, i.e. R is actually a diagonal matrix, each non-zero entry having magnitude 1. Hence, $R, A \in U(n)$. The claim follows with $\det A = 1$.

ii) Let $B = (a_{ij} + \mathrm{i} b_{ij})$ with $a_{ij}, b_{ij} \in \mathbb{R}$. Then

$$\mathrm{tr} BB^* = \sum_{i,j=1}^n (a_{ij}^2 + b_{ij}^2)$$

and

$$\mathrm{tr} B^2 = \sum_i^n (a_{ii}^2 - b_{ii}^2 + 2\mathrm{i} a_{ii} b_{ii}) + \sum_{i,j=1, i<j}^n (2a_{ij}a_{ji} - 2b_{ij}b_{ji}) + \sum_{i,j=1}^n \mathrm{i}(a_{ij}b_{ji} + a_{ji}b_{ij}).$$

The real part of the relation $\mathrm{tr} B^2 = -\mathrm{tr} BB^*$ yields

$$0 = \sum_i 2a_{ii}^2 + \sum_{i<j} ((a_{ij} + a_{ji})^2 + (b_{ij} - b_{ji})^2).$$

Hence, the identities $a_{ij} = -a_{ji}$ and $b_{ij} = b_{ji}$ follow, i.e. $B \in \mathfrak{u}(n)$. With the first relation, $\mathrm{tr} B = 0$, the claim is obtained. □

Thus, with the basis of $\mathrm{Pol}_{SU(n)}(\mathfrak{gl}^{m+k}(n))$ given in subsection 2.1.2 and this $SU(n)$-invariant description all the orbit spaces $SU^m(n) \times \mathfrak{su}^k(n)/SU(n)$ can be calculated, at least theoretically.

Remark 5.2.2. The polynomials occurring in the $SU(n)$-invariant description of proposition 5.2.1 can be written in the form: a set of invariants is given as polynomials in the other invariants (The polynomial $\det A = 1$ can be written as $\mathrm{tr}A^n =$ polynomial$(\mathrm{tr}A^i |\ i < n)$).
Thus, here it is always possible to establish the restriction of $\hat{P}$ to the subset $X = SU^k(n) \times \mathfrak{su}^m(n)$. If the algorithm of Abud and Sartori is applied to $\hat{P}\restriction_X$ and relations r_i and s_j are obtained, the orbit space is immediately given by

$$\theta(X) = \{x \in \mathbb{R}^{m+k+\binom{m+k}{2}+\binom{m+k}{3}} |\ r_i(x) = 0,\ s_j(x) \geq 0 \text{ for all } i, j\}.$$

5.2.2 Again the case of $SU(2)$

Before discussing examples for certain numbers of copies, the basis of the algebra $\mathrm{Pol}_{SU(2)}(\mathfrak{gl}^{m+k}(2,\mathbb{C}))$ given in lemma 4.6.2 shall be restricted to a basis of $\mathrm{Pol}_{SU(2)}(SU^k(2) \times \mathfrak{su}^m(2))$. Using the identities $AA^* = \mathrm{Id}$ for $A \in SU(2)$ and $B = -B^*$ for $B \in \mathfrak{su}(2)$ and the fact that some of the invariants are purely real or imaginary, the basis stated in the next lemma is obtained. The dimension of the orbit space immediately follows from equation (1.3) and the principal orbit type $[\{\pm\mathrm{Id}\}]$ for $m + k \geq 2$ and $[U(1)]$ for $m + k = 1$:

Lemma 5.2.3. Let $SU(2)$ act on $SU^k(2) \times \mathfrak{su}^m(2)$ by diagonal conjugation. A multihomogeneous basis is given by the set

$$\{\mathrm{tr}A_i, \mathrm{tr}B_j^2, \mathrm{tr}C_{l_1}C_{l_2}, \mathrm{tr}C_{l_3}C_{l_4}C_{l_5}\}_{1\leq i\leq k,\ k+1\leq j\leq m+k,\ 1\leq l_1<l_2\leq m+k,\ 1\leq l_3<l_4<l_5\leq m+k},$$

consisting of $m + k + \binom{m+k}{2} + \binom{m+k}{3}$ elements. The dimension of the orbit space is given by $3(m + k) - 3$ for $m + k \geq 2$ else the dimension is 1.

Example 5.2.4.
i) $SU^2(2)$. Since all auxiliary strata were fully described in example 3.1.7 only the principal stratum has to be characterized: A basis of invariants is

$$\theta_1 = \mathrm{tr}A_1, \theta_2 = \mathrm{tr}A_2, \theta_3 = \mathrm{tr}A_1A_2.$$

Since the dimension of the orbit space is 3 (see lemma 5.2.3), there is no relation and the basis is cofree. The inequalities describing the principal stratum are found by using the $\hat{P}$-matrix for $SU(2)$ acting on $\mathfrak{gl}^2(2,\mathbb{C})$ (see section C.3). This matrix is restricted to $SU^2(2)$ and afterwards Lagrange's reduction is used to determine when the matrix is positive definite.
Figure 5.1 shows the orbit space. It is a tetrahedron rounded at the edges; but has still four vertices. The tetrahedron is the union of all auxiliary strata, the inner part

Dim	Orbit space strata
3	$\theta_1^2 \leq 4, \theta_2^2 \leq 4$ (do not saturate simultaneously) $(2\theta_3 - \theta_1\theta_2)^2 < (4-\theta_1^2)(4-\theta_2^2)$
2	$\theta_1^2 \leq 4, \theta_2^2 \leq 4$ (do not saturate simultaneously) $(2\theta_3 - \theta_1\theta_2)^2 = (4-\theta_1^2)(4-\theta_2^2)$
0	$\theta_1^2 = 4,\ \theta_2^2 = 4,\ \theta_3 = \theta_1\theta_2$

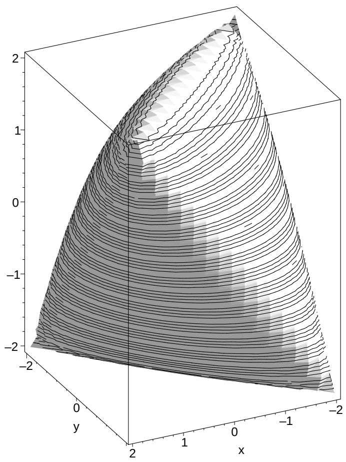

Figure 5.1: The orbit space $SU^2(2)/SU(2)$

of the tetrahedron forms the principal stratum and the four vertices are the lowest dimensional strata.
In this case, the inequalities can also be simply read off from the structure of the auxiliary strata, the surface of the 'tetrahedron'.

ii) $\mathfrak{su}^2(2)$. The auxiliary strata were calculated in appendix C.1 and as in example i) the principal stratum is got. A basis of invariants is given by:

$$\theta_1 = \mathrm{tr} B_1^2,\ \theta_2 = \mathrm{tr} B_2^2,\ \theta_3 = \mathrm{tr} B_1 B_2.$$

The orbit space is shown in figure 5.2. The auxiliary strata form an unbounded shell which pale the inner part, the principal stratum. The origin is the lowest dimensional stratum.

iii) $SU(2) \times \mathfrak{su}(2)$. The auxiliary strata were calculated in appendix C.1 and as in example i) the principal stratum is obtained. A basis of invariants is given by:

$$\theta_1 = \mathrm{tr} A,\ \theta_2 = \mathrm{tr} B^2,\ \theta_3 = \mathrm{tr} AB.$$

Figure 5.3 shows the orbit space. The zero dimensional strata now consists of two points and all the auxiliary strata form an unbounded paper-bag whose contents is the principal stratum.

iv) $SU^3(2)$. A basis of invariants is given by:

$$\mathrm{tr} A,\ \mathrm{tr} B,\ \mathrm{tr} C,\ \mathrm{tr} AB,\ \mathrm{tr} AC,\ \mathrm{tr} BC,\ \mathrm{tr} ABC.$$

Dim	Orbit space strata
3	$\theta_1 \leq 0, \theta_2 \leq 0$ (do not saturate simultaneously) $\theta_1\theta_2 > \theta_3^2$
2	$\theta_1 \leq 0, \theta_2 \leq 0$ (do not saturate simultaneously) $\theta_1\theta_2 = \theta_3^2$
0	$\theta_1 = 0,\ \theta_2 = 0,\ \theta_3 = 0$

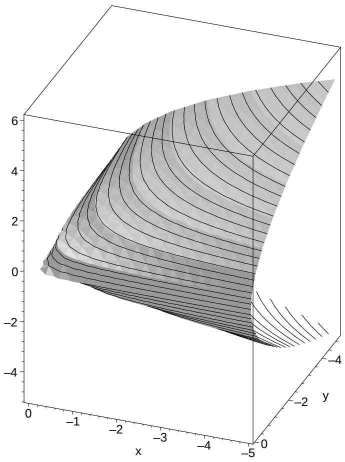

Figure 5.2: The orbit space $\mathfrak{su}^2(2)/SU(2)$

Since the dimension of the orbit space is 6 (cf. lemma 5.2.3), the basis is not cofree anymore and there has to exist a relation. This relation was given in equation (3.3) in subsection 3.2. It is a quadratic equation in trABC. Since for the two sets of matrices $\{A, B, C\}$ given by

$$\left\{ \begin{pmatrix} \mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \right\}$$

and

$$\left\{ \begin{pmatrix} \mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} \right\}$$

all invariants except of trABC take the same value (zero, but tr$ABC = \pm 2$), there cannot be a relation in these invariants that is linear in trABC. Hence, the given basis is already minimal and the relation (3.3) generates the elimination ideal of the orbit space. The inequalities can be established by using the restriction of the $\hat{P}$-matrix for $SU(2)$ acting on $\mathfrak{gl}^3(2, \mathbb{C})$ that is already a 47×47 matrix (since the minimal basis for $\mathrm{Pol}_{SU(2)}(\mathfrak{gl}^3(\mathbb{C}))$ found in lemma 4.6.2 has 47 elements).

