Seminar on Characteristic Classes
Projective Bundle Formula

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Contents

0 Recollections on vector bundles [Skip]

Let F be a topological space. A continuous map $\pi\colon E \to X$ is called a *fibre bundle* with typical fibre F if for every point $x \in X$ there is an open neighborhood U in X and a homeomorphism $\varphi_U\colon \pi^{-1}(U) \to U \times F$ over U, i.e.

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\quad\varphi_U\quad} & U \times F \\ & {\scriptstyle \pi}\searrow \quad \circlearrowleft \quad \swarrow{\scriptstyle \mathrm{pr}_U} & \\ & U & \end{array}$$

We call φ_U a (local) *trivialization* and (U, φ_U) a *bundle chart.* We denote the fibre $\pi^{-1}(x)$ over $x \in X$ by E_x. Fibre bundles (with typical fibre F) correspond to objects in the category of arrows in Top, and a *morphism of fibre bundles* is precisely a morphism between the corresponding arrows. More explicitly, a morphism from a fibre bundle $\pi\colon E \to X$ to a fibre bundle $\pi'\colon E' \to X'$ is a pair of continuous maps $f\colon E \to E'$ and $g\colon X \to X'$ such that

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ \pi\downarrow & \circlearrowleft & \downarrow\pi' \\ X & \xrightarrow[g]{} & X' \end{array}$$

Let G be a topological group acting continuously on the typical fibre F on the left. Moreover, let us assume that this action is faithful, so that we may identify G with a subgroup of the group of self homeomorphisms of F. A *G-atlas* for the bundle $\pi\colon E \to X$ is a collection of charts $(U_\alpha, \varphi_\alpha)$ such that the compositions

$$\varphi_{\alpha\beta} := \varphi_\beta\varphi_\alpha^{-1}\colon (U_\alpha \cap U_\beta) \times F \to (U_\alpha \cap U_\beta) \times F$$

are given by $\varphi_{\alpha\beta}(x, e) = (x, \theta_{\alpha\beta}(x)(e))$, where $\theta_{\alpha\beta}\colon U_\alpha \cap U_\beta \to G$ is a continuous map called the *transition map.* We say that two G-atlases are equivalent if their union is also a G-atlas, and we define a *G-bundle* to be a fibre bundle together with an equivalence class of G-atlases. We call G the *structure group* of the bundle.

Definition 1. Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$. An *rank k vector bundle over* $\mathbb{K}$ is a fibre bundle with typical fibre $\mathbb{K}^k$ and with structure group $\mathrm{GL}_k(\mathbb{K})$.

Lemma 2. Equivalently, a rank k vector bundle over $\mathbb{K}$ is a fibre bundle with typical fibre $\mathbb{K}^k$ in which all fibres are equipped with a $\mathbb{K}$-vector space structure such that for every chart (U, φ_U) the restriction to every fibre $\varphi_U|_{E_x}\colon E_x \to \{x\} \times \mathbb{K}^k$ is a $\mathbb{K}$-linear homeomorphism.

Proof: Let $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) be two bundle charts and let $x \in U_\alpha \cap U_\beta =: U_{\alpha\beta}$. Consider the composition

$$\varphi_{\alpha\beta}\colon U_{\alpha\beta} \times \mathbb{K}^k \to U_{\alpha\beta} \times \mathbb{K}^k$$

Since $\varphi_\alpha|_{E_x}$ and $\varphi_\beta|_{E_x}$ are $\mathbb{K}$-linear, so is $\varphi_\beta|_{E_x}\varphi_\alpha|_{E_x}^{-1}\colon \mathbb{K}^k \to \mathbb{K}^k$. So we can assign to x an element $\theta_{\alpha\beta}(x) \in \mathrm{GL}_k(\mathbb{K})$. We have to check that this defines a continuous transition map $\theta_{\alpha\beta}\colon U_{\alpha\beta} \to \mathrm{GL}_k(\mathbb{K})$.

Let $e_j \in \mathbb{K}^k$ be the j^{th} standard basis vector in $\mathbb{K}^k$ and let $p_i\colon \mathbb{K}^k \to \mathbb{K}$ be the i^{th} coordinate projection. As in any other fibre bundle, the map

$$U_{\alpha\beta} \times \mathbb{K}^k \xrightarrow{\varphi_{\alpha\beta}} U_{\alpha\beta} \times \mathbb{K}^k \xrightarrow{\mathrm{pr}_{\mathbb{K}^k}} \mathbb{K}^k$$

is continuous. In particular, if we fix $e_j \in \mathbb{K}^k$, we get a continuous map which sends $(x, e_j) \mapsto (x, \theta(x)(e_j))$. Composing further with the i^{th} coordinate projection $\mathrm{pr}_i\colon \mathbb{K}^k \to \mathbb{K}$ we get a continuous map which sends $(x, e_j) \mapsto \mathrm{pr}_i(\theta(x)(e_j)) =: a_{ij}(x)$. But $a_{ij}(x) \in \mathbb{K}$ is then ij-entry of the matrix representing $\theta(x)$ with respect to the standard basis, and since $a_{ij}(x)$ varies continuously on x we get that the map $\theta\colon U_{\alpha\beta} \to \mathrm{GL}_k(\mathbb{K})$ is indeed continuous. □

A *morphism of vector bundles* is then a bundle morphism which is linear on each fibre. In particular we can define the category of vector bundles over X, denoted by $\mathrm{Vec}(X)$. If confusion is possible, we will also include the base field in the notation, and write $\mathrm{Vec}_{\mathbb{K}}(X)$ instead.

Lemma 3. Let $\pi_1\colon E_1 \to X$ and $\pi_2\colon E_2 \to X$ be two vector bundles over X. Let $f\colon E_1 \to E_2$ be a continuous map over X which sends each fibre $\pi_1^{-1}(x)$ to $\pi_2^{-1}(x)$ linearly. Then f is a vector bundle isomorphism if and only if it is an isomorphism of vector spaces on each fibre.

Proof: If f is a vector bundle isomorphism, then the map on each fibre is a bijective linear morphism, hence an isomorphism of vector spaces.

Conversely, if f is an isomorphism of vector spaces on each fibre, then it is a continuous bijection. So we have to check whether its inverse f^{-1} is continuous or not. Continuity is a local property, so it suffices to show that $f^{-1}|_U = (f|_U)^{-1}$ is continuous, where U is an open subset of X over which both vector bundles π_1 and π_2 are trivial. Since trivializations are homeomorphisms, the continuity of $f^{-1}|_U$ is equivalent to the continuity of $\varphi_{1,U} \circ f^{-1}|_U \circ \varphi_{2,U}^{-1}\colon U \times \mathbb{K}^k \to U \times \mathbb{K}^k$.

$$\begin{array}{ccc} \pi_2^{-1}(U) & \xrightarrow{(f|_U)^{-1}} & \pi_1^{-1}(U) \\ {\scriptstyle \varphi_{2,U}}\Big\uparrow & \circlearrowleft & \Big\downarrow{\scriptstyle \varphi_{1,U}} \\ U \times \mathbb{K}^k & \dashrightarrow & U \times \mathbb{K}^k \end{array}$$

The inverse of this composition is given by $(x, v) \mapsto (x, \theta(x)(v))$ where $\theta(x) \in \mathrm{GL}_k(\mathbb{K})$ is a linear morphism (composition of three linear morphisms by hypothesis) which varies continuously on x because $f|_U$ is continuous. So this composition is given by $(x, v) \mapsto (x, \theta(x)^{-1}(v))$. The inverse of a matrix can be computed algebraically by a formula that depends continuously on the coefficients of the matrix. Hence, this composition is also continuous, and this finishes the proof. □

As we have seen before, a vector bundle $\pi\colon E \to X$ has a collection of transition functions $\theta_{\alpha\beta}\colon U_{\alpha\beta} \to \mathrm{GL}_k(\mathbb{K})$ which are continuous. They satisfy three straightforward properties:

1. $\theta_{\alpha\alpha}(x) = \mathrm{id}_{\mathbb{K}^k} \in \mathrm{GL}_k(\mathbb{K})$ for all $x \in U_\alpha$.
2. $\theta_{\beta\alpha}(x) = (\theta_{\alpha\beta}(x))^{-1} \in \mathrm{GL}_k(\mathbb{K})$ for all $x \in U_{\alpha\beta}$.
3. $\theta_{\beta\gamma}(x) \circ \theta_{\alpha\beta}(x) = \theta_{\alpha\gamma}(x) \in \mathrm{GL}_k(\mathbb{K})$ for all $x \in U_{\alpha\beta\gamma}$.

Lemma 4. Let X be a topological space and let U_{alpha} be an open cover of X. As before, denote $U_{\alpha\beta} = U_\alpha \cap U_\beta$ and suppose we are given continuous functions $\theta_{\alpha\beta}\colon U_{\alpha\beta} \to \mathrm{GL}_k(\mathbb{K})$ satisfying the three properties above. Then there is an rank k $\mathbb{K}$-vector bundle $\pi\colon E \to X$ which has these functions as transition functions. Moreover, $\pi\colon E \to X$ is uniquely determined up to isomorphism of vector bundles.

Proof: As a set, define

$$E = \bigsqcup_\alpha (U_\alpha \times \mathbb{K}^k)/\sim$$

where $\sim$ is the equivalence relation defined on by

$$(x \in U_\alpha, v) \sim (x' \in U_\beta, v') \quad \Leftrightarrow \quad x = x' \text{ and } \theta_{\alpha\beta}(x)(v) = v'$$

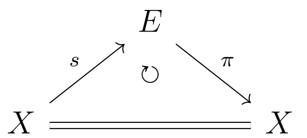
The three properties above guarantee that $\sim$ is an equivalence relation. For every $x \in U_\alpha$ and $v \in \mathbb{K}^k$, the equivalence class $[(x, v)]$ has a unique representative $(x, v_\alpha) \in U_\alpha \times \mathbb{K}^k$: there is at least one because $x \in U_\alpha$ and there is at most one because $\theta_{\alpha\alpha}(x) = \mathrm{id}_{\mathbb{K}^k}$. Therefore the obvious projection $\pi\colon E \to X$ turns this into a vector bundle with trivializations $\varphi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{K}^k$ given by $[(x, v)] \mapsto (x, v_\alpha)$. If $(x, v) \in U_{\alpha\beta} \times \mathbb{K}^k$, then $\varphi_\alpha^{-1}((x, v)) = [(x, v)] = [(x, \theta_{\alpha\beta}(x)(v))]$, thus $\varphi_\beta([(x, v)]) = (x, \theta_{\alpha\beta}(x)(v))$ and therefore the transition functions are indeed the given $\theta_{\alpha\beta}\colon U_{\alpha\beta} \to \mathrm{GL}_k(\mathbb{K}^k)$.

