Invertible Dirac operators and handle attachments

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(joint work with Mattias Dahl)

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Motivation

- Not every closed manifold admits a metric of positive scalar curvature.
- In contrast, on every closed manifold the space of metric wirth negative scalar curvature is nonempty and contractable.
- Topological obstruction for psc-metrics:
  (M, g) closed spin, Dirac operator D^g

Lichnerowicz formula

$$(D^g)^2 = \Delta_g + \frac{\mathrm{scal}_g}{4}$$

$\mathrm{scal}_g > 0 \Rightarrow D^g$ is invertible

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$$(D^g)^2 = \Delta_g + \frac{\mathrm{scal}_g}{4}$$

$\mathrm{scal}_g > 0 \Rightarrow D^g$ is invertible

- $\mathrm{Metr}(M)^{\mathrm{psc}} \subset \mathrm{Metr}(M)^{\mathrm{inv}} \subset \mathrm{Metr}(M)$

Obstruction for psc metrics

From index theory

$$\dim\ker D^g \geq \begin{cases} |\hat{A}(M)| & \text{if } n \equiv 0 \mod 4 \\ 1 & \text{if } n \equiv 1 \mod 8, \quad \alpha(M) \neq 0 \\ 2 & \text{if } n \equiv 2 \mod 8, \quad \alpha(M) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

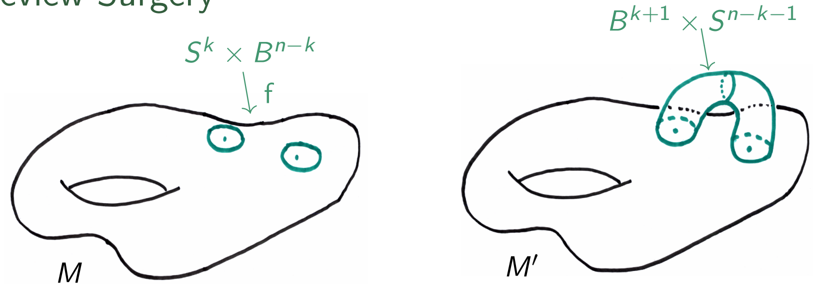
where $\hat{A}$ and α are determined only by the topology of the underlying manifold.

E.g. if $\hat{A}(M^4) \neq 0$, $\mathrm{Metr}(M^4)^{\mathrm{psc}} \subset \mathrm{Metr}(M^4)^{\mathrm{inv}} = \varnothing$.

$\mathrm{Metr}(M)^{\mathrm{psc}} \subset \mathrm{Metr}(M)^{\mathrm{inv}} \subset \mathrm{Metr}(M)$

Construction of manifolds admitting psc-metrics

- Review Surgery



- embedding $f : S^k \times B^{n-k} \to M$
  $S := f(S^k \times \{0\})$ - surgery sphere
- $\partial(M \setminus f(S^k \times B^{n-k})) \cong S^{k-1} \times S^{n-k-1}$
- $M' = (M \setminus f(S^k \times B^{n-k})) \sqcup_\sim B^{k+1} \times S^{n-k-1}$

M' is obtained from M by a surgery of dim k / codim $n-k$.

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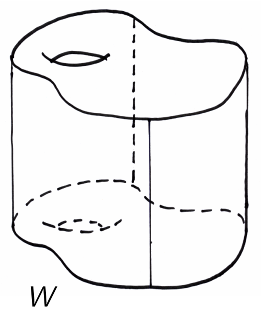
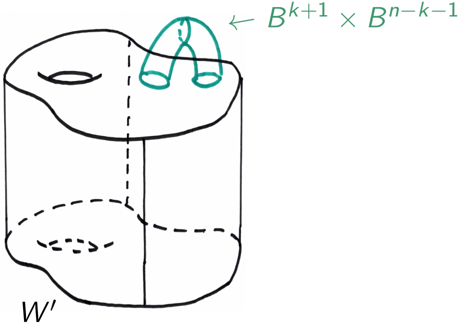




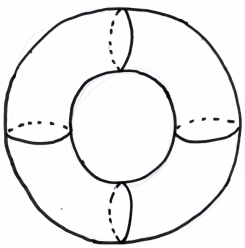
- View the cylinder $W := M \times [0,1]$ as a bordism from M to itself
- Attach $B^{k+1} \times B^{n-k-1}$ to $M \times \{1\}$
- W' is a bordism from M to M' - the trace of the surgery.

W' is obtained from W by attaching a $(k+1)$-handle.