5.3 Summary and remarks on computations

Finally, in this section some remarks on the computational effort and problems will be made.

Assume an orthogonal action of a compact Lie group G on $(\mathbb{R}^z)^{k+l}$ with $G^k \times \mathfrak{g}^l \subseteq (\mathbb{R}^z)^{k+l}$ is given. With a basis for G acting on $(\mathbb{R}^z)^{k+l}$, the algorithm of Abud and

Dim	Orbit space strata
3	$\theta_1^2 \leq 4, \theta_2 \leq 0$ (do not saturate at once) $-2\theta_3^2 < \theta_2(4-\theta_1^2)$
2	$-2\theta_3^2 = \theta_2(4-\theta_1^2)$ $\theta_1^2 \leq 4, \theta_2 \leq 0$ (do not saturate at once)
0	$\theta_1 = 4,\ \theta_2 = 0,\ \theta_3 = 0$

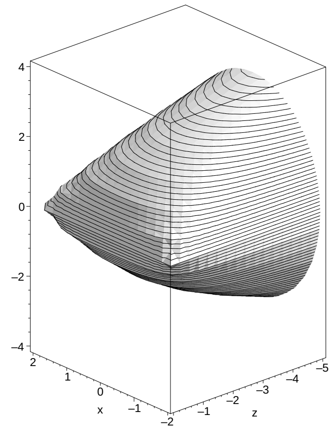

Figure 5.3: The orbit space $(SU(2) \times \mathfrak{su}(2))/SU(2)$

Sartori (see section 4.1) and a G-invariant description of G and $\mathfrak{g}$ (cf. theorem 2.2.4), the orbit space $(G^k \times \mathfrak{g}^l)/G$ is computable.

The steps of the corresponding algorithm are summarized in the following:

1. (Multi)Homogenization of the given basis.
2. Minimization of the (multi)homogeneous basis, see section 4.2.
3. Calculation of the $\hat{P}$-matrix, cf. sections 4.1 and 4.2.
4. Restriction of the $\hat{P}$-matrix to $G^k \times \mathfrak{g}^l$ "as far as possible", see section 5.1.
5. Using Lagrange's reduction to give conditions for positive semidefiniteness of the restricted $\hat{P}$ (cf. section 4.3).
6. Calculation of relations either by evaluating the rank conditions (see 4.1) or by generating ansatz polynomials (section 4.4).
7. Extracting the proper connected component (see section 4.5).

Trying to calculate the orbit space for different numbers of copies leads to the following computational difficulties:

- The number of the elements of the minimal basis already grows polynomially and especially higher order invariants become extremely lengthy. Hence, it turns out to be necessary to encode the polynomials in such a way that it still can be worked with, i.e. creating ansatz polynomials, comparing of coefficients, computation of the entries of the $\hat{P}$-matrix. In the context of this thesis the polynomials were encoded in a text file as a list of the summands of the polynomial where each summand was encoded itself as an array of the form:

leading coefficient, exponent of the first variable, exponent of the second, and so on.

- Until step five of the algorithm summarized above, the different entries of the P-matrix can be treated separately. So, if the algorithm shall be applied to a higher number of copies, it is advisable to switch to parallel programming to save computing time.

Outlook

For further work the following aspects can be considered:

Firstly, some questions raised concerning the invariant theory for actions on semialgebraic subsets. Is it possible to find always a G-invariant description of an algebraic subset that is purely algebraic, i.e. does not contain any inequalities? In the context of the bundle theoretic formulation of classical Yang-Mills theory and its quantization, it is also of interest to find such a description for the subset of $(G \times \mathfrak{g})^k$ corresponding to the momentum zero [9, 13].

Secondly, it is interesting if a common description of the orbit space of diagonal actions can be found as it was proved for the dimension-reducing relations in chapter 3. One possibility is to follow the approach of classifying the relations and to interpret the orbit space as living over the image of a maximal set of algebraically independent invariants. If this can be achieved, it would be worth to find a bound for the number of copies such that, from the knowledge of the orbit spaces belonging to lower number of copies, all higher orbit spaces can be established.

Thirdly and finally, the aspect of computing examples using a computer remains. In particular, the orbit spaces for $SU(3)$ acting on several copies of itself and its Lie algebra are of interest for studying QCD on a lattice. For that purpose, it is necessary to implement the algorithm described in this thesis in such a way that all the problems described in section 5.3 can be handled. Especially, the orbit spaces for diagonal actions on higher numbers of copies need care of RAM and computing time. The use of computer algebra systems may be necessary, but more promising seems to be the usage of a combination of computer algebra and programming languages, e.g. C.

Appendix A

The Hamilton-Cayley polynomial

The characteristic polynomial of a complex matrix A, also called *Hamilton-Cayley polynomial* of A, is given by $\mathcal{C}_A(\lambda) := \det(A - \lambda \mathrm{Id})$. The Hamilton-Cayley theorem states that each matrix satisfies its own characteristic polynomial: $\mathcal{C}_A(A) = 0$. The following proposition gives a construction of this polynomial without explicitly calculating the determinant.

Proposition A.1. [14, p. 79] Define

$$\begin{aligned} F_1(A) &:= A, \\ p_k &:= \frac{1}{k}\mathrm{tr}(F_k(A)), \\ F_{k+1}(A) &:= A(F_k(A) - p_k \mathrm{Id}). \end{aligned}$$

Then, $F_k(A) = A^k - p_1 A^{k-1} - p_2 A^{k-2} - \cdots - p_{k-1}A$ and $F_{n+1}(A) = \mathcal{C}_A(A)A$.

Corollary A.2. Let $\mathcal{C}_A(A) = A^n - p_1 A^{n-1} - \cdots - p_{n-1}A - p_n$ be the Hamilton-Cayley polynomial of $A \in \mathfrak{gl}(n, \mathbb{C})$. Then, every coefficient p_k is a complex polynomial in $\mathrm{tr}A^j$ with $j \in \{1, \ldots, k\}$ of the following form:

$$p_k = \frac{1}{k}\mathrm{tr}A^k + \text{ polynomial } (\mathrm{tr}A^j |\, j \in \{1, \ldots, k-1\}).$$

Proof: The claim follows by induction from $p_1 = \mathrm{tr}A$ and $k \cdot p_k = \mathrm{tr}F_k(A) = \mathrm{tr}A^k - p_1 \mathrm{tr}A^{k-1} - p_2 \mathrm{tr}A^{k-2} - \cdots - p_{k-1}\mathrm{tr}A$. $\square$

Example A.3.
i) The Hamilton-Cayley polynomial of a 2×2 complex matrix A reads:

$$A^2 - \mathrm{tr}A \cdot A + \det A \cdot \mathrm{Id} = 0.$$

If $A \in SU(n)$, this restricts to $A^2 - \mathrm{tr}A \cdot A + \mathrm{Id} = 0$ and if $A \in \mathfrak{su}(n)$, $A^2 = \frac{1}{2}\mathrm{tr}A^2 \cdot \mathrm{Id}$.
ii) Let $A \in SU(n)$ and $\mathcal{C}_A(\lambda) = \lambda^n - p_1\lambda^{n-1} - \cdots - p_{n-1}\lambda - p_n$ be the Hamilton-Cayley polynomial of A. All the roots λ_i of $\mathcal{C}_A$ have magnitude 1 since they are the

eigenvalues of $A \in SU(n)$. The polynomial $\overline{p_n}\lambda^n + \overline{p_{n-1}}\lambda^{n-1} + \cdots + \overline{p_1}\lambda - 1$ shall be now considered. Multiplying by $\overline{\lambda^n}$ shows that it has the same roots as $\mathcal{C}_A$. With $-p_n = (-1)^n \det A = (-1)^n$ both polynomials turn out to be equal up to sign and by comparison of coefficients $p_i = (-1)^n \overline{p_{n-i}}$ holds for all $1 \leq i \leq n$.
For example the Hamilton-Cayley polynomial for $SU(3)$ is

$$A^3 - \mathrm{tr}A \cdot A^2 + \overline{\mathrm{tr}A} \cdot A - \mathrm{Id} = 0.$$