Let now $\pi'\colon E' \to X$ be another vector bundle with bundle charts $(U_\alpha, \varphi'_\alpha)$ and transition functions $\theta_{\alpha\beta}\colon U_{\alpha\beta} \to \mathrm{GL}_k(\mathbb{K})$. Then define

$$\begin{aligned} f\colon \bigsqcup_\alpha U_\alpha \times \mathbb{K}^k &\longrightarrow E' \\ (x \in U_\alpha, v) &\longmapsto \varphi'_\alpha(x, v) \end{aligned}$$

This is a continuous surjection and we have that $(x, v) \sim (x, v')$ if and only if $\theta_{\alpha\beta}(x)(v) = \varphi_\beta|'_{E_x} \circ \varphi_\alpha|'^{-1}_{E_x}(v) = v'$, hence $(x, v) \sim (x, v')$ if and only if $f((x, v)) = f((x, v'))$. By the universal property of the quotient we get a continuous bijection $\bar{f}\colon E \to E'$ over X, and by lemma 3 it must be a vector bundle isomorphism. □

A (global) *section* of a vector bundle $\pi\colon E \to X$ is a continuous map $s\colon X \to E$ such that $\pi \circ s = \mathrm{id}_X$, i.e.



We always have the zero section $s_0\colon X \to E$ which sends each $x \in X$ to the zero vector of the corresponding fibre E_x. We denote by E_0 the complement of the zero section $E \backslash s_0(X)$.

Lemma 5. A rank k vector bundle $\pi\colon E \to X$ is trivial if and only if it admits k sections $(\sigma_1, \ldots, \sigma_k)$ which are linearly independent at each point $x \in X$.

If U is an open set of X, a *local section* of π over U is a section of the vector bundle $\pi^{-1}(U) \to U$. The *sheaf of sections* of a vector bundle determines the vector bundle up to isomorphism. Moreover, there is a bijection between $\mathbb{K}$-vector bundles over X and locally free sheaves of finite type over the sheaf $\mathcal{O}_X$ of $\mathbb{K}$-valued continuous functions on X. If we replace continuous by smooth, holomorphic, algebraic... we get a bijection with vector bundles in the corresponding category.

We can also *pullback* vector bundles (via the usual pullback in the category of topological spaces) and define new vector bundles from old using certain constructions, e.g. the *product* $E \times E' \to X \times X'$ of vector bundles. We won't recall these here.

1 The projective bundle formula

1.1 The Euler class of a vector bundle

In this talk we will work with complex vector bundles and singular cohomology with integer coefficients (see lemma 8 for a justification). Our goal is to introduce Chern classes (definition 24) and study their basic properties (theorem 25). This theory is rather formal, but there is a lot of geometry encoded inside Chern classes. This geometric content is in fact concentrated in the first Chern classes of line bundles, which in our case will be given by the Euler class. The rest of Chern classes of all vector bundles are completely determined by their formal properties and the first Chern classes of line bundles. For this reason we will also spend some time recalling the Euler class and discussing some of its geometric content.

The analogous definition for real vector bundles, Stiefel-Whitney classes, works the same way with $\mathbb{F}_2$ coefficients. All the results in that case are analogous.

Remark 6. All the results are the same for complex vector bundles and singular cohomology with coefficients in an arbitrary ring R, with essentially the same proofs. So we will change the coefficient ring when necessary without making further comments about it. If no coefficients are specify, we are working with $R = \mathbb{Z}$.

Let $\pi\colon E \to X$ be a rank k $\mathbb{C}$-vector bundle and let $E_0 = E\backslash s_0(X)$, where $s_0\colon X \to E$ is the zero section.

Definition 7. A *Thom class* with R coefficients is a class $u = u(E) \in \mathrm{H}^{2k}(E, E_0; R)$ such that the restriction of u to each fibre $F \cong \mathbb{C}^k$ is a generator of $\mathrm{H}^{2k}(\mathbb{C}^k, \mathbb{C}^k\backslash\{0\}; R) \cong R$.

We know from the previous talk that a vector bundle is R-orientable if and only if a Thom class with R coefficients exists. Every vector bundle is $\mathbb{F}_2$-orientable (every family of orientations on the fibres is trivially coherent), but not every vector bundle is $\mathbb{Z}$-orientable. We say that a bundle is simply *orientable* if it is $\mathbb{Z}$-orientable.

Lemma 8. Complex vector bundles have a canonical orientation. In particular, complex manifolds are orientable, because their tangent bundles are.

> Proof: We have to find a coherent family of orientations on the fibres. But this is possible, because every finite dimensional $\mathbb{C}$-vector space V has a canonical orientation as an $\mathbb{R}$-vector space: for any basis $v_1, \ldots, v_k$ of V over $\mathbb{C}$, the (ordered) collection $v_1, iv_1, \ldots, v_k, iv_k$ is a basis of V over $\mathbb{R}$, and its orientation does not depend on the original basis over $\mathbb{C}$ because $\mathrm{GL}_k(\mathbb{C})$ is path connected. □

Recall from Domenico's talk:

Theorem 9 (Leray-Hirsch)**.** Let R be a ring. Let $(F, F') \xrightarrow{i} (E, E') \xrightarrow{\pi} X$ be a fibre bundle pair, i.e. $E \to X$ is a fibre bundle with typical fibre F and $E' \subseteq E$ a subspace such that the restriction of the bundle charts turns $E' \to X$ into a fibre bundle with typical fibre $F' \subseteq F$. Suppose that $\mathrm{H}^l(F, F'; R)$ is a finitely generated free R-module. Let $c_j \in \mathrm{H}^*(E, E'; R)$ be a family of classes whose restrictions $i^*(c_j)$ form a basis of

$\mathrm{H}^*(F, F'; R)$ over R in each fibre (F, F'). Then we have an isomorphism of graded R-modules

$$\begin{aligned} \mathrm{H}^*(X;R) \otimes_R \mathrm{H}^*(F,F';R) &\longrightarrow \mathrm{H}^*(E,E';R) \\ x \otimes i^*(c_j) &\longmapsto \pi^*(x) \cup c_j \end{aligned}$$

In particular, $\mathrm{H}^*(E;R)$ is a free (graded) $H^*(X;R)$-module with basis $\{c_j\}$.

In the case of our vector bundle $(E, E') = (E, E_0) \to X$, we have that $\mathrm{H}^l(F, F') \cong \mathrm{H}^l(\mathbb{C}^k, \mathbb{C}^k \backslash \{0\}) \cong \tilde{H}^l(S^{2k})$ which is $\mathbb{Z}$ if $l = 2k$ and 0 otherwise. Moreover, we know that there exists a Thom class $u \in \mathrm{H}^{2k}(E, E_0)$, so we can apply the Leray-Hirsch theorem to deduce that

$$\mathrm{H}^l(X) \cong \mathrm{H}^{l+2k}(E, E_0)$$

via $x \mapsto \pi^*(x) \cup u$ for all $l \in \mathbb{N}$. This is called the *Thom isomorphism*.

Note also that we can linearly contract the vector bundle E to the zero section $s_0(X) \cong X$, so the fibre bundle projection induces isomorphisms in cohomology $\pi^* \colon \mathrm{H}^*(X) \xrightarrow{\cong} \mathrm{H}^*(E)$. The inverse of this isomorphism is precisely the pullback along the zero section $s_0^* \colon \mathrm{H}^*(E) \xrightarrow{\cong} \mathrm{H}^*(X)$.

Definition 10. The *Euler class* $e = e(E) \in \mathrm{H}^{2k}(X)$ is the image of the Thom class under the composition $\mathrm{H}^{2k}(E, E_0) \xrightarrow{j^*} \mathrm{H}^{2k}(E) \xrightarrow{s_0^*} \mathrm{H}^{2k}(X)$, where $j \colon (E, \varnothing) \hookrightarrow (E, E_0)$ is the natural inclusion.

Remark 11. Note that the Euler class of a vector bundle is a priori only defined up to a sign, and this sign depends on the orientation of the vector bundle (i.e. on the choice of the Thom class). But by lemma 8 every complex vector bundle has a canonical orientation, so we have in fact a canonical choice of Thom and Euler class, and these are what we call the Thom and Euler classes.

Consider the long exact sequence in cohomology of the pair (E, E_0). Using the previous two isomorphisms we can replace $\mathrm{H}^l(E, E_0)$ by $\mathrm{H}^{l-2k}(X)$ and $\mathrm{H}^l(E)$ by $\mathrm{H}^l(X)$ to obtain the *Gysin sequence*

$$\cdots \to \mathrm{H}^{l-1}(E_0) \to \mathrm{H}^{l-2k}(X) \xrightarrow{(-)\cup e} \mathrm{H}^l(X) \xrightarrow{\pi^*} \mathrm{H}^l(E_0) \to \cdots$$

So vanishings on the cohomology of E_0 will yield isomorphisms between the cohomology groups of the base X given by multiplication with the Euler class.

Lemma 12 (Functoriality of the Euler class)**.** Let $\pi \colon E \to X$ and $\pi' \colon E' \to X'$ be two rank k vector bundles with a vector bundle morphism $(g, f) \colon (E', X') \to (E, X)$ such that $g(E'_0) \subseteq E_0$. Then we have $e(E') = f^*(e(E)) \in \mathrm{H}^{2k}(X')$.

<u>Proof:</u> We have a commutative square

$$\begin{array}{ccc} E' & \xrightarrow{g} & E \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ X' & \xrightarrow{f} & X \end{array}$$

By functoriality we get a commutative diagram

$$\begin{array}{ccccc}
H^{2k}(E, E_0) & \longrightarrow & H^{2k}(E) & \xrightarrow{\cong} & H^{2k}(X) \\
{\scriptstyle g^*}\downarrow & & \downarrow{\scriptstyle g^*} & & \downarrow{\scriptstyle f^*} \\
H^{2k}(E', E'_0) & \longrightarrow & H^{2k}(E') & \xrightarrow{\cong} & H^{2k}(X')
\end{array}$$

So it suffices to check that the Thom class $u = u(E) \in H^{2k}(E, E_0)$ is sent to the Thom class $g^*(u) \in H^{2k}(E', E'_0)$. The condition $g(E'_0) \subseteq E_0$ means in particular that the map on the fibres is injective, so it is a $\mathbb{C}$-linear isomorphism and hence the canonical orientation is preserved. So for every fibres $F' = \pi'^{-1}(y)$ and $F = \pi^{-1}(f(y))$ we have a commutative diagram

$$\begin{array}{ccc}
(F, F_0) & \hookrightarrow & (E, E_0) \\
\uparrow{\scriptstyle \cong} & \circlearrowleft & {\scriptstyle g}\uparrow \\
(F', F'_0) & \hookrightarrow & (E', E'_0)
\end{array}$$

Applying cohomology we obtain

$$\begin{array}{ccc}
H^{2k}(E, E_0) & \longrightarrow & H^{2k}(F, F_0) \\
{\scriptstyle g^*}\downarrow & \circlearrowleft & \downarrow{\scriptstyle \cong} \\
H^{2k}(E', E'_0) & \longrightarrow & H^{2k}(F', F'_0)
\end{array}$$

and this shows that $g^*(u)$ also restricts to a generator on each fibre. □

Corollary 13. Isomorphic vector bundles over X have the same Euler class.