Construction of manifolds admitting psc-metrics

- Review Surgery

Each closed manifold has a handle decomposition.



The torus is obtained as:

Construction of manifolds admitting psc-metrics

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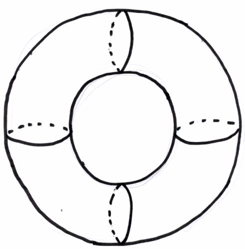

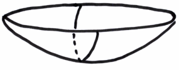

B^2

The torus is obtained as:

$$B^2 \; +$$

Construction of manifolds admitting psc-metrics
- Review Surgery

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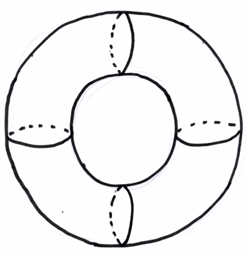

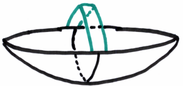

The torus is obtained as:

$$B^2 \; + \text{a 1-handle}$$

Construction of manifolds admitting psc-metrics
- Review Surgery

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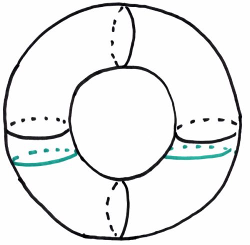

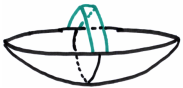

$B^1 \times B^1$

B^2

The torus is obtained as:

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Construction of manifolds admitting psc-metrics

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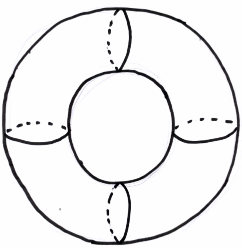

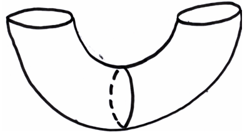

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Construction of manifolds admitting psc-metrics
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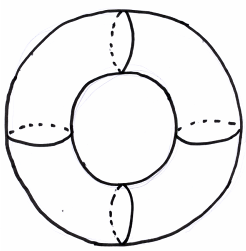

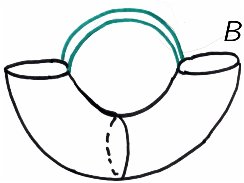

$B^1 \times B^1$

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$$B^2 \; + \text{a 1-handle} \; + \text{a 1-handle}$$

Construction of manifolds admitting psc-metrics
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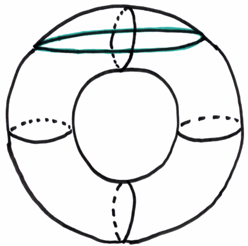

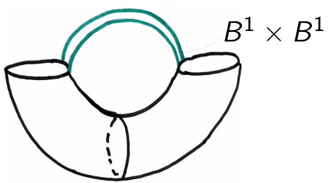

The torus is obtained as:

$$B^2 \ + \text{a 1-handle} \ + \text{a 1-handle}$$

Construction of manifolds admitting psc-metrics
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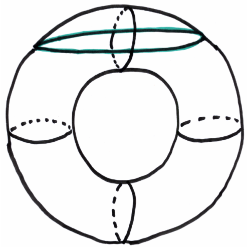
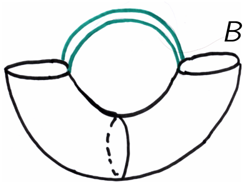




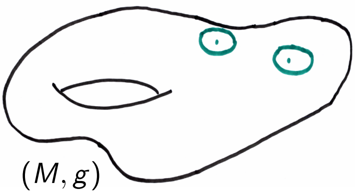
The torus is obtained as:

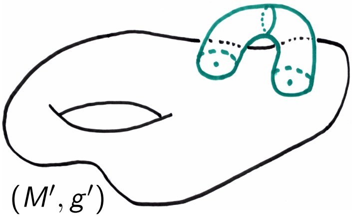
$$B^2 \ + \text{a 1-handle} \ + \text{a 1-handle} \ + B^2 \ = \ T^2$$

Construction of manifolds admitting psc-metrics

Theorem (Gromov, Lawson / Schoen, Yau; '80)

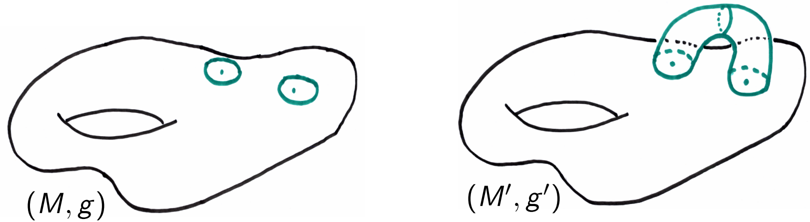
Let (M, g) be a closed Riemannian manifold with $g \in Metr(M)^{psc}$. Let M' be obtained from M by a surgery of codimension ≥ 3. Then, M' admits a psc-metric g'.





g' can be chosen such that it coincides with g outside a small neighbourhood around the surgery sphere.