Beside the "normal" Hamilton-Cayley polynomial there exists a generalization to several matrices. This can be obtained in the following way (An explicit calculation can be found in [18]):
Starting with the Hamilton-Cayley polynomial of an arbitrary $n \times n$ matrix A this equation is polarized, i.e. A is substituted by $\sum_{i=1}^n t_i A_i$ where A_i are again $n \times n$ matrices and t_i are real variables. When evaluating the polynomial and sorting by the different monomials in the t_i

$$\mathcal{C}(A) = \sum_{i_1+\cdots+i_n=n} t_1^{i_1} \cdots t_n^{i_n} \cdot \mathrm{polynomial}(A_i)$$

is obtained. The coefficient of $t_1 \cdots t_n$ is then called the *generalized Hamilton-Cayley polynomial*. Taking the trace over the generalized Hamilton-Cayley polynomial gives the *fundamental trace identity*:

Proposition A.4. [18]

$$0 = \sum_{\mu \in S_{n+1}} \mathrm{sgn}\ \mu \prod_{(i_1,\ldots,i_j)} \mathrm{tr}A_{i_1} \ldots A_{i_j} \tag{A.1}$$

where S_{n+1} is the $n+1$ permutation group and sgn μ denotes the sign of a permutation μ. For each permutation μ the product is taken over the cycles, each denoted by $(i_1, \ldots, i_j)$, of its cycle decomposition.

In the case $n = 2$ the fundamental trace identity reads:

$$\mathrm{tr}ABC + \mathrm{tr}ACB - \mathrm{tr}A\mathrm{tr}BC - \mathrm{tr}B\mathrm{tr}AC - \mathrm{tr}C\mathrm{tr}AB + \mathrm{tr}A\mathrm{tr}B\mathrm{tr}C = 0. \tag{A.2}$$

Appendix B

Aspects of semialgebraic geometry

Definition B.1. [4, pp. 23-25] A subset of $\mathbb{R}^n$ is *algebraic* if it is a set of zeros

$$\mathcal{Z}(B) := \{x \in \mathbb{R}^n \mid f(x) = 0 \text{ for all } f \in B\}$$

for some $B \subseteq \mathrm{Pol}(\mathbb{R}^n)$. A *semialgebraic subset* of $\mathbb{R}^n$ is the finite union of sets of the form:

$$\mathcal{V}(f_i = 0; g_j * 0) := \{x \in \mathbb{R}^n \mid f_1(x) = \cdots = f_l(x) = 0, g_1(x) * 0, \ldots, g_m(x) * 0\}$$

with $f_1, \ldots, f_l, g_1, \ldots, g_m \in \mathrm{Pol}(\mathbb{R}^n)$ and $*$ stands for $>$ or $\geq$. For a given subset $S \subseteq \mathbb{R}^n$ the ideal of polynomials vanishing on S

$$\mathcal{I}(S) := \{f \in \mathrm{Pol}(\mathbb{R}^n) \mid f(x) = 0 \text{ for all } x \in S\}$$

is called the *elimination ideal* of S.

Let $\langle B \rangle$ be the ideal generated by $B \subseteq \mathrm{Pol}(\mathbb{R}^n)$. Then, $\mathcal{Z}(B) = \mathcal{Z}(\langle B \rangle)$. If two subsets $B_1, B_2 \subseteq \mathrm{Pol}(\mathbb{R}^n)$ generate the same ideal, $\langle B_1 \rangle = \langle B_2 \rangle$, they have the same set of zeros. The inverse is in general not true: Having two ideals with the same set of zeros the ideals do not have to coincide as can be seen in example B.2.i.

Example B.2.
i) The ideals generated by x and x^2, respectively, are both subsets of $\mathrm{Pol}(\mathbb{R})$ with the same set of zeros, namely: $\{0\}$. But the ideals do not coincide since $x \notin \langle x^2 \rangle$.
ii) Let $f = (f_1, \ldots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$ where f_i are polynomials on $\mathbb{R}^n$. Then, $Graph(f) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid y = f(x)\}$ and therefore, $Graph(f)$ is semialgebraic.

Due to Hilbert's Basissatz every ideal of $\mathrm{Pol}(\mathbb{R}^n)$ is finitely generated [30, p. 251]. Thus, every algebraic set can be defined by a finite set of polynomials and hence, every algebraic set is semialgebraic.

Theorem B.3 (Tarski-Seidenberg, [4]). Let $X \subseteq \mathbb{R}^{n+m} = \mathbb{R}^n \times \mathbb{R}^m$ be semialgebraic and $\pi : \mathbb{R}^{n+m} \to \mathbb{R}^m$ the corresponding projection. Then, $\pi(X)$ is semialgebraic.

Even if X is algebraic, the projection is in general not. The projection of $\{(x, x^2) | \, x \in \mathbb{R}\}$ to the second coordinate is the set of all non-negative real numbers.

Corollary B.4. Let $f = (f_1, \ldots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$ with $f_i \in \mathrm{Pol}(\mathbb{R}^n)$. Then, it holds:
i) $f(\mathbb{R}^n)$ is semialgebraic.
ii) If further $X \subseteq \mathbb{R}^n$ is semialgebraic, so is $f(X)$.

Proof:
i) $Graph(f) \subseteq \mathbb{R}^{n+m}$ is semialgebraic and therefore, $\pi(Graph(f)) = f(\mathbb{R}^n)$, too.
ii) Since X and $Graph(f)$ are semialgebraic, this also holds for $X \times \mathbb{R}^m$ and $Graph(f) \cap (X \times \mathbb{R}^m)$. Due to theorem B.3 $\pi(Graph(f) \cap (X \times \mathbb{R}^m)) = f(X)$ is semialgebraic.$\square$

Next the dimension of a semialgebraic subset is defined. Firstly, dimension is a local property. Here, a more geometric definition abutting to the definition of a manifold shall be presented. More generally it is also the definition of the dimension of a semianalytic set (see definition B.13).
The algebraic definition for the so-called Krull dimension of a semialgebraic set using maximal chains of prime ideals can be found in [12] page 219 and [4] page 50.

Definition B.5. [19, p. 1582] Let $X \subseteq \mathbb{R}^n$ be a semialgebraic subset. A point x of X is called *regular of dimension* k if there exists a neighbourhood of x in X that is a submanifold of dimension k. The *dimension* of X is then the maximum of the dimensions of its regular points.

Example B.6.
i) Every non-empty, open semialgebraic subset of $\mathbb{R}^n$ has dimension n and for a semialgebraic subset $X \subseteq \mathbb{R}^n$ it holds $\dim X \leq n$.
ii) A square has dimension 2. Two crossing lines have dimension 1; their intersection point is not regular.
iii) Let $p = (p_1, \ldots, p_k) : \mathbb{R}^n \mapsto \mathbb{R}^k$ be a map where p_i is a polynomial on $\mathbb{R}^n$ for every i. Then, $\dim p(\mathbb{R}^n) \leq \min\{n, k\}$.
iv) The dimension of the union of two semialgebraic sets is the maximum of the dimensions of the individual sets.

Lemma B.7. Let $p = (p_1, \ldots, p_k) : \mathbb{R}^n \to \mathbb{R}^k$ with $p_i \in \mathrm{Pol}(\mathbb{R}^n)$ and $l := \dim(p(\mathbb{R}^n))$ be given. Any algebraically independent subset $p_{i_1}, \ldots, p_{i_h}$ can be extended to a maximal algebraically independent subset. A maximal algebraically independent subset of $\{p_1, \ldots, p_k\}$ always consists of l elements.