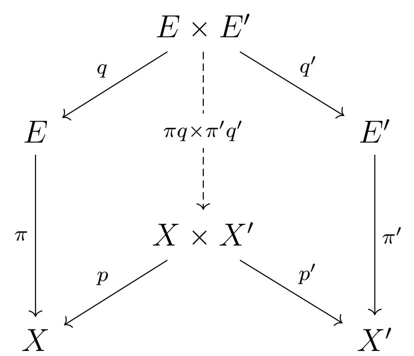
Proof: The map induced on each fibre is an isomorphism, and in particular we can apply lemma 12 with $f = \mathrm{id}_X$. □

Corollary 14. The Euler class of the pullback of a vector bundle $\pi\colon E \to X$ along $f\colon X' \to X$ is given by $e(f^*(E)) = f^*(e(E))$.

Proof: The map induced by the pullback $g\colon f^*E \to E$ is the identity on each fibre, so we can apply lemma 12. □

Lemma 15 (Additivity of the Euler class)**.** Let $\pi\colon E \to X$ and $\pi'\colon E' \to X'$ be two vector bundles of ranks k and k' respectively. Then their product $E \times E' \to X \times X'$ has Euler class $e(E \times E') = e(E) \times e(E') \in H^{2k+2k'}(X \times X')$.

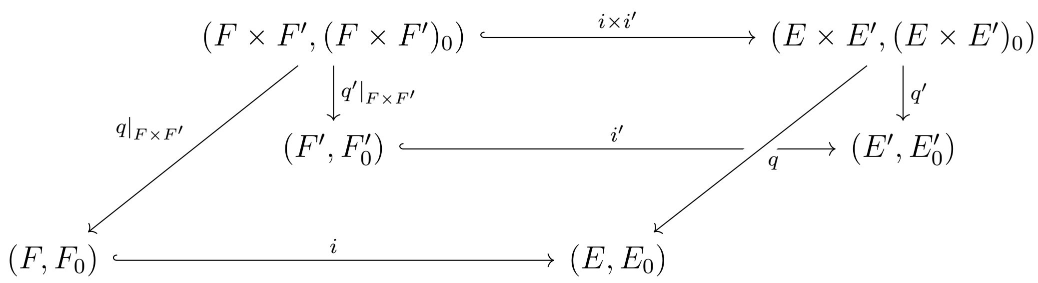
Proof: We have a commutative diagram



Note that $(E \times E')_0 = (E_0 \times E') \cup (E \times E'_0)$. We claim now that

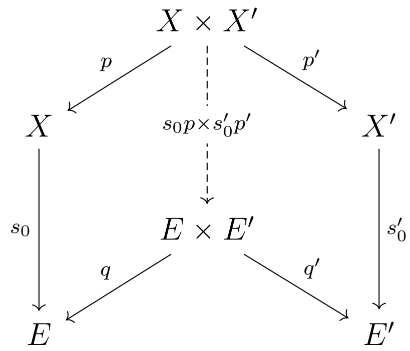
$$u(E \times E') = q^*(u(E)) \cup q'^*(u(E')) = u(E) \times u(E')$$

For this consider the following commutative diagram



Take now the Thom classes $u = u(E)$ and $u' = u(E')$. By definition, they restrict to generators $i^*(u)$ and $i'^*(u')$ on the corresponding fibres. The cross product of these generators is a generator $i^*(u) \times i'^*(u')\,\mathrm{H}^{2k+2k'}(F \times F', (F \times F')_0)$ by the relative Künneth formula (the fibres are nice Euclidean spaces). But by definition $i^*(u) \times i'^*(u') = (q|_{F\times F'})^* i^*(u) \cup (q'|_{F\times F'})^* i'^*(u')$, which by the previous commutative diagram is the same as $(i \times i')^* q^*(u) \cup (i \times i')^* q'^*(u')$, and this in turn is the same as $(i \times i')^*(q^*(u) \cup q'^*(u')) = (i \times i')^*(u \times u')$ by naturality of the cup product. This proves the claim.

To obtain the Euler class of $E \times E'$ we need to pullback $u \times u'$ along the zero section $X \times X' \to E \times E'$. But the zero section is precisely $s_0 p \times s'_0 p' \colon X \times X' \to E \times E'$, fitting into the commutative diagram



So using again naturality of the cup product and commutativity of the previous diagram we obtain

$$(s_0 p \times s'_0 p')^* q^*(u) \cup (s_0 p \times s'_0 p')^* q'^*(u') = p^* s_0^*(u) \cup p'^* s_0'^*(u') = e(E) \times e(E')$$

□

Corollary 16. Let $\pi \colon E \to X$ and $\pi' \colon E' \to X$ be two vector bundles on X of ranks k and k' respectively. Then $e(E \oplus E') = e(E) \cup e(E') \in \mathrm{H}^{2k+2k'}(X)$.

Proof: The sum $E \oplus E' \to X$ is defined as the pullback of the product $E \times E' \to X \times X$ along the diagonal $\Delta \colon X \to X \times X$. By lemma 12, lemma 15 and definition of the cross product we have

$$e(E \oplus E') = \Delta^*(e(E \times E')) = \Delta^*(e(E) \times e(E')) = e(E) \cup e(E')$$

□

1.2 Geometric interpretation of the Euler class

Let X be an n-dimensional smooth orientable closed connected manifold and let $\pi\colon E \to X$ be a smooth rank k real vector bundle and let $i = s_0\colon X \to E$ be the zero section. Up to homotopy we can replace E by the associated disk bundle and E_0 by the corresponding sphere bundle which is the boundary of the fibre of the disk bundle over each point. So we can apply the Poincaré-Lefschetz duality theorem (see [Bre97, Corollary 9.3.])

$$(-) \cap [E]\colon \mathrm{H}^l(E, E_0) \xrightarrow{\cong} \mathrm{H}_{n+k-l}(E)$$

Lemma 17. The Thom class is the unique class $u \in \mathrm{H}^k(E, E_0)$ such that

$$u \cap [E] = i_*[X]$$

Proof: See [Bre97, Lemma 11.5.] for a complete proof.

The idea is first to show that the restriction to any fibre cannot be zero. If this was the case, one can use the bootstrap lemma (see [Bre97, Lemma 7.9.]) to show that $u = 0$ globally, which is not the case.

So the restriction is not zero. Suppose it was not ± 1 and let p be a prime factor of this restriction. Then we can repeat the same argument with coefficients in $\mathbb{Z}/p\mathbb{Z}$ to obtain that $u = 0 \in \mathrm{H}^k(E, E_0; \mathbb{Z}/p\mathbb{Z})$, which is again a contradiction. □

So we have a nice geometric interpretation of the Thom class, namely, the cohomology class in $\mathrm{H}^*(E, E_0)$ whose Poincaré dual is the manifold X embedded as the zero section.

The Euler class was then defined as the pullback $i^*(u) \in \mathrm{H}^k(X)$. Recall that the intersection product of two homology classes $[a] \bullet [b]$ on a manifold is defined by cupping their Poincaré duals.

Proposition 18. The Poincaré dual in X of the Euler class $e = i^*(u) \in \mathrm{H}^k(X)$ corresopnds to the self intersection of X in E under the isomorphism i_*.

Proof: Using the properties in [Bre97, Thm. VI.5.2.] and their relative versions we have

$$i_*[X] \bullet i_*[X] = (u \cup u) \cap [E] = u \cap (u \cap [E]) = u \cap i_*[X] = i_*(e \cap [X])$$

□

This gives a nice geometric interpretation of the Euler class. The homology cycle that we are self intersecting is the image of the zero section. But any other section $s\colon X \to E$ yields a homologous cycle, because both sections induce isomorphisms in homology with the same inverse, namely π_*. The intersection product of two cycles corresponds geometrically to intersecting two representatives which are in general position. So in our case, to compute the Euler class as in proposition 18 we can intersect two sections that meet transversally at each point. We can take one of them to be the zero section and the other one to be any section transversal to the zero section, which we will simply call a transversal section. Computing such an intersection corresponds therefore to computing the zero locus of our section. Hence we get the following:

Corollary 19. The Euler class $e \in \mathrm{H}^k(X)$ is the Poincaré dual of the zero locus of a transversal section.

Proof: A more rigurous proof of this fact can be found in [BT82, Prop. 12.8.]. □

Remark 20. As we will see later, Chern classes are uniquely determined by their formal properties and by the Euler classes of line bundles. Compare corollary 19 with [Har77, Prop. II.7.7.] and [Har77, A.3.C1]. This provides a link between the topological category and the algebraic category.

1.3 The cohomology ring of projective bundles

Let $\pi\colon E \to X$ be a rank k vector bundle over $\mathbb{C}$. Define its *projectivization* $\mathbb{P}(E)$ to be the quotient $E_0/\mathbb{C}^\times$. This has a natural structure of fibre bundle with typical fibre $\mathbb{P}^{k-1}$ given by

$$\begin{aligned} p\colon \mathbb{P}(E) &\longrightarrow X \\ [v] &\longmapsto \pi(v) \end{aligned}$$

To see this, let (U, φ_U) be a chart for $\pi\colon E \to X$. Since $\mathbb{C}^\times$ acts on E_0 fibrewise, the action induces an action on $\pi^{-1}(U)$. Since φ_U is $\mathbb{C}$-linear on each fibre, it is a $\mathbb{C}^\times$-equivariant homeomorphism (with $\mathbb{C}^\times$-equivariant inverse). Quotienting out by the scalar action we get a homeomorphism

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\quad\approx\quad} & U \times \mathbb{P}^{k-1} \\ & {\scriptstyle p}\searrow \quad \circlearrowleft \quad \swarrow{\scriptstyle pr_U} & \\ & U & \end{array}$$

This construction is functorial on injective vector bundle morphisms. Indeed, if $f\colon E_1 \to E_2$ we obtain a $\mathbb{P}^{k-1}$ fibre bundle morphism $\mathbb{P}(f)\colon \mathbb{P}(E_1) \to \mathbb{P}(E_2)$, because vector bundle morphisms are linear on the fibres, and $\mathbb{P}(f)$ is well defined by injectivity.

The space $\mathbb{P}(E)$ in turn comes equipped with a tautological line bundle, namely

$$L_E \to \mathbb{P}(E)$$

where $L_E = \{(l, e) \in \mathbb{P}(E) \times E \mid e \in l\}$ and the projection is given by $(l, e) \mapsto l$. Let $\alpha_E \in \mathrm{H}^2(\mathbb{P}(E))$ be the Euler class associated to this line bundle. We will use the Leray-Hirsch theorem to express the cohomology ring of $\mathbb{P}(E)$ in terms of the cohomology ring of X and the Euler class α_E.