Construction of manifolds admitting psc-metrics



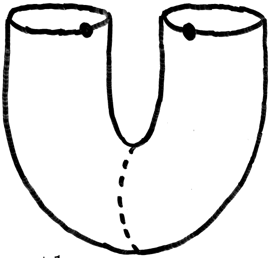
Intuition

- psc is a local property
- codim $n-k \geq 3 \;=\;$ gluing in $B^{k+1} \times S^{n-k-1 \geq 2}$
- standard product structure on $B^{k+1} \times S^{n-k-1 \geq 2}$ has psc

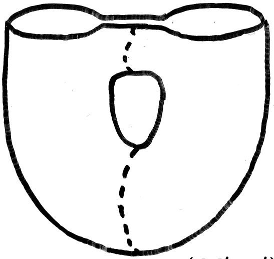
Psc-metrics and handle attachments

Theorem (Carr '88 / Gajer '87)

Let (M^{n+1}, g) be a compact Riemannian manifold with closed boundary ∂M, $g \in Metr(M)^{psc}$ and g having product structure near ∂M. Let M' be obtained from M by adding a $(k+1)$-handle of codimension $n - k \geq 3$. Then, M' admits a psc-metric g' that is again product near the (new) boundary.

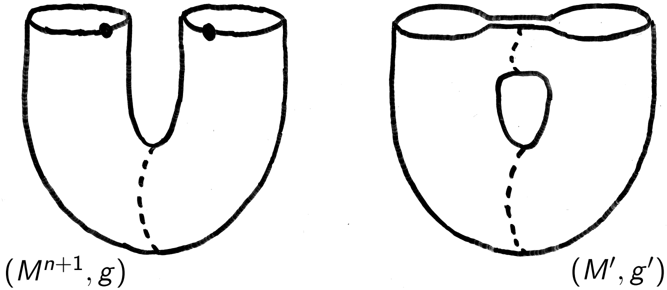


(M^{n+1}, g)



(M', g')

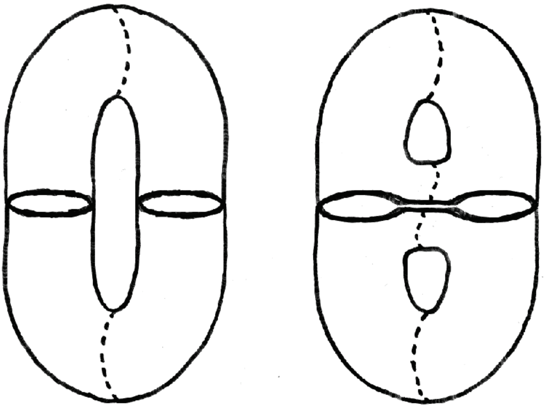
Psc-metrics and handle attachments



Intuition

- On the boundary: surgery of codim $n - k \geq 3$

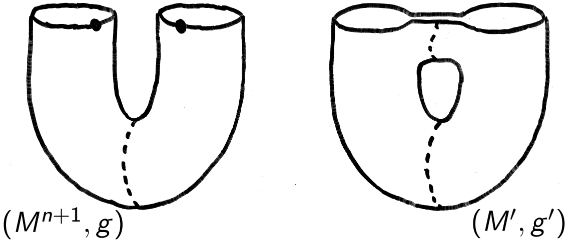
Psc-metrics and handle attachments



Intuition

- On the boundary: surgery of codim $n - k \geq 3$
- On the double: surgery of codim $n - k \geq 3$

Psc-metrics and handle attachments



Intuition

- On the boundary: surgery of codim $n - k \geq 3$
- On the double: surgery of codim $n - k \geq 3$

Implication

- $\mathrm{Metr}^{\mathrm{psc}}(S^{4k-1})$ has infinitely many components ($k \geq 2$) ($\mathrm{Metr}^{\mathrm{psc}}(S^3)$ is connected (Marques, 2011))

Metr$^{\text{inv}}(M)$

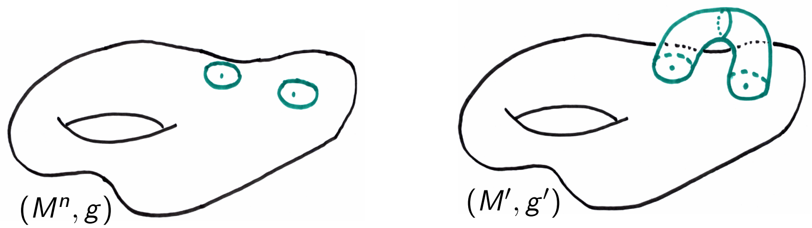
What can be done for metrics with invertible Dirac operators?