Proof: Consider the set of all subsets of algebraically independent polynomials that include the given $\{p_{i_1}, \ldots, p_{i_h}\}$. Choose a maximal subset $\mathcal{P} = \{p_{i_1}, \ldots, p_{i_m}\}$ out of it.
Assume that $m < l$. Then, for every j with $p_j \notin \mathcal{P}$ there exists a polynomial $f_j(x_1, \ldots, x_{m+1})$ such that $f_j(p_{i_1}, \ldots, p_{i_m}, p_j) = 0$. Since l is the dimension of $p(\mathbb{R}^n) \subseteq \mathbb{R}^k$ there exists a regular point $x \in p(\mathbb{R}^n)$, a neighbourhood $U \subseteq p(\mathbb{R}^n)$ of x in $p(\mathbb{R}^n)$ and a homeomorphism from U into an subset of $\mathbb{R}^l$. Now, in U a point

x' is chosen such that the set $Y := \{y \in U \subseteq p(\mathbb{R}^n) | \; y_{i_j} = x'_{i_j}$ for all $1 \leq j \leq m\}$ contains infinitely many points. Since $m < l$ and U is of dimension l such a point x' always exists. Due to the existence of the polynomials f_j, the coordinates $x'_{i_1}, \ldots, x'_{i_m}$ determine all other coordinates up to a finite set of values. Hence, Y only consists of finitely many points. This gives a contradiction. Thus, $m \geq l$.
Assume now $m > l$ and consider the projection of $p(\mathbb{R}^n)$ onto $(p_{i_1}, \ldots, p_{i_m})(\mathbb{R}^n)$. Since $\{p_{i_1}, \ldots, p_{i_m}\}$ is algebraically independent this projection has dimension m. This is a contradiction to the dimension l of $p(\mathbb{R}^n)$. Hence, $m = l$. $\square$

Corollary B.8. Let $p = (p_1, \ldots, p_k) : \mathbb{R}^n \to \mathbb{R}^k$ with $p_i \in \mathrm{Pol}(\mathbb{R}^n)$ be given.
i) Then, $\dim p(\mathbb{R}^n) = k$ if and only if $p_1, \ldots, p_k$ are algebraically independent.
ii) Let the set $\{p_1, \ldots, p_i\}$ be algebraically independent, but the sets $\{p_1, \ldots, p_i\} \cup \{p_j\}$ for $i < j \leq l < k$ algebraically dependent. Further, let there exist a relation in the polynomials $p_1, \ldots, p_{l+1}$ that is not constant in the polynomial p_{l+1}. Then, $\{p_1, \ldots, p_i\} \cup \{p_{l+1}\}$ is algebraically dependent.

Proof:
i) from lemma B.7
ii) Assume $\{p_1, \ldots, p_i\} \cup \{p_{l+1}\}$ is algebraically independent. This means that the dimension of $(p_1, \ldots, p_i, p_{l+1})(\mathbb{R}^n)$ equals $i+1$. Hence, there exists a regular point $x \in \mathbb{R}^{i+1}$ such that when fixing such an $x = (\overline{x}, x_{i+1}) \in \mathbb{R}^{i+1}$ there exist infinitely many points x' being of the form $(\overline{x}, x'_{i+1})$.
However, due to assumption there only exist finitely many points of the form $(\overline{x}, x_j)$ in the image of $(p_1, \ldots, p_i, p_j)(\mathbb{R}^n)$ for every j with $i < j \leq l < k$ and fixed $\overline{x} \in \mathbb{R}^i$. Thus, for fixed $\overline{x} \in \mathbb{R}^i$ there exists only finitely many points y in the set $(p_1, \ldots, p_i, p_{i+1}, \ldots, p_l)(\mathbb{R}^n)$ that have the form $y = (\overline{x}, \overline{y}) \in \mathbb{R}^l$. Due to the assumed relation between the polynomials $p_1, \ldots, p_{l+1}$ for each of the points y there only exist finitely many points in the set $(p_1, \ldots, p_i, p_{i+1}, \ldots, p_l, p_{l+1})(\mathbb{R}^n)$ of the form (y, y_{l+1}). Hence, in total there also exist only finitely many points in the set $(p_1, \ldots, p_i, p_{l+1})(\mathbb{R}^n)$ that are of the form $x' = (\overline{x}, x'_{i+1})$. This gives a contradiction.$\square$

Remark B.9.
i) Let polynomials $\mathfrak{r}_1, \ldots, \mathfrak{r}_k, \mathfrak{r}' \in \mathrm{Pol}(\mathbb{R}^n)$ be given. Assume that $\mathfrak{r}'$ is not included in the ideal generated by the $\mathfrak{r}_i$. Looking at $Z_1 := \mathcal{Z}(\{\mathfrak{r}_i\}_i)$ and $Z_2 := \mathcal{Z}(\{\mathfrak{r}_i\}_i, \mathfrak{r}')$ what is the "effect" of the new polynomial?
The following cases can be distinguished:

1. $\mathfrak{r}$ is *dimension-reducing*, i.e. $\dim Z_2 < \dim Z_1$ (see ii).
   An example is the following situation: If the set of the originally given polynomials $\mathfrak{r}_i$ is empty, $Z_1 = \mathbb{R}^k$ and Z_2 is less dimensional (cf. corollary B.8.i).
2. $\mathfrak{r}$ is not dimension-reducing; hence, $\dim Z_2 = \dim Z_1$, but it is $Z_1 \neq Z_2$.
   Consider a maximal subset $(x_{i_1}, \ldots, x_{i_k})$ of the coordinates such that no polynomial of $\{\mathfrak{r}_i\}$ depends only on these coordinates. $\mathcal{M}$ is defined to be the set

of all m-tuples $(x_{i_1}, \ldots, x_{i_k})$ such that there exists $x \in Z_1$ whose projection to the i_j-th coordinate gives x_{i_j} for all j. Now, Z_1 can be viewed as components "living" over the set $\mathcal{M}$. Due to the continuity of polynomials every component is closed and thus, the projection of every component is closed.

2.1. $\mathfrak{r}$ reduces the set $\mathcal{M}$.

2.2. $\mathfrak{r}$ does not reduce the set $\mathcal{M}$ but the number of discrete values for at least one element of $\mathcal{M}$.

3. $\mathfrak{r}$ is not dimension-reducing and $Z_1 = Z_2$ (see example B.2.i).

ii) A polynomial itself cannot be dimension-reducing. Strictly speaking this notion always refers to the situation where a polynomial is inserted in a set of polynomials already considered, i.e. a polynomial can only be dimension-reducing w.r.t. a set of polynomials, that can also be empty.

For (semi)algebraic sets a special topology is often considered:

Definition B.10. [12, p. 32] A subset of $\mathbb{R}^n$ is *Zariski-closed* if it is an algebraic subset of $\mathbb{R}^n$. The Zariski-closed subsets define a topology on $\mathbb{R}^n$, called *Zariski topology*.

Every Zariski-closed set is also closed w.r.t. the standard topology of $\mathbb{R}^n$.

Proposition B.11. [4, p. 50] Let $A \subseteq \mathbb{R}^n$ a semialgebraic set. Then,

$$\dim(A) = \dim(\mathrm{clos}_{Zar}(A)),$$

where $\mathrm{clos}_{Zar}(A) := \mathcal{Z}(\mathcal{I}(A))$ is the *Zariski closure* of A.

Definition B.12. [4, p. 34] A semialgebraic subset $A \subseteq \mathbb{R}^n$ is *(semialgebraically) connected* if for each pair of semialgebraic sets F_1 and F_2, that are disjoint, closed in A and satisfy $F_1 \cup F_2 = A$, it holds $F_1 = A$ or $F_2 = A$.

Every semialgebraic subset A is the union of finitely many connected semialgebraic sets [4, p. 50]. The connected components of an algebraic set are semialgebraic, but in general not algebraic.
Further remarks on the connected components of semialgebraic sets can be found in section 4.5.

Finally, a type of stratification for semialgebraic sets into semianalytic sets will be introduced. But first the notion of a semianalytic set is needed. More about semianalytic sets can be e.g. found in [3].

Definition B.13. [3, p. 3] Let U be an open subset of $\mathbb{R}^n$, let $\mathcal{O}(U)$ be the ring of real analytic functions on U and let $S(\mathcal{O}(U))$ denote the smallest family of subsets of U that contains all $\{x \in U \mid f(x) > 0,\ f \in \mathcal{O}(U)\}$ and is stable under finite union, finite intersection and complement. A subset $X \in \mathbb{R}^n$ is called *semianalytic* if each $x \in \mathbb{R}^n$ has a neighbourhood U such that $X \cap U \in S(\mathcal{O}(U))$.

Remark B.14. Each semialgebraic set is semianalytic.

Definition B.15. [29, p. 43] A *primary stratification* of a closed semianalytic set is a locally finite collection E_i of connected semianalytic submanifolds of $\mathbb{R}^d$, called strata, such that $S = \cup_i E_i$ and $\bar{E}_i \setminus E_i$ is a union of lower dimensional strata for all i.

Such a primary stratification always exists [19].