Example 21. We may regard $\mathbb{P}^n$ as the projectivization of the trivial vector bundle $\mathbb{C}^{n+1} \to \{*\}$ over a point. Then L is the usual tautological line bundle on $\mathbb{P}^n$ and the Euler class $\alpha \in \mathrm{H}^2(\mathbb{P}^n)$ generates the cohomology ring:

$$\mathrm{H}^*(\mathbb{P}^n) \cong \mathbb{Z}[\alpha]/(\alpha^{n+1})$$

Indeed, in this case $L_0 \simeq S^{2n+1}$, so we have $\mathrm{H}^l(S^{2n+1}) = 0$ for all $l \in \{1, \dots, 2n\}$, so by the Gysin sequence multiplication with α yields isomorphisms

$$\mathbb{Z} \cong \mathrm{H}^0(\mathbb{P}^n) \cong \mathrm{H}^2(\mathbb{P}^n) \cong \cdots \cong \mathrm{H}^{2n}(\mathbb{P}^n)$$

For odd l we know by cellular cohomology that $\mathrm{H}^l(\mathbb{P}^n) = 0$.

Consider now

$$\mathbb{P}^\infty = \varinjlim_n \mathbb{P}^n$$

with the direct limit taken over the inclusions $\mathbb{P}^n \hookrightarrow \mathbb{P}^{n+1}$ given by the usual CW structure, or in coordinates given by $[(v_1, \dots, v_n)] \mapsto [(v_1, \dots, v_n, 0)]$. Since the cohomology ring functor commutes with direct limits of topological spaces, we get

$$\mathrm{H}^*(\mathbb{P}^\infty) = \varprojlim_n \mathbb{Z}[\alpha]/(\alpha^{n+1}) = \mathbb{Z}[\alpha]$$

where the restrictions to each skeleton $i_n^*\colon \mathrm{H}^*(\mathbb{P}^\infty) \twoheadrightarrow \mathrm{H}^*(\mathbb{P}^n)$ are the maps induced by the inverse limit. Let $q\colon L_\infty \to \mathbb{P}^\infty$ be the tautological line bundle on $\mathbb{P}^\infty$. Since $i_n^* L_\infty \cong q^{-1}(\mathbb{P}^n) \cong L_n$ is the tautological line bundle on $\mathbb{P}^n$, we have $i_n^*(e(E)) = e(i_n^* E) = \alpha$ by example 21. But $\alpha \in \mathrm{H}^2(\mathbb{P}^\infty)$ is by definition the (only) element in the inverse limit such that $i_n^* \alpha = \alpha$ for all $n \in \mathbb{N}$. Hence $e(E) = \alpha$.

Remark 22. The generator α found in example 21 is not the usual generator, which is the Poincaré dual of the linear subspace $[\mathbb{P}^{n-1}] \in \mathrm{H}_{2n-2}(\mathbb{P}^n)$ given by the usual CW complex. We can see this in two different ways. Algebraically, the tautological bundle corresopnds to $\mathcal{O}(-1)$, and hence its first Chern class is given by $-H$. Topologically, we will see in example 30 that $e(T\mathbb{P}^n) = (-1)^n(n+1)\alpha^n$. We can compute the Euler characteristic of $\mathbb{P}^n$ as the evaluation of the Euler class $e(T\mathbb{P}^n) \in \mathrm{H}^{2n}(\mathbb{P}^n)$ at the ground class $[\mathbb{P}^n] \in \mathrm{H}_0(\mathbb{P}^n)$, which is then

$$e(T\mathbb{P}^n)([\mathbb{P}^n]) = (-1)^n(n+1)\alpha([\mathbb{P}^n]) = e(\mathbb{P}^n) = n+1$$

We need therefore $\alpha([\mathbb{P}^n]) = (-1)^n$. So α is $(-1)^n$ times the usual generator of the cohomology ring of projective space.

Now for the general case:

Proposition 23 (Projective bundle formula)**.** In the situation above, there is an $\mathrm{H}^*(X)$-module isomorphism

$$(\mathrm{H}^*(X)[\alpha])/(\alpha^k) \longrightarrow \mathrm{H}^*(\mathbb{P}(E))$$

given by $\alpha^l \mapsto \alpha_E^l$, i.e. $\beta\alpha^l \mapsto p^*(\beta) \cup \alpha_E^l$.

Proof: Consider $p\colon \mathbb{P}(E) \to X$ as above, with typical fibre $\mathbb{P}^{k-1}$. We have a pullback square

$$\begin{array}{ccc} q^{-1}(\mathbb{P}^{k-1}) = L & \hookrightarrow & L_E \\ \downarrow & & \downarrow q \\ \mathbb{P}^{k-1} & \overset{i}{\hookrightarrow} & \mathbb{P}(E) \end{array}$$

So by lemma 12 we have that $i^*(\alpha_E) = \alpha$ is the Euler class of example 21. But we have seen in example 21 that

$$\mathrm{H}^*(\mathbb{P}^{k-1}) \cong \mathbb{Z}[\alpha]/(\alpha^k)$$

Hence each power α^l is a free generator of the corresponding cohomology of the fibre $\mathrm{H}^{2l}(\mathbb{P}^{k-1})$. And in the odd degrees we have zero cohomology groups. We are thus in the hypothesis of the (absolute) Leray-Hirsch theorem for the fibration

$$\mathbb{P}^{k-1} \xrightarrow{i} \mathbb{P}(E) \xrightarrow{p} X$$

And we obtain therefore

$$\mathrm{H}^*(X) \otimes \mathrm{H}^*(\mathbb{P}^{k-1}) \xrightarrow{\cong} \mathrm{H}^*(\mathbb{P}(E))$$

via $\beta \otimes i^*(\alpha_E^l) = \beta \otimes \alpha^l \mapsto p^*(\beta) \cup \alpha_E^l$. By example 21 again, we have that the left hand side is

$$\mathrm{H}^*(X) \otimes_{\mathbb{Z}} \mathrm{H}^*(\mathbb{P}^{k-1}) \cong \mathrm{H}^*(X) \otimes_{\mathbb{Z}} \mathbb{Z}[\alpha]/(\alpha^k) \cong (\mathrm{H}^*(X)[\alpha])/(\alpha^k)$$

□

2 Chern classes

2.1 Definition and first properties

Let $\pi\colon E \to X$ be a rank k complex vector bundle and let $\alpha_E \in \mathrm{H}^2(\mathbb{P}(E))$ be its Euler class. From proposition 23 we know that $\mathrm{H}^*(\mathbb{P}(E))$ is a free $\mathrm{H}^*(X)$-module with basis $1, \alpha_E, \ldots, \alpha_E^{k-1}$. The scalar multiplication on $\mathrm{H}^*(\mathbb{P}(E))$ is given by $\beta \cdot \gamma := p^*(\beta) \cup \gamma$. So writing $\alpha_E^k \in \mathrm{H}^{2k}(\mathbb{P}(E))$ as a linear combination of these basis elements (modulo signs) we can find unique classes $c_i = c_i(E) \in \mathrm{H}^{2i}(X)$ for $i \in \mathbb{N}$ such that $c_0 = 1$, $c_i = 0$ for all $i > k$ and

$$\sum_{i=0}^{k} (-1)^i p^*(c_i) \cup \alpha_E^{k-i} = 0 \in \mathrm{H}^{2k}(\mathbb{P}(E))$$

Definition 24. The classes $c_i(E) \in \mathrm{H}^{2i}(X)$ are called the *Chern classes* of the vector bundle $\pi\colon E \to X$. The *total Chern class* is defined as their sum

$$c(E) = 1 + c_1(E) + c_2(E) + \cdots + c_k(E) \in \mathrm{H}^*(X)$$

The *Stiefel-Whitney classes* of a real vector bundle $\pi\colon E \to X$, denoted $w_i(E) \in \mathrm{H}^i(X, \mathbb{F}_2)$, and the *total Stiefel-Whitney class* of the vector bundle, denoted $w(E) \in \mathrm{H}^*(X, \mathbb{F}_2)$, are defined in the exact same way.

Theorem 25. The Chern classes $c_i(E)$ only depend on E up to isomorphism. Moreover, they satisfy the following three properties:

1. *Functoriality:* if $f\colon Y \to X$ is a continuous map, then

$$c(f^*E) = f^*(c(E))$$

2. *Normalization:* for any line bundle $\pi\colon L \to X$, we have

$$c(L) = 1 + e(L)$$

3. *Additivity:* for every pair of complex vector bundles $E_1, E_2 \in \mathrm{Vec}(X)$ we have

$$c(E_1 \oplus E_2) = c(E_1) \cup c(E_2)$$

In other words, $c_i(E_1 \oplus E_2) = \sum_{l+m=i} c_l(E_1) \cup c_m(E_2) \in \mathrm{H}^{2i}(X)$.

Proof: Let $f\colon E_1 \xrightarrow{\cong} E_2$ be an isomorphism of vector spaces over X. By functoriality of $\mathbb{P}(E)$ we get an isomorphism $\mathbb{P}(f)\colon \mathbb{P}(E_1) \xrightarrow{\cong} \mathbb{P}(E_2)$ of $\mathbb{P}^{k-1}$ fibre bundles over X with $\mathbb{P}(f)^*(L_2) \cong L_1$, because f induces a vector bundle morphism $L_1 \to \mathbb{P}(f)^*(L_2)$ which is an isomorphism on every fibre. Hence we have $\mathbb{P}(f)^*(\alpha_2) = \alpha_1$, and so $\sum_{i=0}^k (-1)^i p_2^*(c_i(E_2)) \cup \alpha_2^{k-i} = 0$ implies that

$$\mathbb{P}(f)^* \left(\sum_{i=0}^{k} (-1)^i p_2^*(c_i(E_2)) \cup \alpha_2^{k-i} \right) = \sum_{i=0}^{k} (-1)^i p_1^*(c_i(E_2)) \cup \alpha_1^{k-i} = 0$$

This shows that the Chern classes $c_i(E_2)$ are also a solution to the defining equation of the Chern classes $c_i(E_1)$. By uniqueness of the solutions, we must have $c_i(E_1) = c_i(E_2)$ for all i, so Chern classes are invariant under isomorphism of vector bundles.