$\mathbf{Metr}^{\mathbf{inv}}(M)$

What can be done for metrics with invertible Dirac operators?

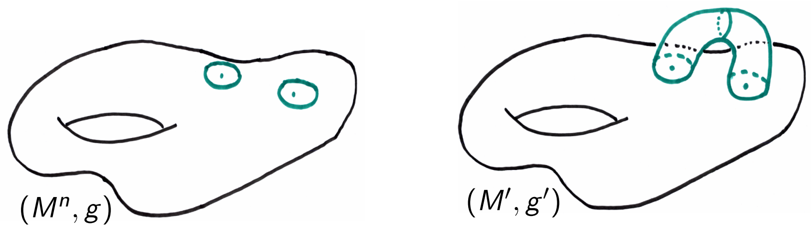
From now on: Let all manifolds be spin.

Surgery for $\mathbf{Metr}^{\mathbf{inv}}(M)$



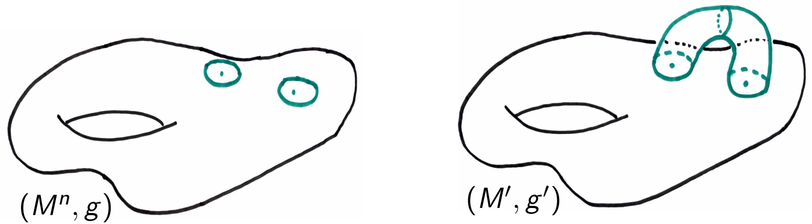
- After the surgery the manifold should still be spin!

Surgery for $\mathrm{Metr}^{\mathrm{inv}}(M)$



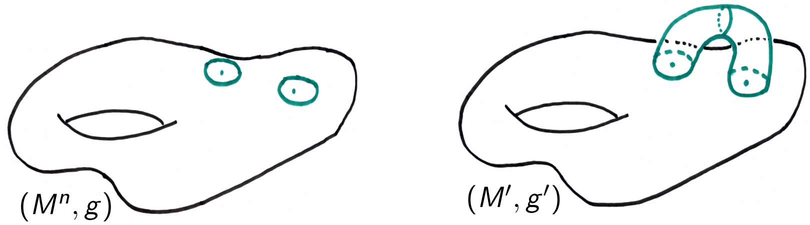
- After the surgery the manifold should still be spin!
  - $S^k \times B^{n-k}$ - induces spin structure on $S^k \times S^{n-k-1}$
  - glue in $B^{k+1} \times S^{n-k-1}$
    Its boundary should carry same spin structure.
  - For $k > 1$, the spin structure on S^k is unique and bounds the disk. - No Problem here.
  - For $k = 1$, two spin structures on S^1 - we only allow the one that bounds the disk.

Surgery for $\mathrm{Metr}^{\mathrm{inv}}(M)$



- $f : S^k \times B^{n-k} \to M$ spin-preserving embedding.

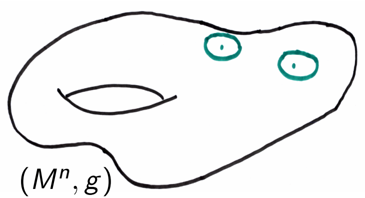
Surgery for $\mathrm{Metr}^{\mathrm{inv}}(M)$

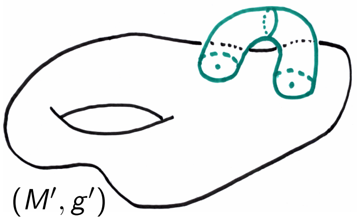


Intuition

- Invertible Dirac operator is a global condition.
- codim $n - k \geq 3 \;=\;$ gluing in $B^{k+1} \times S^{n-k-1 \geq 2}$
- standard product structure on $\mathbb{R}^{k+1} \times S^{n-k-1 \geq 2}$ has invertible Dirac operator