Appendix C

Further examples

C.1 The auxiliary strata of $(SU^m(2) \times \mathfrak{su}^k(2))/SU(2)$

Let $SU(2)$ act on $\widetilde{SU}^m(2) \times \widetilde{\mathfrak{su}}^k(2) := \{(A_1, A_2, \dots, A_m, B_{m+1}, \dots, B_{m+k}) |\ [C_i, C_j] = 0$ for all i, j where C stands for A or $B\}$. Due to subsection 5.2.2 only the invariants

$$\mathrm{tr}A_{i_1},\ \mathrm{tr}B_{i_2}^2,\ \mathrm{tr}C_{i_3}C_{i_4},\ \mathrm{tr}C_{i_5}C_{i_6}C_{i_7},$$

where $1 \le i_1 \le m$, $m{+}1 \le i_2 \le m{+}k$, $1 \le i_3 < i_4 \le m{+}k$, $1 \le i_5 < i_6 < i_7 \le m{+}k$, have to be considered. Since the matrices commute, the fundamental trace identity (A.2) gives a relation for every tr $C_{i_5}C_{i_6}C_{i_7}$:

$$\begin{aligned} \mathrm{tr}B_{i_5}B_{i_6}B_{i_7} &= 0, \\ 2\mathrm{tr}A_{i_5}B_{i_6}B_{i_7} &= \mathrm{tr}A_{i_5}\mathrm{tr}B_{i_6}B_{i_7}, \\ 2\mathrm{tr}A_{i_5}A_{i_6}B_{i_7} &= \mathrm{tr}A_{i_5}\mathrm{tr}A_{i_6}B_{i_7} + \mathrm{tr}A_{i_6}\mathrm{tr}A_{i_5}B_{i_7}. \end{aligned}$$

The relation for $\mathrm{tr}A_{i_5}A_{i_6}A_{i_7}$ was given in equation (3.1).
Thus, a basis of invariants is given by $\mathrm{tr}A_i, \mathrm{tr}B_j^2, \mathrm{tr}\ C_iC_j$. Due to their commutativity $A_i,\ B_j$ can be assumed to be diagonal:

$$A_i = \begin{pmatrix} e^{\mathrm{i}\alpha_i} & 0 \\ 0 & e^{-\mathrm{i}\alpha_i} \end{pmatrix},\ B_j = \begin{pmatrix} \mathrm{i}x_j & 0 \\ 0 & -\mathrm{i}x_j \end{pmatrix},\ \text{with } \alpha_i \in [0, 2\pi),\ x_j \in \mathbb{R}.$$

Hence, $\mathrm{tr}A_i = 2\cos\alpha_i$, $\mathrm{tr}B_j^2 = -2x_j^2$, $\mathrm{tr}A_iA_j = 2\cos(\alpha_i + \alpha_j)$, $\mathrm{tr}B_iB_j = -2x_ix_j$ and $\mathrm{tr}A_iB_j = -2\sin\alpha_i x_j$. This immediately gives the inequalities

$$\mathrm{tr}^2 A_i \le 4,\ \mathrm{tr}B_j^2 \le 0 \tag{C.1}$$

and the relations

$$\begin{aligned} \mathrm{tr}^2 B_iB_j &= \mathrm{tr}B_i^2\mathrm{tr}B_j^2, \\ 2\mathrm{tr}^2 A_iB_j &= -\mathrm{tr}B_j^2(4 - \mathrm{tr}^2 A_i), \\ (2\mathrm{tr}A_iA_j - \mathrm{tr}A_i\mathrm{tr}A_j)^2 &= (4 - \mathrm{tr}^2 A_i)(4 - \mathrm{tr}^2 A_j). \end{aligned} \tag{C.2}$$

Dim	Orbit space strata	Orbit type
m+k	$\mathrm{tr}^2 A_i \leq 4,\ \mathrm{tr}B_p^2 \leq 0$ (do not saturate simultaneously) (C.2), (C.3)	$[\{\mathrm{diag}(e^{\mathrm{i}\alpha}, e^{-\mathrm{i}\alpha})\}_{\alpha\in[0,2\pi)}]$
0	$\mathrm{tr}^2 A_i = 4,\ \mathrm{tr}B_p^2 = 0,\ \mathrm{tr}A_iA_j = \mathrm{tr}A_i\mathrm{tr}A_j,$ $\mathrm{tr}A_iB_p = \mathrm{tr}B_pB_q = 0$	$[SU(2)]$

Table C.1: The auxiliary strata for $SU(2)$ acting on $SU^m(2) \times \mathfrak{su}^k(2)$

Hence, if $\mathrm{tr}A_i$ and $\mathrm{tr}B_j^2$ are chosen, such that (C.1) holds, $\mathrm{tr}C_iC_j$ can only take two values which have to fulfill the consistency relations:

$$\begin{aligned}
\mathrm{tr}B_iB_j\mathrm{tr}B_iB_l\mathrm{tr}B_iB_l + \mathrm{tr}B_i^2\mathrm{tr}B_j^2\mathrm{tr}B_l^2 &= 0,\\
2\mathrm{tr}A_iB_j\mathrm{tr}A_iB_l + \mathrm{tr}B_jB_l(4-\mathrm{tr}^2A_i) &= 0,\\
2\mathrm{tr}A_iB_l\mathrm{tr}A_jB_l + \mathrm{tr}A_i\mathrm{tr}A_j\mathrm{tr}B_l^2 &= 2\mathrm{tr}B_l^2\mathrm{tr}A_iA_j,\\
(2\mathrm{tr}A_iA_j - \mathrm{tr}A_i\mathrm{tr}A_j)(2\mathrm{tr}A_iA_l - \mathrm{tr}A_i\mathrm{tr}A_l) &= (4-\mathrm{tr}^2A_i)(2\mathrm{tr}A_jA_l - \mathrm{tr}A_j\mathrm{tr}A_l).
\end{aligned} \tag{C.3}$$

Thus, one of the two possible values for $\mathrm{tr}C_1C_i$ can be freely chosen and then due to the consistency relations all other $\mathrm{tr}C_iC_j$ are uniquely determined. Summarizing, $\{\mathrm{tr}A_i, \mathrm{tr}B_p^2\}$ is a maximal set of algebraically independent basis elements. Hence, the dimension of $\widetilde{SU}^m(2) \times \widetilde{\mathfrak{su}}^k(2)/SU(2)$ is $m+k$ (see corollary B.8). In table C.1 the description for the orbit type strata is given. The zero dimensional strata consists of 2^{m+k} points.

The orbit types of $SU(2)$ acting on one copy of $\mathfrak{su}(2)$ can be examined analogously as in the case of $SU(2)$ acting on itself (see example 1.1.1) by only considering its diagonal form $A := \mathrm{diag}(\mathrm{i}x, -\mathrm{i}x)$. Then, there are also only two orbit types possible, namely the same as in the case $SU(2)/SU(2)$.
Calculating the orbit types of $SU(2)$ acting on several copies of $\mathfrak{su}(2)$ and $SU(2)$, the intersections of the orbit types obtained for one copy have to be calculated. Thus, the orbit types are the same as in example 3.1.1: $[\{\pm\mathrm{Id}\}]$, $[\{\mathrm{diag}(e^{\mathrm{i}\alpha}, e^{-\mathrm{i}\alpha})\}_{\alpha\in[0,2\pi)}]$ and $[\{SU(2)\}]$.

C.2 Maximal set of algebraically independent invariants

In this section the argumentation of example 3.2.5 will be generalized to figure out a maximal algebraically independent set of invariants for $SU(2)$ acting on $SU^k(2) \times \mathfrak{su}^m(2)$.

Lemma C.2.1. A set of maximal algebraically independent invariants for $SU(2)$

acting on $SU^k(2) \times \mathfrak{su}^m(2)$is given by $S^{(k,m)} = \{\mathrm{tr}A_{i_1},\ \mathrm{tr}B_{i_2}^2,\ \mathrm{tr}C_1C_{i_3},\ \mathrm{tr}C_2C_{i_4}\}$ with $1 \leq i_1 \leq k; m+1 \leq i_2 \leq m+k; 2 \leq i_3 \leq m+k; 3 \leq i_4 \leq m+k$.

Proof: The number of elements of $S^{(k,m)}$ is $3(m+k)-3$ and coincides with the dimension of the orbit space (see equation (1.3)). Due to lemma B.7 it suffices to show that the set $S^{(k,m)}$ is algebraically independent:

a) For the case $m = 0$ the claim is already known from example 3.2.5.

b) For the cases $(k,m) = (0,2),(1,1)$ the sets $S^{(m,k)}$ are even cofree bases (cf. subsection 5.2.2).

c) The next relations verify the claim for the cases $(k,m) = (0,3),(1,2),(2,1)$:

$$\begin{aligned}\mathrm{tr}^2B_1B_2B_3 = {} & \mathrm{tr}B_1^2\mathrm{tr}^2B_2B_3 + \mathrm{tr}B_2^2\mathrm{tr}^2B_1B_3 + \mathrm{tr}B_3^2\mathrm{tr}^2B_1B_2 \\ & - \mathrm{tr}B_1^2\mathrm{tr}B_2^2\mathrm{tr}B_3^2 - 2\mathrm{tr}B_1B_2\mathrm{tr}B_1B_3\mathrm{tr}B_2B_3\end{aligned}$$

$$\begin{aligned}(2\mathrm{tr}A_iB_jB_k - \mathrm{tr}A_i\mathrm{tr}B_jB_k)^2 = {} & (4-\mathrm{tr}^2A_i)(\mathrm{tr}B_j^2\mathrm{tr}B_k^2 - \mathrm{tr}^2B_jB_k) \\ & + 2\mathrm{tr}^2A_iB_j\mathrm{tr}B_k^2 + 2\mathrm{tr}^2A_iB_k\mathrm{tr}B_j^2 - 4\mathrm{tr}A_iB_j\mathrm{tr}A_iB_k\mathrm{tr}B_jB_k\end{aligned}$$

$$\begin{aligned}(2\mathrm{tr}A_iA_jB_k - \mathrm{tr}A_iB_k\mathrm{tr}A_j - \mathrm{tr}A_jB_k\mathrm{tr}A_i)^2 = {} & 2\mathrm{tr}B_k^2(\tfrac{1}{2}\mathrm{tr}A_i\mathrm{tr}A_j - \mathrm{tr}A_iA_j)^2 \\ & + 4(\tfrac{1}{2}\mathrm{tr}A_i\mathrm{tr}A_j - \mathrm{tr}A_iA_j)\mathrm{tr}A_iB_k\mathrm{tr}A_jB_k + 8\mathrm{tr}B_k^2 \\ & + 2(2\mathrm{tr}^2A_iB_k - \mathrm{tr}^2A_i\mathrm{tr}B_k^2 + 2\mathrm{tr}^2A_jB_k - \mathrm{tr}^2A_j\mathrm{tr}B_k^2) \\ & - \mathrm{tr}^2A_j\mathrm{tr}^2A_iB_k - \mathrm{tr}^2A_i\mathrm{tr}^2A_jB_k + \tfrac{1}{2}\mathrm{tr}^2A_i\mathrm{tr}^2A_j\mathrm{tr}B_k^2\end{aligned}$$

