Invertible Dirac operators and handle attachments

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Motivation

- Not every closed manifold admits a metric of positive scalar curvature.
- In contrast, on every closed manifold the space of metric with negative scalar curvature is nonempty and contractable.
- Topological obstruction for psc-metrics: $(M, g)$ closed spin, Dirac operator $D^g$

**Lichnerowicz formula**

\[(D^g)^2 = \Delta_g + \frac{\text{scal}_g}{4}\]

$\text{scal}_g > 0 \Rightarrow D^g$ is invertible
Motivation

- Not every closed manifold admits a metric of positive scalar curvature.
- Topological obstruction for psc-metrics: 
  \[(M, g)\] closed spin, Dirac operator \(D^g\)

**Lichnerowicz formula**

\[
(D^g)^2 = \Delta_g + \frac{\text{scal}_g}{4}
\]

\(\text{scal}_g > 0 \Rightarrow D^g \text{ is invertible}\)

- \(\text{Metr}(M)^{\text{psc}} \subset \text{Metr}(M)^{\text{inv}} \subset \text{Metr}(M)\)
Obstruction for psc metrics

From index theory

\[
\dim \ker D^g \geq \begin{cases} 
|\hat{A}(M)| & \text{if } n \equiv 0 \pmod{4} \\
1 & \text{if } n \equiv 1 \pmod{8}, \quad \alpha(M) \neq 0 \\
2 & \text{if } n \equiv 2 \pmod{8}, \quad \alpha(M) \neq 0 \\
0 & \text{otherwise}
\end{cases}
\]

where \(\hat{A}\) and \(\alpha\) are determined only by the topology of the underlying manifold.

E.g. if \(\hat{A}(M^4) \neq 0\), \(\text{Metr}(M^4)^{\text{psc}} \subset \text{Metr}(M^4)^{\text{inv}} = \emptyset\).

\(\text{Metr}(M)^{\text{psc}} \subset \text{Metr}(M)^{\text{inv}} \subset \text{Metr}(M)\)
Construction of manifolds admitting psc-metrics
- Review Surgery

- embedding $f : S^k \times B^{n-k} \to M$
  $S := f(S^k \times \{0\})$ - surgery sphere
- $\partial(M \setminus f(S^k \times B^{n-k})) \cong S^{k-1} \times S^{n-k-1}$
- $M' = (M \setminus f(S^k \times B^{n-k})) \sqcup \sim B^{k+1} \times S^{n-k-1}$

$M'$ is obtained from $M$ by a surgery of dim $k$ / codim $n - k$. 
View the cylinder $W := M \times [0, 1]$ as a bordism from $M$ to itself

Attach $B^{k+1} \times B^{n-k-1}$ to $M \times \{1\}$

$W'$ is a bordism from $M$ to $M'$ - the trace of the surgery.

$W'$ is obtained from $W$ by attaching a $(k + 1)$-handle.
Each closed manifold has a handle decomposition.

The torus is obtained as:
Construction of manifolds admitting psc-metrics
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\[ B^2 + \]
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The torus is obtained as:

\[ B^2 + \text{a 1-handle} \]
Construction of manifolds admitting psc-metrics
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Each closed manifold has a handle decomposition.

The torus is obtained as:

\[ B^2 \times B^1 \]

\[ B^2 + \text{a 1-handle} \]
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The torus is obtained as:

\[ B^2 + \text{a 1-handle} + \text{a 1-handle} \]
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Each closed manifold has a handle decomposition.

The torus is obtained as:

\[ B^2 + \text{a 1-handle} + \text{a 1-handle} + B^2 = T^2 \]
Construction of manifolds admitting psc-metrics

Theorem (Gromov, Lawson / Schoen, Yau; ’80)

Let \((M,g)\) be a closed Riemannian manifold with \(g \in \text{Metr}(M)^{psc}\). Let \(M'\) be obtained from \(M\) by a surgery of codimension \(\geq 3\). Then, \(M'\) admits a psc-metric \(g'\).