Surgery for $\mathbf{Metr}^{\mathbf{inv}}(M)$





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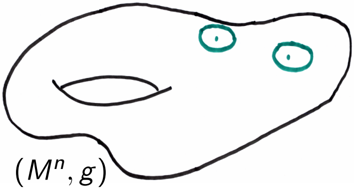
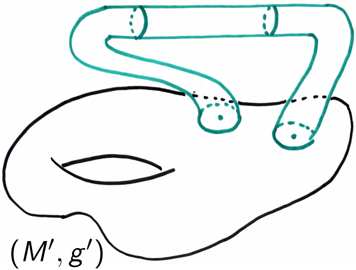
Surgery for $\mathrm{Metr}^{\mathrm{inv}}(M)$





Intuition

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- standard product structure on $\mathbb{R}^{k+1} \times S^{n-k-1 \geq 1}$ has invertible Dirac operator ('When taking the right S^1')
- 'If the inserted cylinder is large enough, invertibility survives.'

Construction for manifolds admitting inv-metrics

Theorem (Ammann, Dahl, Humbert; 2009)

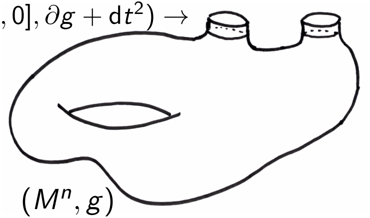
Let (M^n, g) be a closed Riemannian spin manifold with $g \in Metr(M)^{inv}$. Let M' be obtained from M by a surgery of codimension ≥ 2. Then, M' admits an inv-metric g'. Moreover, g' can be chosen such that it coincides with g outside a small neighbourhood around the surgery sphere.

Consequences (Ammann, Dahl, Humbert; 2009)

For a generic metric g, $\dim \ker D^g$ is no larger than forced by the index theorem.

Inv-metrics on manifolds with boundary

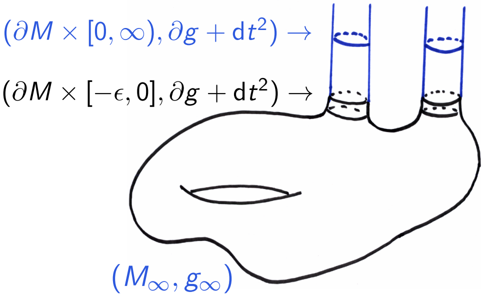
$(\partial M \times [-\epsilon, 0], \partial g + \mathrm{d}t^2) \to$



(M^n, g)

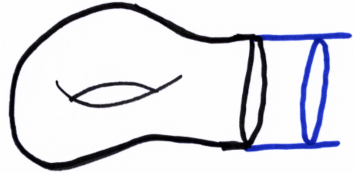
When do we call D^g invertible?

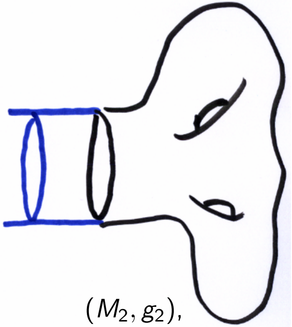
Inv-metrics on manifolds with boundary



$g \in \mathrm{Metr}(M)^{\mathrm{inv}}$ iff D^{g_∞} is invertible as operator on $L^2(M_\infty, S)$

Inv-metrics on manifolds with boundary





(M_1, g_1), $(\partial M_1, \partial g_1) = (N^+, h)$,

(M_2, g_2),

$g_1 \in \mathrm{Metr}(M_1)^{\mathrm{inv}}$

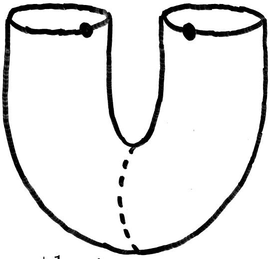
$(\partial M_2, \partial g_2) = (N^-, h)$, $g_2 \in \mathrm{Metr}(M_2)^{\mathrm{inv}}$

If M_1 and M_2 are glued together using a large enough cylinder $(N \times [-R, R], h + \mathrm{d}t^2)$, the resulting metric has again invertible Dirac operator.