All these relations were first found by guessing which was still feasible since all relations have to be homogeneous w.r.t. each element of $\mathfrak{su}(n)$, B_i. But these relations can also be found, similarly to relation (3.3), by the restriction of the relations determining $\mathfrak{gl}^3(2,\mathbb{C})/SU(2)$ established by an ansatz function of multihomogeneous degree $(2,2,2)$. The first relation was also computed in example 4.4.1.

d) With analogous arguments as in example 3.2.5 the claim is now proved for $(k,m) = (0,4),(1,3),(2,2),(3,1)$ and then with a generalization of remark 3.2.4.ii to subsets $X^k \times Y^m \subseteq (\mathbb{R}^n)^{k+m}$ the claim holds for all (k,m). □

C.3 The $\hat{P}$-matrix for $SU(2)$ acting on $\mathfrak{gl}^2(2,\mathbb{C})$

Due to lemma 4.6.2 the algebra $\mathrm{Pol}_{SU(2)}(\mathfrak{gl}^2(\mathbb{C}))$ is generated by:

$t_1 =$ Re tr$A$	$t_2 =$ Im tr$A$	$t_3 =$ Re trA^2	$t_4 =$ Im trA^2
$t_5 =$ trAA^*	$t_6 =$ Re tr$B$	$t_7 =$ Im tr$B$	$t_8 =$ Re trB^2
$t_9 =$ Im trB^2	$t_{10} =$ trBB^*	$t_{11} =$ Re trAB	$t_{12} =$ Im trAB
$t_{13} =$ Re trAB^*	$t_{14} =$ Im trAB^*	$t_{15} =$ Re trAA^*B	$t_{16} =$ Im trAA^*B
$t_{17} =$ Re trABB^*	$t_{18} =$ Im trABB^*.		

Using the algorithms of chapter 4 the $\hat{P}$-matrix can be calculated using a computer. This matrix is stated in the following where all entries $*$ are given by its symmetry and where the following substitutions are used:

$$
\begin{aligned}
P_{3,6} = {} & -3/2\,t_1t_3{}^2 - 1/2\,t_1t_4{}^2 + 1/2\,t_1t_{13} + 1/2\,t_2t_{14} + 2\,t_3t_6 - 1/2\,t_1t_2{}^2 \\
& +1/2\,t_1t_8 + 1/2\,t_2t_9 - t_{17} - 1/2\,t_1{}^3 + t_1t_{12} \\
& +t_3t_{10} + t_4t_{11} + t_1t_5 - t_2t_3t_4 \\
P_{4,6} = {} & -t_{18} + 1/2\,t_1t_{14} - 1/2\,t_2t_3{}^2 - 3/2\,t_2t_4{}^2 - 1/2\,t_2t_{13} + 2\,t_4t_6 - t_4t_{10} \\
& -1/2\,t_1{}^2t_2 + 1/2\,t_1t_9 - 1/2\,t_2t_8 - 1/2\,t_2{}^3 + t_2t_{12} + t_3t_{11} + t_2t_5 - t_1t_3t_4 \\
P_{5,6} = {} & -1/2\,t_3{}^3 - 3/2\,t_1{}^2t_3 + 2\,t_1t_6 - 1/2\,t_2{}^2t_3 + 1/2\,t_3t_8 + 1/2\,t_4t_9 - 1/2\,t_3t_4{}^2 \\
& +1/2\,t_3t_{13} + 1/2\,t_4t_{14} - t_{15} + t_1t_{10} + t_2t_{11} + t_3t_5 + t_3t_{12} - t_1t_2t_4 \\
P_{6,6} = {} & -1/2\,t_4{}^3 - 1/2\,t_3{}^2t_4 + 1/2\,t_3t_{14} - 1/2\,t_4t_{13} - 1/2\,t_1{}^2t_4 - 3/2\,t_2{}^2t_4 \\
& +2\,t_2t_6 - t_2t_{10} + 1/2\,t_3t_9 - 1/2\,t_4t_8 - t_{16} + t_4t_{12} + t_1t_{11} + t_4t_5 - t_1t_2t_3 \\
P_{4,7} = {} & t_{18} + 1/2\,t_1t_{14} + 3/2\,t_2t_3{}^2 + 1/2\,t_2t_4{}^2 - t_2t_{12} - 1/2\,t_2t_{13} + 2\,t_3t_7 - t_3t_{11} \\
& +1/2\,t_1{}^2t_2 - 1/2\,t_1t_9 - t_2t_5 + 1/2\,t_2t_8 + 1/2\,t_2{}^3 + t_4t_{10} - t_1t_3t_4 \\
P_{5,7} = {} & -1/2\,t_1t_3{}^2 - 3/2\,t_1t_4{}^2 - 1/2\,t_1t_{13} - 1/2\,t_2t_{14} + 2\,t_4t_7 - 1/2\,t_1t_2{}^2 \\
& +1/2\,t_1t_8 + 1/2\,t_2t_9 + t_2t_3t_4 - t_{17} - 1/2\,t_1{}^3 + t_1t_{12} + t_3t_{10} + t_4t_{11} + t_1t_5 \\
P_{6,7} = {} & -1/2\,t_4{}^3 - 1/2\,t_3{}^2t_4 + 1/2\,t_3t_{14} - 1/2\,t_4t_{13} - 3/2\,t_1{}^2t_4 + 2\,t_1t_7 - 1/2\,t_2{}^2t_4 \\
& -t_2t_{10} - 1/2\,t_3t_9 + 1/2\,t_4t_8 + t_1t_2t_3 - t_{16} + t_4t_{12} + t_1t_{11} + t_4t_5 \\
P_{7,7} = {} & 1/2\,t_3{}^3 + 1/2\,t_1{}^2t_3 - t_1t_{10} + 3/2\,t_2{}^2t_3 + 2\,t_2t_7 - t_2t_{11} - t_3t_5 + 1/2\,t_3t_8 \\
& +1/2\,t_4t_9 + 1/2\,t_3t_4{}^2 - t_3t_{12} - 1/2\,t_3t_{13} - 1/2\,t_4t_{14} + t_{15} - t_1t_2t_4 \\
P_{5,8} = {} & -2\,t_1{}^2t_3 + 2\,t_1t_6 + 2\,t_1t_{10} + 2\,t_2{}^2t_3 + 2\,t_2t_7 - 2\,t_2t_{11} + 2\,t_3t_8 \\
P_{6,8} = {} & -2\,t_1{}^2t_4 + 2\,t_1t_7 + 2\,t_1t_{11} + 2\,t_2{}^2t_4 - 2\,t_2t_6 + 2\,t_2t_{10} + 2\,t_4t_8 \\
P_{6,9} = {} & -4\,t_1t_2t_3 - 2\,t_1t_7 + 2\,t_1t_{11} + 2\,t_2t_6 + 2\,t_2t_{10} + 2\,t_3t_9 \\
P_{7,9} = {} & -4\,t_1t_2t_4 + 2\,t_1t_6 - 2\,t_1t_{10} + 2\,t_2t_7 + 2\,t_2t_{11} + 2\,t_4t_9 \\
P_{7,10} = {} & -1/2\,t_1t_3{}^2 + 1/2\,t_1t_4{}^2 + 1/2\,t_1t_{13} - 1/2\,t_2t_{14} - t_4t_{11} - 1/2\,t_1t_2{}^2 \\
& +1/2\,t_1t_8 + 1/2\,t_2t_9 + t_2t_3t_4 + t_{17} - 1/2\,t_1{}^3 + t_3t_{10} + t_1t_5 \\
P_{8,10} = {} & -t_{18} + 1/2\,t_1t_{14} - 1/2\,t_2t_3{}^2 + 1/2\,t_2t_4{}^2 + 1/2\,t_2t_{13} + 1/2\,t_1{}^2t_2 \\
& -1/2\,t_1t_9 - t_2t_5 + 1/2\,t_2t_8 + 1/2\,t_2{}^3 + t_3t_{11} + t_4t_{10} - t_1t_3t_4 \\
P_{9,10} = {} & -1/2\,t_3{}^3 - 1/2\,t_1{}^2t_3 + 1/2\,t_2{}^2t_3 - t_2t_{11} + 1/2\,t_3t_8 - 1/2\,t_4t_9 - 1/2\,t_3t_4{}^2 \\
& +1/2\,t_3t_{13} + 1/2\,t_4t_{14} + t_1t_2t_4 + t_{15} + t_1t_{10} + t_3t_{12} \\
P_{10,10} = {} & -t_{16} - 1/2\,t_1{}^2t_4 + 1/2\,t_2{}^2t_4 + 1/2\,t_3t_9 + 1/2\,t_4t_8 + 1/2\,t_3{}^2t_4 - 1/2\,t_3t_{14} \\
& -t_4t_{12} + 1/2\,t_4t_{13} + 1/2\,t_4{}^3 + t_1t_{11} + t_2t_{10} - t_1t_2t_3 \\
P_{8,11} = {} & t_{18} + 1/2\,t_1t_{14} - 1/2\,t_2t_3{}^2 + 1/2\,t_2t_4{}^2 + 1/2\,t_2t_{13} - 1/2\,t_1{}^2t_2 + 1/2\,t_1t_9 \\
& -1/2\,t_2t_8 - 1/2\,t_2{}^3 + t_3t_{11} + t_4t_{10} + t_2t_5 - t_1t_3t_4 \\
P_{9,11} = {} & 1/2\,t_1t_3{}^2 - 1/2\,t_1t_4{}^2 - 1/2\,t_1t_{13} + 1/2\,t_2t_{14} - t_3t_{10} - 1/2\,t_1t_2{}^2 \\
& +1/2\,t_1t_8 + 1/2\,t_2t_9 + t_{17} - 1/2\,t_1{}^3 + t_4t_{11} + t_1t_5 - t_2t_3t_4 \\
P_{10,11} = {} & -1/2\,t_4{}^3 - 1/2\,t_3{}^2t_4 + 1/2\,t_3t_{14} - 1/2\,t_4t_{13} - 1/2\,t_1{}^2t_4 \\
& +1/2\,t_2{}^2t_4 + 1/2\,t_3t_9 + 1/2\,t_4t_8 + t_{16} + t_4t_{12} + t_1t_{11} + t_2t_{10} - t_1t_2t_3 \\
P_{11,11} = {} & -1/2\,t_3{}^3 + 1/2\,t_1{}^2t_3 - t_1t_{10} - 1/2\,t_2{}^2t_3 - 1/2\,t_3t_8 + 1/2\,t_4t_9 - 1/2\,t_3t_4{}^2 \\
& +1/2\,t_3t_{13} + 1/2\,t_4t_{14} + t_{15} + t_2t_{11} + t_3t_{12} - t_1t_2t_4 \\
P_{12,13} = {} & -2\,t_1t_3{}^2 + 2\,t_1t_4{}^2 + 2\,t_1t_{13} + 2\,t_3t_6 + 2\,t_3t_{10} - 2\,t_4t_7 - 2\,t_4t_{11} \\
P_{13,13} = {} & -2\,t_2t_3{}^2 + 2\,t_2t_4{}^2 + 2\,t_2t_{13} - 2\,t_3t_7 + 2\,t_3t_{11} - 2\,t_4t_6 + 2\,t_4t_{10} \\
P_{13,14} = {} & -4\,t_1t_3t_4 + 2\,t_1t_{14} + 2\,t_3t_7 + 2\,t_3t_{11} + 2\,t_4t_6 + 2\,t_4t_{10} \\
P_{14,14} = {} & -4\,t_2t_3t_4 + 2\,t_2t_{14} + 2\,t_3t_6 - 2\,t_3t_{10} - 2\,t_4t_7 + 2\,t_4t_{11}
\end{aligned}
$$