\(g'\) can be chosen such that it coincides with \(g\) outside a small neighbourhood around the surgery sphere.
Construction of manifolds admitting psc-metrics

Intuition

- psc is a local property
- $\text{codim } n - k \geq 3 = \text{gluing in } B^{k+1} \times S^{n-k-1} \geq 2$
- standard product structure on $B^{k+1} \times S^{n-k-1} \geq 2$ has psc
Psc-metrics and handle attachments

Theorem (Carr ’88 / Gajer ’87 )

Let \((M^{n+1}, g)\) be a compact Riemannian manifold with closed boundary \(\partial M\), \(g \in \text{Metr}(M)^{psc}\) and \(g\) having product structure near \(\partial M\). Let \(M'\) be obtained from \(M\) by adding a \((k + 1)\)-handle of codimension \(n - k \geq 3\). Then, \(M'\) admits a psc-metric \(g'\) that is again product near the (new) boundary.
Psc-metrics and handle attachments

\[(M^{n+1}, g) \quad (M', g')\]

Intuition
- On the boundary: surgery of codim \( n - k \geq 3 \)
Psc-metrics and handle attachments

Intuition

- On the boundary: surgery of codim $n - k \geq 3$
- On the double: surgery of codim $n - k \geq 3$
Psc-metrics and handle attachments

Intuition

- On the boundary: surgery of codim $n - k \geq 3$
- On the double: surgery of codim $n - k \geq 3$

Implication

- $\text{Metr}^\text{psc}(S^{4k-1})$ has infinitely many components ($k \geq 2$)
  $(\text{Metr}^\text{psc}(S^3)$ is connected (Marques, 2011))
What can be done for metrics with invertible Dirac operators?
What can be done for metrics with invertible Dirac operators?

From now on: Let all manifolds be spin.
Surgery for $\text{Metr}^{\text{inv}}(M)$

$(M^n, g)$

$(M', g')$

- After the surgery the manifold should still be spin!
Surgery for $\text{Metr}^{\text{inv}}(M)$

- After the surgery the manifold should still be spin!
  - $S^k \times B^{n-k}$ - induces spin structure on $S^k \times S^{n-k-1}$
  - glue in $B^{k+1} \times S^{n-k-1}$
    - Its boundary should carry same spin structure.
  - For $k > 1$, the spin structure on $S^k$ is unique and bounds the disk. - No Problem here.
  - For $k = 1$, two spin structures on $S^1$ - we only allow the one that bounds the disk.
Surgery for $\text{Metr}^{\text{inv}}(M)$

$\begin{align*}
(M^n, g) \\
(M', g')
\end{align*}$

$f : S^k \times B^{n-k} \to M$ spin-preserving embedding.
Surgery for Metr$^{\operatorname{inv}}(M)$

Intuition

- Invertible Dirac operator is a global condition.
- $\operatorname{codim} n - k \geq 3 = \text{gluing in } B^{k+1} \times S^{n-k-1} \geq 2$
- Standard product structure on $\mathbb{R}^{k+1} \times S^{n-k-1} \geq 2$ has invertible Dirac operator
Surgery for $\text{Metr}^{\text{inv}}(M)$

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Surgery for $\text{Metr}^{\text{inv}}(M)$

\[(M^n, g)\]

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**Intuition**

- Invertible Dirac operator is a **global** condition.
- $\text{codim } n - k \geq 2 = \text{gluing in } B^{k+1} \times S^{n-k-1} \geq 1$
- Standard product structure on $\mathbb{R}^{k+1} \times S^{n-k-1} \geq 1$ has invertible Dirac operator (‘When taking the right $S^1$’)
- ‘If the inserted cylinder is large enough, invertibility survives.’
Construction for manifolds admitting inv-metrics

Theorem (Ammann, Dahl, Humbert; 2009)

Let $(M^n, g)$ be a closed Riemannian spin manifold with $g \in \text{Metr}(M)^{\text{inv}}$. Let $M'$ be obtained from $M$ by a surgery of codimension $\geq 2$. Then, $M'$ admits an inv-metric $g'$. Moreover, $g'$ can be chosen such that it coincides with $g$ outside a small neighbourhood around the surgery sphere.