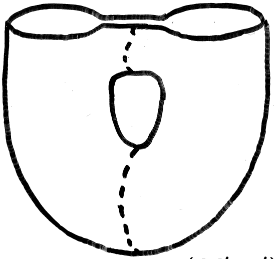
Inv-metrics and handle attachments

Theorem (Dahl, G. 2012)

Let (M^{n+1}, g) be a compact Riemannian spin manifold with closed boundary ∂M, $g \in Metr(M)^{inv}$ and g having product structure near ∂M. Let M' be obtained from M by adding a $(k+1)$-handle of codimension $n-k \geq 2$. Then, M' admits an inv-metric g' that is again product near the (new) boundary.



(M^{n+1}, g)



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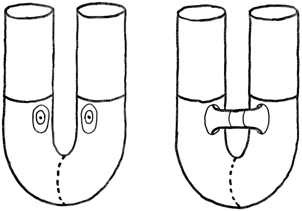
Implication

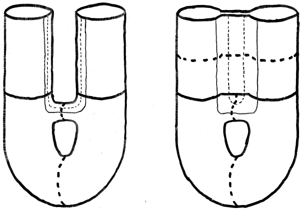
- $\mathrm{Metr}(S^{4k-1})^{\mathrm{inv}}$ has infinitely many components for all $k \geq 1$

Strategy and Methods

- 'Topological strategy' - Decompose the handle attachment



surgery of codim $n-k$



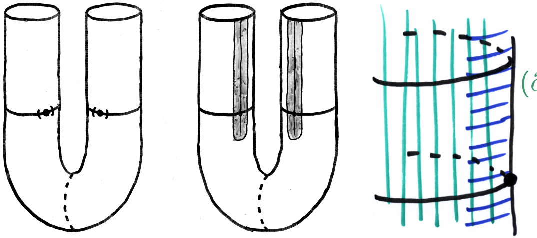
'half' surgery of codim $n-k+1$

glue in $'\frac{1}{2}B^{k+1} \times S^{n-k}'$

Metric strategy

- Approximation by 'double' product metrics near the surgery sphere

$(U_\delta(S \times [-\epsilon, \infty)),$
$g_S + \xi_{\mathbb{R}^{n-k}} + \mathrm{d}t^2)$



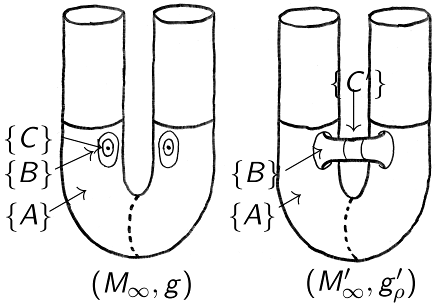
$(\partial M \times [-\epsilon, \infty),$
$, \partial g + \mathrm{d}t^2)$

$\leftarrow \partial M \times \{0\}$

If δ small enough, still $g_\delta \in \mathrm{Metr}(M)^{\mathrm{inv}}$.
('C^1-continuity of the spectrum')

Metric strategy

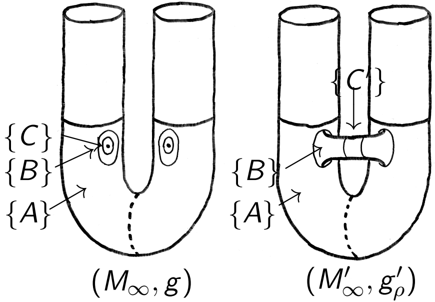
- First surgery



  - 'Parameter for tuning': ρ - 'diameter of $\{B\}$'
  - For ρ small enough, $g_\rho \in \mathrm{Metr}(M')^{\mathrm{inv}}$ - proof by contradiction

Metric strategy

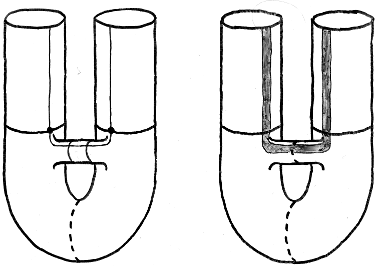
- First surgery



- For ρ small enough, $g_\rho \in \mathrm{Metr}(M')^{\mathrm{inv}}$ - proof by contradiction
- $\rho_i \to 0$, $g_{\rho_i} \notin \mathrm{Metr}(M')^{\mathrm{inv}}$
  $\curvearrowright$ g_{ρ_i} has a harmonic spinor: $D^{g_{\rho_i}}\varphi_i = 0$, $\|\varphi_i\|_{L^2(M', g_{\rho_i})} = 1$
  (regularity) $\curvearrowright$ $\phi_i \to \phi$ in $C^1_{\mathrm{loc}}(M \setminus (S \times [-\epsilon, \infty))$
  (removal of singularities) $\curvearrowright$ $D^g\phi = 0$ on M, $\|\phi\|_{L^2(M,g)} \leq 1$
  a priori estimates on the L^2-norm of ϕ_i on $\{A\}$ vs $\{C\}$.