$$
\begin{aligned}
P_{12,15} = \;& -2\,t_3{}^2 t_5 + 4\,t_3 t_{15} - t_1{}^2 t_2{}^2 - t_2{}^2 t_8 - 1/2\,t_1{}^4 - 1/2\,t_2{}^4 \\
& + t_5{}^2 - 1/2\,t_8{}^2 - 1/2\,t_9{}^2 + t_5 t_{12} + t_5 t_{13} + t_1{}^2 t_8 + 2\,t_1 t_2 t_9 \\
P_{13,15} = \;& 2\,t_3 t_{16} + 2\,t_4 t_{15} + t_5 t_{14} - 2\,t_3 t_4 t_5 \\
P_{14,15} = \;& 1/2\,t_1 t_4 t_{14} - t_3 t_4 t_{11} + 1/2\,t_1{}^2 t_2 t_4 - t_1 t_2 t_7 + t_3 t_4 t_7 - 1/2\,t_1 t_2{}^2 t_3 \\
& + 1/2\,t_1 t_3 t_8 - 1/2\,t_2 t_3 t_{14} + 1/2\,t_2 t_3{}^2 t_4 - t_1 t_2 t_{11} - 1/2\,t_1 t_4 t_9 + 1/2\,t_2 t_4 t_8 \\
& + 1/2\,t_2 t_3 t_9 - 1/2\,t_1 t_3 t_4{}^2 + 1/2\,t_1 t_3 t_{13} + t_1 t_{15} + t_2 t_{16} + t_3 t_{17} + t_4 t_{18} \\
& + t_{10} t_{12} - 1/2\,t_1{}^3 t_3 + 1/2\,t_1{}^2 t_6 - 1/2\,t_1{}^2 t_{10} + 1/2\,t_2{}^3 t_4 - 1/2\,t_2{}^2 t_6 \\
& + 1/2\,t_2{}^2 t_{10} - 1/2\,t_6 t_8 + 1/2\,t_7 t_9 + 1/2\,t_8 t_{10} + 1/2\,t_9 t_{11} - 1/2\,t_1 t_3{}^3 \\
& + 1/2\,t_2 t_4{}^3 - 1/2\,t_3{}^2 t_{10} - 1/2\,t_4{}^2 t_6 + 1/2\,t_4{}^2 t_{10} + 1/2\,t_3{}^2 t_6 - 1/2\,t_6 t_{13} \\
& - 1/2\,t_7 t_{14} + 1/2\,t_{10} t_{13} + 1/2\,t_{11} t_{14} + t_5 t_{10} + 1/2\,t_2 t_4 t_{13} \\
P_{15,15} = \;& t_3 t_4 t_{10} + 1/2\,t_1 t_3 t_9 - 1/2\,t_1 t_2{}^2 t_4 + 1/2\,t_1 t_3 t_{14} + t_3 t_4 t_6 + 1/2\,t_2 t_4 t_{14} \\
& - 1/2\,t_1 t_4 t_{13} - 1/2\,t_1{}^2 t_2 t_3 - t_1 t_2 t_{10} + 1/2\,t_2 t_4 t_9 - 1/2\,t_2 t_3 t_4{}^2 + 1/2\,t_2 t_3 t_{13} \\
& - 1/2\,t_1 t_3{}^2 t_4 + 1/2\,t_1 t_4 t_8 + t_1 t_2 t_6 - 1/2\,t_2 t_3 t_8 - t_4 t_{17} - t_1 t_{16} + t_{11} t_{12} \\
& + t_2 t_{15} + t_5 t_{11} - 1/2\,t_1 t_4{}^3 - 1/2\,t_2 t_3{}^3 - 1/2\,t_3{}^2 t_7 - 1/2\,t_3{}^2 t_{11} + 1/2\,t_4{}^2 t_7 \\
& + 1/2\,t_4{}^2 t_{11} - 1/2\,t_6 t_{14} + 1/2\,t_7 t_{13} - 1/2\,t_{10} t_{14} + 1/2\,t_{11} t_{13} - 1/2\,t_1{}^3 t_4 \\
& + 1/2\,t_1{}^2 t_7 + 1/2\,t_1{}^2 t_{11} - 1/2\,t_2{}^3 t_3 - 1/2\,t_2{}^2 t_7 - 1/2\,t_2{}^2 t_{11} \\
& - 1/2\,t_6 t_9 - 1/2\,t_7 t_8 - 1/2\,t_8 t_{11} + 1/2\,t_9 t_{10} + t_3 t_{18} \\
P_{14,16} = \;& -2\,t_4{}^2 t_5 + 4\,t_4 t_{16} - t_5 t_{13} - t_1{}^2 t_2{}^2 - t_2{}^2 t_8 - 1/2\,t_1{}^4 - 1/2\,t_2{}^4 + t_5{}^2 \\
& - 1/2\,t_8{}^2 - 1/2\,t_9{}^2 + t_5 t_{12} + t_1{}^2 t_8 + 2\,t_1 t_2 t_9 \\
P_{15,16} = \;& -t_3 t_4 t_{10} + 1/2\,t_1 t_3 t_9 - 1/2\,t_1 t_2{}^2 t_4 + 1/2\,t_1 t_3 t_{14} + t_3 t_4 t_6 + 1/2\,t_2 t_4 t_{14} \\
& - 1/2\,t_1 t_4 t_{13} - 1/2\,t_1{}^2 t_2 t_3 + t_1 t_2 t_{10} + 1/2\,t_2 t_4 t_9 - 1/2\,t_2 t_3 t_4{}^2 + 1/2\,t_2 t_3 t_{13} \\
& - 1/2\,t_1 t_3{}^2 t_4 + 1/2\,t_1 t_4 t_8 + t_1 t_2 t_6 - 1/2\,t_2 t_3 t_8 + t_4 t_{17} + t_1 t_{16} + t_{11} t_{12} \\
& - t_2 t_{15} + t_5 t_{11} - 1/2\,t_1 t_4{}^3 - 1/2\,t_2 t_3{}^3 - 1/2\,t_3{}^2 t_7 + 1/2\,t_3{}^2 t_{11} + 1/2\,t_4{}^2 t_7 \\
& - 1/2\,t_4{}^2 t_{11} - 1/2\,t_6 t_{14} + 1/2\,t_7 t_{13} + 1/2\,t_{10} t_{14} - 1/2\,t_{11} t_{13} - 1/2\,t_1{}^3 t_4 \\
& + 1/2\,t_1{}^2 t_7 - 1/2\,t_1{}^2 t_{11} - 1/2\,t_2{}^3 t_3 - 1/2\,t_2{}^2 t_7 + 1/2\,t_2{}^2 t_{11} - 1/2\,t_6 t_9 \\
& - 1/2\,t_7 t_8 + 1/2\,t_8 t_{11} - 1/2\,t_9 t_{10} - t_3 t_{18} \\
P_{16,16} = \;& -1/2\,t_1 t_4 t_{14} - t_3 t_4 t_{11} - 1/2\,t_1{}^2 t_2 t_4 + t_1 t_2 t_7 - t_3 t_4 t_7 + 1/2\,t_1 t_2{}^2 t_3 \\
& - 1/2\,t_1 t_3 t_8 + 1/2\,t_2 t_3 t_{14} - 1/2\,t_2 t_3{}^2 t_4 - t_1 t_2 t_{11} + 1/2\,t_1 t_4 t_9 - 1/2\,t_2 t_4 t_8 \\
& - 1/2\,t_2 t_3 t_9 + 1/2\,t_1 t_3 t_4{}^2 - 1/2\,t_1 t_3 t_{13} + t_1 t_{15} + t_2 t_{16} + t_3 t_{17} + t_4 t_{18} \\
& - t_{10} t_{12} + 1/2\,t_1{}^3 t_3 - 1/2\,t_1{}^2 t_6 - 1/2\,t_1{}^2 t_{10} - 1/2\,t_2{}^3 t_4 + 1/2\,t_2{}^2 t_6 \\
& + 1/2\,t_2{}^2 t_{10} + 1/2\,t_6 t_8 - 1/2\,t_7 t_9 + 1/2\,t_8 t_{10} + 1/2\,t_9 t_{11} + 1/2\,t_1 t_3{}^3 \\
& - 1/2\,t_2 t_4{}^3 - 1/2\,t_3{}^2 t_{10} + 1/2\,t_4{}^2 t_6 + 1/2\,t_4{}^2 t_{10} - 1/2\,t_3{}^2 t_6 + 1/2\,t_6 t_{13} \\
& + 1/2\,t_7 t_{14} + 1/2\,t_{10} t_{13} + 1/2\,t_{11} t_{14} - t_5 t_{10} - 1/2\,t_2 t_4 t_{13} \\
P_{16,17} = \;& -2\,t_1{}^2 t_{12} + 4\,t_1 t_{17} - t_3{}^2 t_4{}^2 - t_4{}^2 t_{13} - 1/2\,t_3{}^4 - 1/2\,t_4{}^4 + t_{12}{}^2 - 1/2\,t_{13}{}^2 \\
& - 1/2\,t_{14}{}^2 + t_5 t_{12} + t_8 t_{12} + t_3{}^2 t_{13} + 2\,t_3 t_4 t_{14} \\
P_{17,17} = \;& 2\,t_1 t_{18} + 2\,t_2 t_{17} + t_9 t_{12} - 2\,t_1 t_2 t_{12} \\
P_{18,18} = \;& -2\,t_2{}^2 t_{12} + 4\,t_2 t_{18} - t_8 t_{12} - t_3{}^2 t_4{}^2 - t_4{}^2 t_{13} - 1/2\,t_3{}^4 - 1/2\,t_4{}^4 + t_{12}{}^2 \\
& - 1/2\,t_{13}{}^2 - 1/2\,t_{14}{}^2 + t_5 t_{12} + t_3{}^2 t_{13} + 2\,t_3 t_4 t_{14}
\end{aligned}
$$