Consequences (Ammann, Dahl, Humbert; 2009)

For a generic metric $g$, $\dim \ker D^g$ is no larger than forced by the index theorem.
Inv-metrics on manifolds with boundary

When do we call $D^g$ invertible?
Inv-metrics on manifolds with boundary

\[ (\partial M \times [0, \infty), \partial g + dt^2) \rightarrow (\partial M \times [-\epsilon, 0], \partial g + dt^2) \rightarrow (M_\infty, g_\infty) \]

\[ g \in \text{Metr}(M)^{\text{inv}} \text{ iff } Dg_\infty \text{ is invertible as operator on } L^2(M_\infty, S) \]
Inv-metrics on manifolds with boundary

\[(M_1, g_1), (\partial M_1, \partial g_1) = (N^+, h),\]

\[g_1 \in \text{Metr}(M_1)^{\text{inv}}\]

\[(\partial M_2, \partial g_2) = (N^-, h), g_2 \in \text{Metr}(M_2)^{\text{inv}}\]

If \(M_1\) and \(M_2\) are glued together using a large enough cylinder \((N \times [-R, R], h + dt^2)\), the resulting metric has again invertible Dirac operator.
Inv-metrics and handle attachments

Theorem (Dahl, G. 2012)

Let \((M^{n+1}, g)\) be a compact Riemannian spin manifold with closed boundary \(\partial M\), \(g \in \text{Metr}(M)^{\text{inv}}\) and \(g\) having product structure near \(\partial M\). Let \(M'\) be obtained from \(M\) by adding a \((k + 1)\)-handle of codimension \(n - k \geq 2\). Then, \(M'\) admits an inv-metric \(g'\) that is again product near the (new) boundary.

\[(M^{n+1}, g) \quad (M', g')\]
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Implication

- \(\text{Metr}(S^{4k-1})^{\text{inv}}\) has infinitely many components for all \(k \geq 1\).
Strategy and Methods

- 'Topological strategy' - Decompose the handle attachment

surgery of codim $n - k$

'half' surgery of codim $n - k + 1$

glue in $\frac{1}{2} B^{k+1} \times S^{n-k}$
Metric strategy

- Approximation by 'double' product metrics near the surgery sphere

\[(U_\delta(S \times [-\epsilon, \infty)), \ g_S + \xi_{\mathbb{R}^{n-k}} + dt^2)\]

If \(\delta\) small enough, still \(g_\delta \in \text{Metr}(M)^{\text{inv}}\).

('\(C^1\)-continuity of the spectrum')
First surgery

'Parameter for tuning': \( \rho \) - 'diameter of \( \{B\} \)'

For \( \rho \) small enough, \( g_\rho \in \text{Metr}(M')^{\text{inv}} \) - proof by contradiction
Metric strategy

▶ First surgery

For $\rho$ small enough, $g_\rho \in \text{Metr}(M')^{\text{inv}}$ - proof by contradiction

$\rho_i \to 0$, $g_{\rho_i} \not\in \text{Metr}(M')^{\text{inv}}$

$\sim g_{\rho_i}$ has a harmonic spinor: $D^{g_{\rho_i}} \varphi_i = 0$, $\| \varphi_i \|_{L^2(M', g_{\rho_i})} = 1$
(regularity) $\sim \varphi_i \to \phi$ in $C^1_{\text{loc}}(M \setminus (S \times [-\epsilon, \infty)))$

(removal of singularities) $\sim D^g \phi = 0$ on $M$, $\| \phi \|_{L^2(M, g)} \leq 1$

a priori estimates on the $L^2$-norm of $\phi_i$ on $\{A\}$ vs $\{C\}$. 
Again approximating by 'double' product metrics
Metric strategy

- Second surgery
Theorem (Dahl, G.; 2012)

Let $M$ be a closed 3-dimensional Riemannian spin manifold and $g \in \text{Metr}(M)^{\text{inv}}$. Then there are metrics $g^i \in \text{Metr}(M)^{\text{inv}}$, $i \in \mathbb{Z}$, such that $g^i$ is bordant to $g$ but $g^i$ is not concordant to $g^j$ for $i \neq j$.