For the functoriality, consider the pullback square

$$\begin{array}{ccc} E_Y & \xrightarrow{g} & E \\ {\scriptstyle \pi_Y}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{f} & X \end{array}$$

Since g is an isomorphism on the fibres we get a commutative square

$$\begin{array}{ccc} \mathbb{P}(E_Y) & \xrightarrow{\mathbb{P}(g)} & \mathbb{P}(E) \\ {\scriptstyle p_Y}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{f} & X \end{array}$$

and as before g induces an isomorphism $L_Y \cong \mathbb{P}(g)^*(L_E)$. Thus $\alpha_Y = \mathbb{P}(g)^*(\alpha_E)$, so arguing as before we get from the equality

$$\mathbb{P}(g)^* \left(\sum_{i=0}^{k} (-1)^i p^*(c_i(E)) \cup \alpha_E^{k-i} \right) = \sum_{i=0}^{k} (-1)^i p_Y^* f^*(c_i(E)) \cup \alpha_Y^{k-i}$$

that $c_i(E_Y) = f^*(c_i(E))$.

To see that the normalization property also holds, note that $\mathbb{P}(L) = X$ and $L_L \to \mathbb{P}(L)$ is again $L \to X$. So we have $\alpha_L = e(L)$. The defining equation of $c_1(L)$ is then

$$e(L) - p^*(c_1(L)) = 0$$

But $p\colon \mathbb{P}(L) = X \to X$ is the identity, so the equation is just

$$e(L) = c_1(L)$$

It remains to show additivity. Let $\pi_1\colon E_1 \to X$ and $\pi_2\colon E_2 \to X$ be two vector bundles of ranks k_1 and k_2 on X. We need to show that

$$c_i(E_1 \oplus E_2) = \sum_{l+m=i} c_l(E_1) \cup c_m(E_2)$$

for all $i \in \mathbb{N}$. Denote $q\colon E_1 \oplus E_2 \twoheadrightarrow \mathbb{P}(E_1 \oplus E_2)$ the quotient map and $p\colon \mathbb{P}(E_1 \oplus E_2) \to X$ the projection. The total spaces E_1 and E_2 are closed subsets in $E_1 \oplus E_2$. Identify $\mathbb{P}(E_1)$ with the subspace $\{[(e_1, 0)]\} \subseteq \mathbb{P}(E_1 \oplus E_2)$ and similarly for $\mathbb{P}(E_2)$. Then $E_1 = q^{-1}(\mathbb{P}(E_1))$ and $E_2 = q^{-1}(\mathbb{P}(E_2))$, so by definition of the quotient topology we have that $\mathbb{P}(E_1)$ and $\mathbb{P}(E_2)$ are closed subspaces of $\mathbb{P}(E_1 \oplus E_2)$. Moreover, since $E_1 \cap E_2 = s_0(X) \subseteq E_1 \oplus E_2$, we have $\mathbb{P}(E_1) \cap \mathbb{P}(E_2) = \varnothing$ inside $\mathbb{P}(E_1 \oplus E_2)$. So these two closed subspaces are disjoint. If we define open sets $U_1 = \mathbb{P}(E_1 \oplus E_2) \backslash \mathbb{P}(E_1)$ and $U_2 = \mathbb{P}(E_1 \oplus E_2) \backslash \mathbb{P}(E_2)$, we get an open cover

$$\mathbb{P}(E_1 \oplus E_2) = U_1 \cup U_2$$

The subspace $\mathbb{P}(E_1) = \{[(e_1, e_2)] \in \mathbb{P}(E_1 \oplus E_2) \mid e_2 = 0\}$ is a deformation retract of $U_2 = \{[(e_1, e_2)] \in \mathbb{P}(E_1 \oplus E_2) \mid e_1 \neq 0\}$ via

$$([(e_1, e_2)], t) \longmapsto [(e_1, (1-t)e_2)]$$

Denote by $j_1\colon E_1 \to E_1 \oplus E_2$ the inclusion and let $\mathbb{P}(j_1)\colon \mathbb{P}(E_1) \to \mathbb{P}(E_1 \oplus E_2)$ be the induced inclusion on projectivizations. Under the previous identification $\mathbb{P}(E_1) = \{[(e_1, 0)]\} \subseteq \mathbb{P}(E_1 \oplus E_2)$ we can also identify $L_{E_1} = \{([(e_1, 0)], (\lambda e_1, 0))\} \subseteq L_{E_1 \oplus E_2}$. Under this identification, L_{E_1} is the restriction of $L_{E_1 \oplus E_2}$ to the subspace $\mathbb{P}(E_1)$, so by lemma 12 we get that $\alpha_{E_1} = \mathbb{P}(j_1)^*(\alpha_{E_1 \oplus E_2})$. This implies that the class

$$\gamma_1 = \sum_{l=0}^{k_1} (-1)^l p^*(c_l(E_1)) \cup \alpha_{E_1 \oplus E_2}^{k_1 - l} \in \mathrm{H}^{2k_1}(\mathbb{P}(E_1 \oplus E_2))$$

restricts to 0 over the subspace $\mathbb{P}(E_1)$. But because $\mathbb{P}(E_1)$ is a deformation retract of U_2, the class γ_1 must also restrict to 0 over U_2. Similarly, the class

$$\gamma_2 = \sum_{m=0}^{k_2} (-1)^m p^*(c_m(E_2)) \cup \alpha_{E_1 \oplus E_2}^{k_2 - m} \in \mathrm{H}^{2k_2}(\mathbb{P}(E_1 \oplus E_2))$$

restricts to 0 over U_1. Since U_1 and U_2 form an open cover of $\mathbb{P}(E_1 \oplus E_2)$ the cup product $\gamma_1 \cup \gamma_2 \in \mathrm{H}^{2(k_1+k_2)}(\mathbb{P}(E_1 \oplus E_2)$ is zero. Since the cup product is $\mathbb{Z}$-bilinear we get

$$0 = \gamma_1 \cup \gamma_2 = \sum_{i=0}^{k_1+k_2} (-1)^i p^* \left(\sum_{l+m=i} c_l(E_1) \cup c_m(E_2) \right) \cup \alpha_{E_1 \oplus E_2}^{k_1+k_2-i}$$

By uniqueness of the solutions as before we deduce finally that

$$c_i(E_1 \oplus E_2) = \sum_{l+m=i} c_l(E_1) \cup c_m(E_2)$$

□

Example 26. The trivial vector bundle $X \times \mathbb{C}^k \to X$, denoted $\underline{\mathbb{C}}^k$, is the pullback of the trivial vector bundle $\{*\} \times \mathbb{C}^k \to \{*\}$ along the unique map $X \to \{*\}$. So its Chern classes are the pullback of the Chern classes of the trivial vector bundle over a point. Since a point has no cohomology on degree $i > 0$, we get

$$c(\underline{\mathbb{C}}^k) = 1$$

Example 27. Let $E \to X$ and $E' \to X'$ be two complex vector bundles. For all $i \in \mathbb{N}$ we have

$$c_i(E \times E') = \sum_{l+m=i} c_l(E) \times c_m(E')$$

To see this, write $E \times E'$ as $(p^*E) \oplus (p'^*E')$, where $p\colon X \times X' \to X$ and $p'\colon X \times X' \to X'$ are the respective projections. Then by additivity we have

$$c_i(E \times E') = \sum_{l+m=i} c_l(p^*E) c_m(p'^*E')$$

And by naturality this is equal to

$$\sum_{l+m=i} p^*(c_l(E)) p'^*(c_m(E')) = \sum_{l+m=i} c_l(E) \times c_m(E')$$

2.2 Stability of Chern classes

An immediate consequence of example 26 together with the additivity from theorem 25 is that taking the direct sum of a vector bundle with a trivial vector bundle does not affect its Chern classes. This easy observation turns out to be very useful in many situations.

Example 28. The sphere S^n embedded in $\mathbb{R}^{n+1}$ has a rank 1 normal vector bunlde which is trivial, because taking the outward pointing unitary vector at each point gives a nowhere vanishing section. We have

$$TS^n \oplus \underline{\mathbb{R}} \cong \underline{\mathbb{R}}^{n+1}$$

Hence $w(TS^n) = w(TS^n \oplus \underline{\mathbb{R}}) = w(\underline{\mathbb{R}}^{n+1}) = 1$ and in particular

$$w_i(TS^n) = 0$$

for all $i > 0$.

Example 29. (See [MS74, Lemma 4.4.]) The real projective n-space $\mathbb{P}^n_{\mathbb{R}}$ is the quotient of S^n by the antipodal map. The differential of this map sends the tangent vector $v \in T_xS^n$ and the tangent vector $-v \in T_{-x}S^n$ to the same tangent vector in $T\mathbb{P}^n_{\mathbb{R}}$, so we can identify the tangent bundle $T\mathbb{P}^n_{\mathbb{R}}$ with the set of pairs $\{\{(x,v),(-x,-v)\} \mid x \in S^n, v \in T_x S^n\}$. So an elmenet in $T_{[x]}\mathbb{P}^n_{\mathbb{R}}$ yields a linear map

$$\begin{aligned} L_{[x]} &\longrightarrow L^{\perp}_{[x]} \\ x &\longmapsto v \end{aligned}$$

where $L_{[x]}$ is by definition the fibre of the tautological line bundle $L \to \mathbb{P}^n_{\mathbb{R}}$ over the point $[x]$. Moreover, any such linear morphism determines a unique elmenet in $T_{[x]}\mathbb{P}^n_{\mathbb{R}}$, because if we started with the point $-x \in S^n$ we would only get the same morphism by picking $-v$ instead of v. This construction gives a canonical isomorphism

$$T\mathbb{P}^n_{\mathbb{R}} \cong \mathrm{Hom}(L, E)$$

where E has the property that $E \oplus L \cong \underline{\mathbb{R}}^{n+1}$. The line bundle $\mathrm{Hom}(L, L)$ is trivial, because the identity gives a non vanishing global section. Hence

$$T\mathbb{P}^n_{\mathbb{R}} \oplus \underline{\mathbb{R}} \cong \mathrm{Hom}(L, E) \oplus \mathrm{Hom}(L, L)$$

The canonical isomorphisms from linear algebra yield globally defined morphisms of vector bundles (they glue because they are canonical) which are also isomorphisms (because this can be checked on the fibres). Hence the last bundle is isomorphic to

$$\mathrm{Hom}(L, E \oplus L) \cong \mathrm{Hom}(L, \underline{\mathbb{R}}^{n+1}) \cong L^{\oplus n+1}$$

By normalization in the analogous of theorem 25 for Stiefel-Whitney classes, the Euler class $e(L)$ is the generator $\alpha \in \mathrm{H}^1(\mathbb{P}_{\mathbb{R}}P^n, \mathbb{F}_2)$ such that $\mathrm{H}^*(\mathbb{P}^n_{\mathbb{R}}, \mathbb{F}_2) \cong \mathbb{F}_2[\alpha]/(\alpha^{n+1})$. By additivity $w(L^{\oplus n+1}) = (1+\alpha)^{n+1}$. Since $w(T\mathbb{P}^n_{\mathbb{R}} \oplus \underline{\mathbb{R}}) = w(T\mathbb{P}^n_{\mathbb{R}})$ we deduce finally

$$w(T\mathbb{P}^n_{\mathbb{R}}) = (1+\alpha)^{n+1} \in \mathrm{H}^*(\mathbb{P}^n_{\mathbb{R}}, \mathbb{F}_2)$$