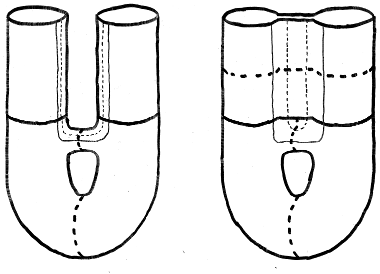
Metric strategy

- Again approximating by 'double' product metrics



Metric strategy

- Second surgery



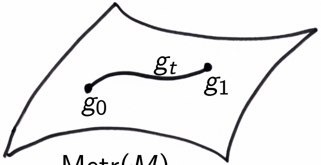
One Application

Theorem (Dahl, G.; 2012)

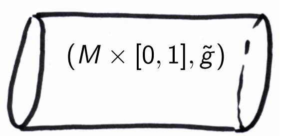
Let M be a closed 3-dimensional Riemannian spin manifold and $g \in Metr(M)^{inv}$. Then there are metrics $g^i \in Metr(M)^{inv}$, $i \in \mathbb{Z}$, such that g^i is bordant to g but g^i is not concordant to g^j for $i \neq j$.
In particular, $Metr(M)^{inv}$ has infinitely many connected components.

Notations

$g_0, g_1 \in \mathrm{Metr}(M)^{\mathrm{inv}}$ are *isotopic* *if* $\exists$ *smooth family* $g_t \in \mathit{Metr}(M)^{\mathit{inv}}$ *with* $g_t = g_0$ *for* $t \leq 0$, $g_t = g_1$ *for* $t \geq 1$.



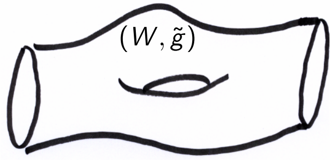
$\mathrm{Metr}(M)$



(M, g_0) (M, g_1)

$g_0, g_1 \in \mathrm{Metr}(M)^{\mathrm{inv}}$ are *concordant* if $\exists$ $\tilde{g} \in \mathrm{Metr}(M \times [0,1])^{\mathrm{inv}}$ with $\tilde{g}|_{M\times\{i\}} = g_i$.

$g_i \in \mathrm{Metr}(M_i)^{\mathrm{inv}}$ $(i = 0, 1)$ are *bordant* if $\exists$ $(W, \tilde{g})$ with $\partial W = M_0 \sqcup (M_1)^-$, $\tilde{g} \in \mathrm{Metr}(W)^{\mathrm{inv}}$, $\tilde{g}|_{M_i} = g_i$.



(M_0, g_0) (M_1, g_1)

An application

Theorem (Dahl, G.; 2012)

Let M be a closed 3-dimensional Riemannian spin manifold and $g \in Metr(M)^{inv}$. Then there are metrics $g^i \in Metr(M)^{inv}$, $i \in \mathbb{Z}$, such that g^i is bordant to g but g^i is not concordant to g^j for $i \neq j$.
In particular, $Metr(M)^{inv}$ has infinitely many connected components.

Lemma

There exist 4-manifolds $(Y^i, \tilde{h}^i)$ ($i \in \mathbb{Z}$) with $\tilde{h}^i \in Metr(Y^i)^{inv}$, $\partial Y^i = S^3$ such that $\alpha(Y^i \cup_{S^3} (Y^j)^-) = c(i-j)$ for a constant $c \neq 0$.

An application

Lemma

There exist 4-manifolds $(Y^i, \tilde{h}^i)$ $(i \in \mathbb{Z})$ *with* $\tilde{h}^i \in \mathit{Metr}(Y^i)^{\mathit{inv}}$, $\partial Y^i = S^3$ *such that* $\alpha(Y^i \cup_{S^3} (Y^j)^-) = c(i-j)$ *for a constant* $c \neq 0$.

