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WS1718

Seminar Microlocal Analysis

Je nach Teilnehmern findet das Seminar auf deutsch oder englisch statt. In jedem Fall können Sie aber Ihren Vortrag auf deutsch halten, wenn Sie zu große Bedenken mit dem Englischen haben. Sie müssten dann nur ggf. die anderen Vorträge auf Englisch hören.

This seminar gives a first introduction to microlocal analysis. The knowledge of functional analysis and Fourier analysis is not required. We cover pseudodifferential operators, wavefront sets, oscillatory integrals and the stationary phase lemma and their use in elliptic and hyperbolic partial differential equations.

Dates: 16.10, 23,10, 6.11, 13.11, 20.11, 27.11, 4.12, 11.12, 18.12, 8.1, 15.1, 22.1, 29.1, 5.2 (14 sessions)

Plan:

1. **(16.10) Introductory examples and historical treatment of hyperbolic equations.**
   Give an hystorical introduction on hyperbolic equations following [4] and show one example of geometric optics (e.g. following [1, Appendix IV].)

2. **(23.10) Distributions and Fourier Analysis.**
   Introduce the space of distributions and define product, derivative and convolution [11, §4.2]. Recall basic results of Fourier Analysis [11, §4.2] (see also [6, §7.1]) and sketch the proof of Theorem 6 in [11, §4.3].

3. **(06.11) The Wavefront set of a distribution I.**
   Introduce the wavefront set using Fourier analysis [6, §8.1]. State and prove the theorem of pullback and product of distributions [6, §8.2]. Finally study the wavefront set of solutions of PDEs [6, §8.3].

4. **(13.11) Oscillary integrals.**
   Introduce the oscillatory integral as generalization of Fourier Transform [10, §1.1-1.2] and define the Fourier Integral Operator [10, §2.1]. Define the operator phase function and explain Example 1 and 2 in [10, §2.3-4].

5. **(20.11) Stationary phase approximation.**
   Explain the method of the stationary phase [5, §2] (see also [2, §4]). Calculate as application the asymptotics of some special function (e.g. Bessel function [2, §4.15]) and sketch the calculation of the the angle of the Kelvin wedge (see [8] and [2, §4.13]).

6. **(27.11) Pseudodifferential operators.**
   Introduce the notion of a pseudodifferential operator (ΨDO) as in Example 3 in [10, §2.5]. Characterize the properly supported ΨDOs and study the adjoint [5, §3] (for more details see also [10, §3]).

7. **(4.12) Application to elliptic operators** Define ellipticity of a ΨDO and give examples [10, §5.1]. Introduce the parametrix of an elliptic ΨDO and sketch its relation with Sobolev space (Theorem 4.7 in [5, §4]) and local solvability ([5, §4] and [10, §5]).

8. **(11.12) Hamilton-Jacobi Theory.**
   Recall the definition of tangent and cotangent vector, canonical 1- and 2- forms, Lie derivatives and Hamilton Jacobi equations [5, §5].

9. (18.12) Strictly hyperbolic Cauchy problem.
Sketch the asymptotic behavior of solutions of linear partial differential equations with highly oscillatory initial values [5, §6] (for more details see also [7]).

10. (8.1) The wave front set of a distribution II.
Revisit the definition of wavefront set using pseudodifferential operator [5, §7], in particular prove that this definition is equivalent to the one gave in talk 3 (see Proposition 7.4). Present examples of wave front sets ([5, §7], [11, §4.3.3])

11. (15.1) Propagation of singularities for operators of real principal type.
Prove Theorem 8.1 in [5, §8]

12. (22.1) Fredholm operator, index and spectrum.
Define the index of a Fredholm operator and study its properties [10, §8.1]. Analyze the index and the spectrum of an elliptic operator on a closed manifold [10, §8.2-8.3].

13. (29.1) Spectral theory for elliptic operators.
Prove a theorem of Hörmander on the Weyl asymptotics with a small remainder for self-adjoint elliptic operators of arbitrary order on a compact manifold [5, §12].

14. (5.2) Micro-Local Approach to QFT on curved space-times.
Introduce the distributional approach to quantized fields on curved space-time and the Klein-Gordon quantum field model [9, Sect. 2]. Explain the Hadamard condition and indicate the relation to micro-analysis.[9].

References

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