Example 30. (See [MS74, Theorem 14.10.]) Let now $\mathbb{P}^n$ be the complex projective n-space. Let $L\colon \mathbb{P}^n$ be the tautological line bundle and let $E \to \mathbb{P}^n$ be its orthogonal complement inside the trivial bundle $\underline{\mathbb{C}}^{n+1} \to \mathbb{P}^n$. Define a vector bundle morphism $\varphi\colon \mathrm{Hom}(L, E) \to T\mathbb{P}^n$ as follows. For $[x] \in \mathbb{P}^n$ with we are given a linear map $f\colon L_{[x]} \to L^{\perp}_{[x]}$ on the fibres. Together with the identity on $L_{[x]}$, this map yields a linear map into the direct sum

$$L_{[x]} \xrightarrow{\mathrm{id} \sqcup f} L_{[x]} \oplus L^{\perp}_{[x]} \cong \underline{\mathbb{C}}^{n+1}$$

The image of this linear map is again a line L_f, which is closer to $L_{[x]}$ the closer $f(x)$ is to zero. If $t \in \mathbb{R}$, then $tf\colon L_{[x]} \to L^{\perp}_{[x]}$ is a new linear map, and we get a path $t \mapsto L_{tf}$ passing through $L_{[x]}$ when $t = 0$. Therefore we get an element $\varphi(f) \in T_{[x]}\mathbb{P}^n$, and the resulting vector bundle morphisms is an isomorphism on each fibre. Now add $\mathrm{Hom}(L, L) \cong \underline{\mathbb{C}}$ as before to obtain

$$T\mathbb{P}^n \oplus \underline{\mathbb{C}} \cong \mathrm{Hom}(L, E \oplus L) \cong \mathrm{Hom}(L, \underline{\mathbb{C}})^{\oplus n+1}$$

In proposition 44 we will see that $c_1(\mathrm{Hom}(L, \underline{\mathbb{C}})) = -c_1(L)$, so we get

$$c(T\mathbb{P}^n) = (1-\alpha)^{n+1} \in \mathrm{H}^*(\mathbb{P}^n)$$

where α is the Euler class from example 21.

Remark 31. In algebraic geometry, example 30 corresponds to the *Euler sequence*

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(1)^{\oplus n+1} \to \mathcal{T}_X \to 0$$

of sheaves on $\mathbb{P}^n$ (dual of the short exact sequence in [Har77, Theorem II.8.13.]).

2.3 The splitting principle

This principle will help us to reduce questions about arbitrary vector bundles to the case of sum of line bundles, which is much easier to handle thanks to the normalization and additivity properties in theorem 25.

Proposition 32. Let $\pi\colon E \to X$ be a complex vector bundle over a paracompact base X. Then there is a space $F(E)$ and a map $f\colon F(E) \to X$ such that the map $f^*\colon \mathrm{H}^*(X) \to \mathrm{H}^*(F(E))$ is injective and the vector bundle f^*E is a direct sum of complex line bundles.

Proof: By induction on the rank of E it is enough to find a space $F(E)$ and a map $f\colon F(E) \to X$ such that f^* is injective and $f^*E \cong E' \oplus L$ for some complex line bundle L, because composition of injective maps is injective, pullbacks of line bundles are line bundles and pullbacks commute with direct sums of vector bundles. The last statement holds because the vector bundle morphism $f^*(E_1 \oplus E_2) \to f^*(E_1) \oplus f^*(E_2)$ given by $(y, (e_1, e_2)) \mapsto ((y, e_1), (y, e_2))$ is an isomorphism on each fibre.

Consider $F(E) = \mathbb{P}(E)$ and $f = p\colon \mathbb{P}(E) \to X$. Consider the injective bundle morphism

$$\begin{aligned} \varphi\colon L_E &\longrightarrow p^*(E) \\ ([e], \lambda e) &\longmapsto ([e], \lambda e) \end{aligned}$$

Since X is paracompact, we may choose an Hermitian inner product on E (see [Hat13, Prop. 1.2.]). But this induces one on p^*E, so we can take $E' = \varphi(L_E)^\perp \subseteq p^*E$. Then we have

$$p^*E \cong E' \oplus L$$

By proposition 23 the map $p^*\colon \mathrm{H}^*(X) \to \mathrm{H}^*(\mathbb{P}(E))$ is injective, because it is the inclusion of the subring of polynomials of degree 0 in α. □

Corollary 33. Let $\pi\colon E \to X$ be a rank k complex vector bundle over a paracompact base X. Then $c_k(E) = e(E) \in \mathrm{H}^{2k}(X)$.

Proof: Let $f\colon F(E) \to X$ be the map from proposition 32, so that $f^*E \cong L_1 \oplus \cdots \oplus L_k$ is a direct sum of complex vector bundles. Then by theorem 25 we have

$$f^*(c(E)) = c(L_1 \oplus \cdots \oplus L_k) = (1 + e(L_1)) \cup \cdots \cup (1 + e(L_k))$$

In particular we have

$$f^*(c_k(E)) = e(L_1) \cup \cdots \cup e(L_k)$$

But on the other hand by lemma 12 and lemma 15 we have

$$f^*(e(E)) = e(f^*E) = e(L_1 \oplus \cdots \oplus L_k) = e(L_1) \cup \cdots \cup e(L_k)$$

The corollary follows then from injectivity of f^*. □

Corollary 34. Let X be a paracompact space. The Chern classes on X are uniquely determined by the properties in theorem 25. More precisely, every sequence of functions $\{c'_i\}_{i\in\mathbb{N}}$ assigning to each complex vector bundle $E \to X$ a class $c'_i(E) \in \mathrm{H}^{2i}(X)$ which depends on E only up to isomorphism and which verifies the properties 1 to 3 coincides with the Chern classes $\{c_i\}_{i\in\mathbb{N}}$.

Proof: The normalization property determines the Chern classes of all complex line bundles on X. The additivity property determines the Chern classes of all sums of line bundles on X. Let $E \to X$ be any rank k vector bundle and let $f\colon F(E) \to X$ be the map given by proposition 32. Then the Chern classes of f^*E are already determined, because it is isomorphic to a sum of line bundles and they are isomorphism invariant. But since f^* is injective, the Chern classes of E are also determined as the only classes whose pullback under f are the Chern classes of f^*E. □

Remark 35. The previous result also holds for arbitrary base spaces X, because we can always find CW approximations (see [Hat02, Prop. 4.13.]) and because every CW complex is paracompact (see [Hat13, Prop. 1.20.]). But the paracompactness assumption is very mild in any case: compact Hausdorf spaces, CW complexes and metric spaces are all paracompact spaces.

3 Computation of Chern classes

In this section we will deduce some formulas to compute Chern classes. From now on we will omit the $\cup$ symbol when multiplying cohomology classes.

As we briefly mentioned in example 29, canonical morphisms of vector spaces such as those induced by a universal property yield well defined vector bundle morphisms. Once we have globally defined vector bundle morphisms we can check if they are an isomorphism on the fibres. So canonical isomorphisms of vector spaces give canonical isomorphisms of vector bundles. For example, we have the usual tensor-hom adjunction

$$E \otimes (-) \dashv \mathrm{Hom}(E, -)$$

3.1 The Picard group and the first Chern class

Tensor product of line bundles corresponds to multiplication of transition functions and product of sections corresponds to a section of the tensor product. Isomorphism classes of complex line bundles over X form an abelian group with respect to the tensor product. The neutral element is the trivial line bundle $\underline{\mathbb{C}}$ and the inverse of a line bundle L is given by its dual $L^\vee = \mathrm{Hom}(L, \underline{\mathbb{C}})$, because the canonial map

$$L \otimes L^\vee \to \underline{\mathbb{C}}$$

is the corresonding canonical isomorphism of complex vector spaces over each fibre. We call this group the *Picard group* of X, denoted $\mathrm{Pic}(X)$.

Lemma 36. Let $L_1 \to X$ and $L_2 \to X$ be two line bundles. Then we have

$$e(L_1 \otimes L_2) = e(L_1) + e(L_2)$$

Proof: Let us first show the universal case. Consider

$$\begin{array}{ccc} & \mathbb{P}^\infty \times \mathbb{P}^\infty & \\ {\scriptstyle p_1} \swarrow & & \searrow {\scriptstyle p_2} \\ \mathbb{P}^\infty & & \mathbb{P}^\infty \end{array}$$

Let $L \to \mathbb{P}^\infty$ be the tautological line bundle and let $L_1 = p_1^* L$ and $L_2 = p_2^* L$. Consider the tensor product $L_1 \otimes L_2 \to \mathbb{P}^\infty \times \mathbb{P}^\infty$.

We know that $e(L) = \alpha$ is a generator of $\mathrm{H}^2(\mathbb{P}^\infty)$ and that $\mathrm{H}^*(\mathbb{P}^\infty) \cong \mathbb{Z}[\alpha]$ (see example 21). Let $\alpha_1 = p_1^*\alpha = e(L_1)$ and $\alpha_2 = p_2^*\alpha = e(L_2)$. Then $\mathrm{H}^*(\mathbb{P}^\infty \times \mathbb{P}^\infty) = \mathbb{Z}[\alpha_1, \alpha_2]$ by the Künneth theorem. The inclusion $\mathbb{P}^\infty \vee \mathbb{P}^\infty \subset \mathbb{P}^\infty \times \mathbb{P}^\infty$ induces an isomorphism on the second cohomology group (which can be seen geometrically via cellular cohomology). So we can compute $e(L_1 \otimes L_2)$ by restricting this line bundle over the wedge sum. Over each copy of $\mathbb{P}^\infty$, one of the two factors becomes trivial, so the tensor product is isomorphic to the remaining factor, which is isomorphic to the tautological line bundle over $\mathbb{P}^\infty$. This means that $e(L_1 \otimes L_2)$ restricts to α_1 over the corresponding copy of $\mathbb{P}^\infty$ and to α_2 over the other one. Hence $e(L_1 \otimes L_2)$ and $\alpha_1 + \alpha_2$ have the same restrictions, but since the inclusion induces an isomorphism they must be the same. Therefore

$$e(L_1 \otimes L_2) = e(L_1) + e(L_2)$$

The general case follows now by naturality of the Euler class. The two line bundles on X are the pullback of the tautological line bundle $L \to \mathbb{P}^\infty$ under some maps $X \to \mathbb{P}^\infty$. Take the product of these maps to obtain a map $f \colon X \to \mathbb{P}^\infty \times \mathbb{P}^\infty$ so that the two line bundles on X are the pullbacks of the line bundles L_1 and L_2 respectively. Then use that pullbacks commute with tensor products to conclude:

$$e(f^*L_1 \otimes f^*L_2) = e(f^*(L_1 \otimes L_2)) = f^*(e(L_1) + e(L_2)) = e(f^*L_1) + e(f^*L_2)$$