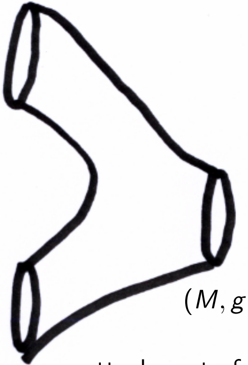
Construction:

- Y^0 - B^4 with a 'torpedo metric' $\tilde{h}^0 \in \mathrm{Metr}(B^4)^{\mathrm{psc}}$ and $\tilde{h}^0|_{S^3} =$ standard metric
- $Y^i = \underbrace{(K3\#K3\#\cdots\#K3)}_{i \text{ times}} \setminus B^4 = Y^0 +$ several 2-handles
- $\alpha(Y^i \cup_{S^3} (Y^j)^-) = \alpha(\#_{(i-j)} K3) = (i-j)\alpha(K3) \neq 0$ for $i \neq j$

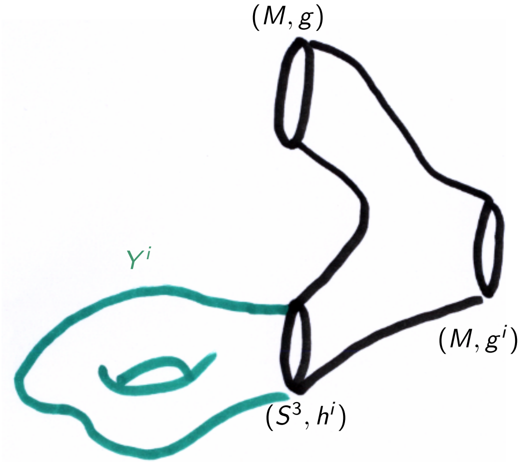
$h^i := \tilde{h}^i|_{S^3}$

Constructions of g^i

Start with $(M^3, g \in \mathrm{Metr}(M)^{\mathrm{inv}})$ - construct g^i bordant to g.

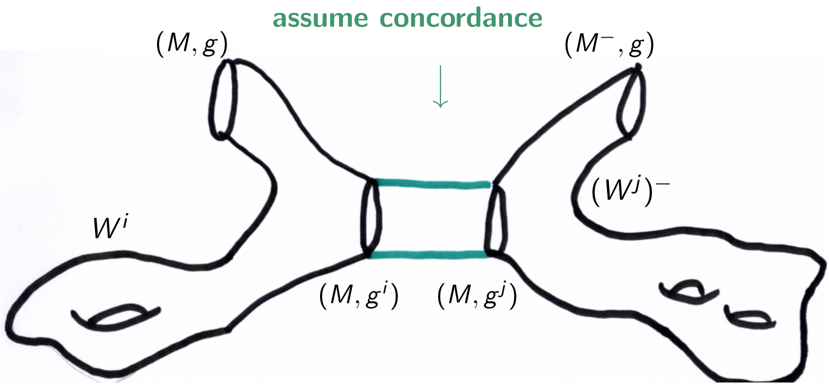


Constructions of g^i

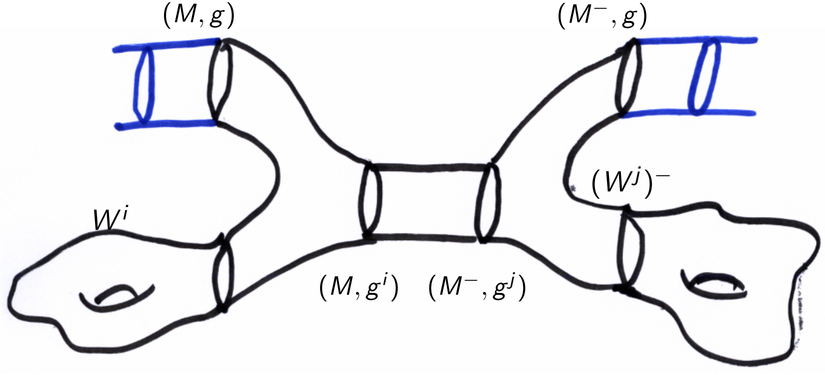


(M, g) and (M, g^i) are bordant.
Bordism $(W^i, \tilde{g}^i) \in \mathrm{Metr}(W^i)^{\mathrm{inv}}$

Constructions of g^i



Constructions of g^i



Closed manifold $(W, \tilde{g})$ with $\tilde{g} \in \mathrm{Metr}(W)^{\mathrm{inv}}$ and $\alpha(W) = c(i - j)$.

One Application

Theorem (Dahl, G.; 2012)

Let M be a closed 3-dimensional Riemannian spin manifold and $g \in Metr(M)^{inv}$. Then there are metrics $g^i \in Metr(M)^{inv}$, $i \in \mathbb{Z}$, such that g^i is bordant to g but g^i is not concordant to g^j for $i \neq j$.
In particular, $Metr(M)^{inv}$ has infinitely many connected components.

Thank you for your attention.