$$
\begin{bmatrix}
2 & 0 & 0 & 0 & 2\,t_1 & t_3 & t_4 & 2\,t_1 & 2\,t_2 & t_3 & t_4 & 0 & 0 & 0 & t_6+t_{10} & t_7+t_{11} & t_{12} & 0 \\
* & 2 & 0 & 0 & 2\,t_2 & t_4 & -t_3 & -2\,t_2 & 2\,t_1 & -t_4 & t_3 & 0 & 0 & 0 & -t_7+t_{11} & t_6-t_{10} & 0 & t_{12} \\
* & * & 2 & 0 & 0 & t_1 & -t_2 & 0 & 0 & t_1 & t_2 & 2\,t_3 & 2\,t_3 & 2\,t_4 & t_5 & 0 & t_6+t_{10} & -t_7+t_{11} \\
* & * & * & 2 & 0 & t_2 & t_1 & 0 & 0 & -t_2 & t_1 & 2\,t_4 & -2\,t_4 & 2\,t_3 & 0 & t_5 & t_7+t_{11} & t_6-t_{10} \\
* & * & * & * & 4\,t_5 & 2\,t_6 & 2\,t_7 & 4\,t_8 & 4\,t_9 & 2\,t_{10} & 2\,t_{11} & 0 & 0 & 0 & 4\,t_{15} & 4\,t_{16} & 2\,t_{17} & 2\,t_{18} \\
* & * & * & * & * & t_{12}+t_5 & 0 & 2\,t_{10} & 2\,t_{11} & t_{13}+t_8 & t_{14}+t_9 & 2\,t_6 & 2\,t_{10} & 2\,t_{11} & P_{3,6} & P_{4,6} & P_{5,6} & P_{6,6} \\
* & * & * & * & * & * & t_{12}+t_5 & 2\,t_{11} & -2\,t_{10} & t_{14}-t_9 & -t_{13}+t_8 & 2\,t_7 & -2\,t_{11} & 2\,t_{10} & P_{4,7} & P_{5,7} & P_{6,7} & P_{7,7} \\
* & * & * & * & * & * & * & 4\,t_5 & 0 & 2\,t_6 & 2\,t_7 & 0 & 0 & 0 & P_{5,8} & P_{6,8} & 2\,t_{17} & -2\,t_{18} \\
* & * & * & * & * & * & * & * & 4\,t_5 & -2\,t_7 & 2\,t_6 & 0 & 0 & 0 & P_{6,9} & P_{7,9} & 2\,t_{18} & 2\,t_{17} \\
* & * & * & * & * & * & * & * & * & t_{12}+t_5 & 0 & 2\,t_{10} & 2\,t_6 & 2\,t_7 & P_{7,10} & P_{8,10} & P_{9,10} & P_{10,10} \\
* & * & * & * & * & * & * & * & * & * & t_{12}+t_5 & 2\,t_{11} & -2\,t_7 & 2\,t_6 & P_{8,11} & P_{9,11} & P_{10,11} & P_{11,11} \\
* & * & * & * & * & * & * & * & * & * & * & 4\,t_{12} & 4\,t_{13} & 4\,t_{14} & 2\,t_{15} & 2\,t_{16} & 4\,t_{17} & 4\,t_{18} \\
* & * & * & * & * & * & * & * & * & * & * & * & 4\,t_{12} & 0 & 2\,t_{15} & -2\,t_{16} & P_{12,13} & P_{13,13} \\
* & * & * & * & * & * & * & * & * & * & * & * & * & 4\,t_{12} & 2\,t_{16} & 2\,t_{15} & P_{13,14} & P_{14,14} \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & P_{12,15} & P_{13,15} & P_{14,15} & P_{15,15} \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & P_{14,16} & P_{15,16} & P_{16,16} \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & P_{16,17} & P_{17,17} \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & P_{18,18}
\end{bmatrix}
$$

Figure C.1: The $\hat{P}$-matrix for $SU(2)$ acting on $\mathfrak{gl}^2(2,\mathbb{C})$

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