□

Corollary 37. The first Chern class induces a group homomorphism

$$c_1 \colon \operatorname{Pic}(X) \to \mathrm{H}^2(X)$$

Proof: We have already seen in example 26 that the trivial line bundle has zero first Chern class. lemma 36 shows that the first Chern class of the tensor product of line bundles is the sum of their first Chern classes. Hence c_1 is indeed a group homomorphism. □

Remark 38. Let X be smooth projective variety over $\mathbb{C}$. Consider the analytic exponential sequence on X_h (see [Har77, Appendix B])

$$0 \to \mathbb{Z} \to \mathcal{O}_{X_h} \to \mathcal{O}^*_{X_h} \to 0$$

On the level of global sections, the map $\mathbb{C} \to \mathbb{C}^*$ is given by the exponential function, hence is surjective. So we get a long exact sequence

$$0 \to \mathrm{H}^1(X_h, \mathbb{Z}) \to \mathrm{H}^1(X_h, \mathcal{O}_{X_h}) \to \mathrm{H}^1(X_h, \mathcal{O}^*_{X_h}) \to \\ \to \mathrm{H}^2(X_h, \mathbb{Z}) \to \mathrm{H}^2(X_h, \mathcal{O}_{X_h}) \to \cdots$$

Using Čech cohomology one sees that $\mathrm{Pic}(X_h) \cong \mathrm{H}^1(X_h, \mathcal{O}^*_{X_h})$. Since X_h is a closed orientable manifold, singular cohomology with integer coefficients agrees with sheaf cohomology of the constant sheaf of integers (see [Spa66]). So the boundary map of the exponential sequence yields a group homomorphism

$$\mathrm{Pic}(X_h) \to \mathrm{H}^2(X_h, \mathbb{Z})$$

which coincides with the first Chern class homomorphism. If we apply the GAGA theorems we can recover algebraic information from this analytic information, because $\mathrm{H}^1(X, \mathcal{O}_X) \cong \mathrm{H}^1(X_h, \mathcal{O}_{X_h})$, $\mathrm{H}^2(X, \mathcal{O}_X) \cong \mathrm{H}^2(X_h, \mathcal{O}_{X_h})$ and $\mathrm{Pic}(X) \cong \mathrm{Pic}(X_h)$.

3.2 The Chern character

The formulas for the Chern classes of the tensor product of vector bundles of higher rank get a bit uglier. The *Chern character* will allow us to express the information given by the Chern classes in a more convenient way and get nicer formulas. To define the Chern character of a vector bundle we will work with coefficients in $\mathbb{Q}$ (see remark 6). Let $E \to X$ be a vector bundle. We are looking for a class in $\mathrm{H}^*(X, \mathbb{Q})$, so by the splitting principle it suffices to define Chern characters for direct sums of line bundles.

Definition 39. Let $L \to X$ be a line bundle and let $a = c_1(L) \in \mathrm{H}^2(X, \mathbb{Q})$ be its first Chern class. Define

$$\mathrm{ch}(L) = e^a = 1 + a + \frac{a^2}{2} + \frac{a^3}{6} + \cdots = \sum_{n \in \mathbb{N}} \frac{a^n}{n!} \in \mathrm{H}^*(X, \mathbb{Q})$$

Let now $E = L_1 \oplus \cdots \oplus L_k \to X$ be a direct sum of line bundles and $a_i = e(L_i) \in \mathrm{H}^2(X, \mathbb{Q})$ be their first Chern classes. Define

$$\mathrm{ch}(E) = \sum_{i=1}^{k} e^{a_i}$$

Let now $E \to X$ be an arbitrary vector bundle on X. Pick a map $f \colon F(E) \to X$ as in proposition 32. We would like to define $\mathrm{ch}(E)$ to be the unique class in $\mathrm{H}^*(X, \mathbb{Q})$ such that $f^*(\mathrm{ch}(E)) = \mathrm{ch}(F(E))$, but this is not well defined a priori. So we need to do some work. If $E = L_1 \oplus \cdots \oplus L_k$ is a sum of line bundles, we can recover its Chern class as

$$c(E) = \prod_{i=1}^{k} (1 + a_i)$$

By the Cardano-Vieta formulas, the Chern classes $c_l(E)$ are the elementary degree l symmetric polynomials in the variables $a_1, \dots, a_k$. On the other hand, we can expand the Chern character as follows

$$\operatorname{ch}(E) = k + (a_1 + \cdots + a_k) + \frac{a_1^2 + a_2^2 + \cdots + a_k^2}{2} + \cdots + \frac{a_1^n + \cdots + a_k^n}{n!} + \cdots$$

By the fundamental theorem on symmetric polynomials, every degree l symmetric polynomial can be expressed as a unique polynomial in the elementary symmetric polynomials with degrees less or equal to l. In particular, $a_1^l + \cdots a_k^l$ can be written as a unique polynomial $s_l(c_1(E), c_2(E), \dots, c_l(E))$ (which is called the l^{th} Newton polynomial). Hence

$$\operatorname{ch}(E) = k + s_1(c_1(E)) + \frac{s_2(c_1(E), c_2(E))}{2} + \cdots = k + \sum_{l>0} \frac{s_l(c_1(E), \dots, c_l(E))}{l!}$$

This shows that the definition does not depend on $f\colon F(E) \to X$ and provides an explicit formula for the Chern character in terms of the Chern classes. The first terms of the Chern character of a rank k vector bundle $E \to X$ are

$$\operatorname{ch}(E) = k + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_1c_3 + 3c_3) + \frac{1}{24}(c_1^4 - 4c_1^2c_2 + 4c_1c_3 + 2c_2^2 - 4c_4) + \cdots$$

Remark 40. We will usually work with spaces X that do not have cohomology in arbitrarily high degrees, e.g. manifolds. In those cases, the sum in the previous definition is actually a finite sum.

3.3 Formulas to compute Chern classes

We are ready now to start going through the usual constructions with vector bundles.

Proposition 41 (Pullback)**.** Let $f\colon Y \to X$ be a continuous map and $E \to X$ a vector bundle. Then

$$\operatorname{ch}(f^*E) = f^* \operatorname{ch}(E)$$

Proof: Follows from functoriality in theorem 25 and from the fact that the pullback is $\mathbb{Q}$-linear. □

Proposition 42 (Direct sum)**.** Let $E \to X$ and $E' \to X$ be two vector bundles. Then

$$\operatorname{ch}(E \oplus E') = \operatorname{ch}(E) + \operatorname{ch}(E')$$

Proof: It suffices to check this for direct sums of line bundles, because we can pullback along a first map f turning f^*E into a direct sum of line bundles and then along a second map g turning $g^*(f^*E')$ into a direct sum of line bundles (direct sums commute with pullbacks). But

$$(L_1 \oplus \cdots \oplus L_k) \oplus (L_1' \oplus \cdots \oplus L_{k'}') = L_1 \oplus \cdots \oplus L_k \oplus L_1' \oplus \cdots \oplus L_{k'}'$$

and so the Chern character of the direct sum is the sum of the Chern characters by definition. □

Proposition 43 (Tensor product)**.** Let $E \to X$ and $E' \to X$ be two vector bundles. Then

$$\operatorname{ch}(E \otimes E') = \operatorname{ch}(E) \operatorname{ch}(E')$$

Proof: Tensor products also commute with pullbacks, so it suffices again to show the result for direct sums of line bundles. By lemma 36 we know already the result for the tensor product of line bundles, because $e^{c_1(L \otimes L')} = e^{c_1(L)+c_1(L')} = e^{c_1(L)} e^{c_1(L')}$. For higher ranks we use proposition 42 to obtain

$$\operatorname{ch}((\oplus_i L_i) \otimes (\oplus_j L'_j)) = \operatorname{ch}(\oplus_{i,j} L_i \otimes L'_j) = \sum_{i,j} \operatorname{ch}(L_i \otimes L'_j)$$

And if we expand this expression we get

$$\sum_{i,j} \operatorname{ch}(L_i) \operatorname{ch}(L'_j) = (\sum_i \operatorname{ch}(L_i))(\sum_j \operatorname{ch}(L'_j)) = \operatorname{ch}(\oplus_i L_i) \operatorname{ch}(\oplus_j L'_j)$$

□

Proposition 44 (Dual)**.** Let $E \to X$ be a vector bundle and $E^\vee \to X$ be its dual. Then

$$c_i(E^\vee) = (-1)^i c_i(E)$$

Proof: As usual it suffices to show this for a direct sum of line bundles. If $L \to X$ is a single line bundle, $c_1(L) = -c_1(L^\vee)$ follows from $L^\vee$ being the inverse of L in $\operatorname{Pic}(X)$ and $c_1 \colon \operatorname{Pic}(X) \to \mathrm{H}^2(X)$ being a group homomorphism. For a sum of line $E = L_1 \oplus \cdots L_k$, note that $E^\vee = L_1^\vee \oplus \cdots \oplus L_k^\vee$, so that

$$c(E^\vee) = \prod_i (1 - c_1(L))$$

And therefore $c_i(E^\vee) = (-1)^i c_i(E)$. □

Combining these basic results and the splitting principle one can deduce formulas for more involved constructions.

Example 45 (Hom)**.** For the hom bundle one can combine proposition 43 and proposition 44 with the canonical isomorphism

$$E^\vee \otimes E' \cong \operatorname{Hom}(E, E')$$

Example 46 (Determinant)**.** The determinant of a rank k vector bundle E is defined as the line bundle $\bigwedge^k E \to X$. If $E = L_1 \oplus \cdots \oplus L_k$, we get $\bigwedge^k E = L_1 \otimes \cdots \otimes L_k$, hence

$$c_1(\det E) = \sum_i c_1(L_i) = c_1(E)$$

Example 47 (Symmetric squares)**.** Let $E = L_1 \oplus L_2$ be a rank 2 vector bundle. Then

$$S^2(E) = L_1^{\otimes 2} \oplus (L_1 \otimes L_2) \oplus L_2^{\otimes 2}$$

and thus if we define $\alpha_1 = c_1(L_1)$ and $\alpha_2 = c_1(L_2)$ we get

$$c(S^2(E)) = (1 + 2\alpha_1)(1 + \alpha_1 + \alpha_2)(1 + 2\alpha_2)$$

which can be rewritten as

$$1 + 2(\alpha_1 + \alpha_2) + (2\alpha_1^2 + 8\alpha_1\alpha_2 + 2\alpha_2^2) + 4\alpha_1\alpha_2(\alpha_1\alpha_2)$$

Since $c_1(E) = \alpha_1 + \alpha_2$ and $c_2(E) = \alpha_1\alpha_2$ we can rewrite this as

$$1 + 2c_1(E) + (2c_1(E)^2 + 4c_2(E)) + 4c_1(E)c_2(E)$$

By the splitting principle, this formula for $c(S^2(E))$ is valid for any rank 2 vector bundle.