In particular, $\text{Metr}(M)^{\text{inv}}$ has infinitely many connected components.
Notations

\( g_0, g_1 \in \text{Metr}(M)^{\text{inv}} \) are isotopic if \( \exists \) smooth family \( g_t \in \text{Metr}(M)^{\text{inv}} \) with \( g_t = g_0 \) for \( t \leq 0 \), \( g_t = g_1 \) for \( t \geq 1 \).

\[ (M \times [0, 1], \tilde{g}) \]

\((M, g_0) \quad (M, g_1)\)

\( g_i \in \text{Metr}(M_i)^{\text{inv}} \) (\( i = 0, 1 \)) are bordant if \( \exists (W, \tilde{g}) \) with \( \partial W = M_0 \sqcup (M_1)^{-} \), \( \tilde{g} \in \text{Metr}(W)^{\text{inv}}, \tilde{g}|_{M_i} = g_i \).

\[ (M_0, g_0) \quad (M_1, g_1) \]

\( g_0, g_1 \in \text{Metr}(M)^{\text{inv}} \) are concordant if \( \exists \tilde{g} \in \text{Metr}(M \times [0, 1])^{\text{inv}} \) with \( \tilde{g}|_{M \times \{i\}} = g_i \).

\[ (W, \tilde{g}) \]
Theorem (Dahl, G.; 2012)

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Lemma

There exist 4-manifolds $(Y^i, \tilde{h}^i)$ ($i \in \mathbb{Z}$) with $\tilde{h}^i \in \text{Metr}(Y^i)^{\text{inv}}$, $\partial Y^i = S^3$ such that $\alpha(Y^i \cup_{S^3} (Y^j)^-) = c(i - j)$ for a constant $c \neq 0$. 
An application

Lemma

There exist 4-manifolds $(Y^i, \tilde{h}^i) \ (i \in \mathbb{Z})$ with $\tilde{h}^i \in \text{Metr}(Y^i)^{\text{inv}}$, $\partial Y^i = S^3$ such that $\alpha(Y^i \cup_{S^3} (Y^j)^-) = c(i - j)$ for a constant $c \neq 0$.

Construction:

- $Y^0 - B^4$ with a 'torpedo metric' $\tilde{h}^0 \in \text{Metr}(B^4)^{\text{psc}}$ and $\tilde{h}^0|_{S^3} = \text{standard metric}$
- $Y^i = (K3\#K3\# \cdots \#K3) \setminus B^4 = Y^0 + \text{several 2-handles}$ $i$ times
- $\alpha(Y^i \cup_{S^3} (Y^j)^-) = \alpha(\#(i-j)K3) = (i-j)\alpha(K3) \neq 0$ for $i \neq j$

$h^i := \tilde{h}^i|_{S^3}$
Constructions of $g^i$

Start with $(M^3, g \in \text{Metr}(M)^{\text{inv}})$ - construct $g^i$ bordant to $g$.

$(M, g)$

attachment of a 1-handle

$(S^3, h^i)$

$(M, g^i)$
Constructions of $g^i$

$(M, g)$ and $(M, g^i)$ are bordant.
Bordism $(W^i, \tilde{g}^i) \in \text{Metr}(W^i)^{\text{inv}}$
Constructions of $g^i$
Constructions of $g^i$

Closed manifold $(W, \tilde{g})$ with $\tilde{g} \in \text{Metr}(W)^\text{inv}$ and $\alpha(W) = c(i-j)$.
Theorem (Dahl, G.; 2012)

Let $M$ be a closed 3-dimensional Riemannian spin manifold and $g \in \text{Metr}(M)^{\text{inv}}$. Then there are metrics $g^i \in \text{Metr}(M)^{\text{inv}}$, $i \in \mathbb{Z}$, such that $g^i$ is bordant to $g$ but $g^i$ is not concordant to $g^j$ for $i \neq j$.

In particular, $\text{Metr}(M)^{\text{inv}}$ has infinitely many connected components.
Thank you for your attention.