And similarly we can compute other symmetric powers and exterior powers.

4 Applications

4.1 Nonimmersions of projective spaces

Following the ideas in example 28, we will see that the stability of Chern and Stiefel-Whitney classes allow us to determine in some cases the non existence of an immersion of a smooth manifold in some euclidean space.

Definition 48. Let $f\colon M \to N$ be a smooth map between smooth manifolds. We say that f is an *immersion* if the induced map on tangent spaces is injective at each point, i.e. if for all $x \in M$ we have

$$(Df)_x\colon T_xM \hookrightarrow T_{f(x)}N$$

We denote an immersion by $f\colon M \looparrowright N$.

Being an immersion is not directly related to injectivity, as the following two examples show.

Example 49. The map $f\colon \mathbb{R} \to \mathbb{R}$ sending $t \mapsto t^3$ is a smooth injective map which is not an immersion, because the derivative vanishes at 0.

Example 50. A map $i\colon S^1 \looparrowright \mathbb{R}^2$ with image a shape ∞ and with constant speed is an immersion which is not injective. The Boy surface is an immersion of $\mathbb{P}^2_{\mathbb{R}}$ in $\mathbb{R}^3$ which is not injective.

Let M be a smooth manifold of dimension n. We want to know if we can find an immersion $i\colon M \looparrowright \mathbb{R}^N$ for some $N \geqslant n$. So suppose we have such an immersion, giving hence an injective map of vector bundles

$$Di\colon TM \to \underline{\mathbb{R}}^N$$

Then we can look at the normal bundle $NM \to M$ which has the property that $TM \oplus NM \cong \underline{\mathbb{R}}^N$. This implies the *Whitney duality* theorem:

$$w(TM)w(NM) = 1$$

Since the rank of NM is $k = N - n$, we know that $w_j(NM) = 0$ for all $j > k$.

Example 51. As a particular example, we can use example 29 to discard the possibility of existence of immersions

$$\mathbb{P}^n_{\mathbb{R}} \looparrowright \mathbb{R}^N$$

for different values of n and N. We know from example 21 that the cohomology ring of $\mathbb{P}^n_{\mathbb{R}}$ with coefficients in $\mathbb{F}_2$ is

$$\mathrm{H}^*(\mathbb{P}^n_{\mathbb{R}}, \mathbb{F}_2) = \mathbb{F}_2[\alpha]/(\alpha^{n+1})$$

We have also computed the Stiefel-Whitney class of the tangent space

$$w(T\mathbb{P}^n_{\mathbb{R}}) = (1+\alpha)^{n+1}$$

The Stiefel-Whitney class of the normal bundle in an immersion $\mathbb{P}^n_{\mathbb{R}} \looparrowright \mathbb{R}^N$ is by the previous discussion equal to

$$\frac{1}{(1+\alpha)^{n+1}} = a_0 + a_1\alpha + a_2\alpha^2 + \cdots + a_k\alpha^k$$

where $k = N - n$ and $a_i \in \mathbb{F}_2$.

For example, for $n = 2$ we have $(1+\alpha)^4 = (1+\alpha^2)^2 = 1 + \alpha^4 = 1$, so $1+\alpha$ is the inverse of $w(T\mathbb{P}^n_{\mathbb{R}})$. For $N = 2$ we get $k = 0$, so in order to have an immersion we need this inverse to be a polynomial of degree 0. But it has degree 1, so no such immersion is possible, i.e.

$$\nexists\ \mathbb{P}^2_{\mathbb{R}} \looparrowright \mathbb{R}^2$$

4.2 Real division algebras

We have seen in example 29 that

$$w(T\mathbb{P}^n_{\mathbb{R}}) = (1+\alpha)^{n+1} \in \mathbb{F}_2[\alpha]/(\alpha^{n+1})$$

Recall that a manifold is called *parallelizable* if its tangent bundle is trivial. In particular, if $\mathbb{P}^n_{\mathbb{R}}$ is parallelizable, then $w(T\mathbb{P}^n_{\mathbb{R}}) = 1$, that is

$$(1+\alpha)^{n+1} = 1 = 1 + \alpha^{n+1}$$

For this to happen, $n+1$ must be a power of two. Indeed, if $n + 1 = 2^k m$ with m odd, we have

$$(1+\alpha)^{2^k m} = (1+\alpha^{2^k})^m = 1 + m\alpha^{2^k} + \binom{m}{2}\alpha^{2^k+2^k} + \cdots = 1 + \alpha^{2^k} + \cdots$$

And if $m > 1$ then $\alpha^{2^k} \neq 0$ and this is the only term in degree 2^k in the sum.

Corollary 52. If $\mathbb{P}^n_{\mathbb{R}}$ is paralellizable, then $n+1$ is a power of 2.

Definition 53. A not necessarily associative algebra over $\mathbb{R}$ is called a *real division algebra* if every equation of the form $ax = b$ and $xa = b$ with $a \neq 0$ has a unique solution.

Note that a finite dimensional real division algebra has no zero divisors, because multiplication with an element yields an endomorphism which is by definition surjective (and hence by dimension argument it is also injective).

Corollary 54. If a real division algebra of dimension n exits, then n is a power of 2.

Proof: Up to isomorphism, our real division algebra looks like $(\mathbb{R}^n, +)$ equipped with an $\mathbb{R}$-bilinear product $p\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ without zero divisors. Let e_i be the i^{th} vector of the standard basis on $\mathbb{R}^n$. The map $p(-, e_i)$ is then an automorphism, so we can define

$$v_i = p(-, e_i) \circ p(-, e_1)^{-1}$$

We have $v_1 = \mathrm{id}_{\mathbb{R}^n}$ and for any $x \in \mathbb{R}^n \backslash \{0\}$, the collection

$$\{x, v_2(x), \ldots, v_n(x)\}$$

is linearly independent over $\mathbb{R}$. Indeed, $\sum_i \lambda_i v_i(x) = 0$ means that $(\sum_i \lambda_i p(-, e_i)) \circ p(-, e_1)^{-1}(x) = 0$, which by injectivity and $\mathbb{R}$-bilinearity implies

$$p(x, \sum_i \lambda_i e_i) = 0$$

But $x \neq 0$ is not a zero divisor, so we get $\sum_i \lambda_i e_i = 0$, hence all λ_i are zero.

Recall now from example 29 that we had an isomorphism $T\mathbb{P}_{\mathbb{R}}^{n-1} \cong \mathrm{Hom}(L, E)$, where $L \to \mathbb{P}_{\mathbb{R}}^{n-1}$ is the tautological line bundle and E is such that $E \bigoplus L \cong \underline{\mathbb{R}}^n$, with the fibre of E over a point $[x] \in \mathbb{P}_{\mathbb{R}}^{n-1}$ being the orthogonal complement of the line $L_{[x]}$ in $\mathbb{R}^n$.

Let $L_{[x]}$ be a line through the origin and through $0 \neq x \in \mathbb{R}^n$. Each v_i defines a linear map

$$\bar{v_i}\colon L_{[x]} \longrightarrow L_{[x]}^{\perp}$$

by sending $y \in L_{[x]}$ to the orthogonal projection of $v_i(y)$ onto $L_{[x]}^{\perp}$. Since $v_1(y) = y$, we have $\bar{v_1} = 0$. But since $v_1, v_2, \ldots, v_n$ are everywhere linearly independent and $\bar{v_1} = 0$, none of the other $\bar{v_i}$ is in the line $L_{[x]}$ and so their projections $\bar{v_2}, \ldots, \bar{v_n}$ remain linearly independent. This holds for every point $x \in \mathbb{R}^n \backslash \{0\}$, and since everything is linear, everything is also continuous. We get $n-1$ linearly independent continuous sections of $\mathrm{Hom}(L, E) \cong T\mathbb{P}_{\mathbb{R}}^{n-1}$, so $\mathbb{P}_{\mathbb{R}}^{n-1}$ is parallelizable. By corollary 52, n must be a power of 2. □

This restricts already a lot the candidates for real division algebras. But in fact we know that the only possibilities are $n \in \{2, 4, 8\}$.

4.3 How many lines are there on a cubic surface?

This question concerns algebraic objects: a line is meant to be an actual straight line in (projective) space, not just some curve homeomorphic to a line. The topological category is therefore too flexible for our purposes, and we need more rigid morphisms. We will thus work on the algebraic category. The objects involved are algebraic objects, given locally by the zero locus of a family of polynomials in some affine space $\mathbb{C}^n$. The morphisms between these objects are continuous maps which locally look like quotients of polynomials.

We will work in complex projective space $\mathbb{P}^3$ with homogeneous coordinates x_0, x_1, x_2, x_3. A *line* will mean a linear subspace of $\mathbb{P}^3$ of dimension 1. A *cubic surface* is a subspace $S \subseteq \mathbb{P}^3$ which is described as the zero locus of a degree 3 homogeneous polynomial in $\mathbb{C}[x_0, x_1, x_2, x_3]$. For example we could consider the Fermat cubic

$$S = \{x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0\} \subseteq \mathbb{P}^3$$

The zero locus of a homogeneous polynomial $f \in \mathbb{C}[x_0, x_1, x_2, x_3]$ will be denoted by $Z(f)$, and depending on the context we regard this as a locus in $\mathbb{P}^3$ or as a vector subspace in $\mathbb{C}^4$.

Such a surface is said to be *smooth* if the partial derivatives of the polynomial defining it do not vanish all at once in any of the points of the surface.

In the algebraic category, vector bundles are better understood in terms of their sheaves of sections, which are locally free sheaves of finite rank over the structure sheaf $\mathcal{O}$. These sheaves are objects in the larger category of sheaves of abelian groups, which is an abelian category. So we have exact sequences, derived functors, etc.

Remark 55. In the topological category, every short exact sequence of vector bundles splits, because we can always put an Hermitian inner product in the middle term and the right term is then the orthogonal complement of the left term. But in the algebraic category this is not true anymore, because we cannot always find a holomorphic metric on a holomorphic vector bundle (complex analytic and algebraic categories are equivalent by GAGA).

Definition 56. The line bundle $\mathcal{O}(1)$ on $\mathbb{P}^3$ is the line bundle with sections which are locally degree 1 quotients of polynomials in $\mathbb{C}[x_0, x_1, x_2, x_3]$. Over the open set $U_i = \{x_i \neq 0\}$ this line bundle is trivial, with a non-zero section given by $[(x_0, x_1, x_2, x_3)] \mapsto x_i$. The transition functions from U_i to U_j are given by multiplication with $\frac{x_i}{x_j}$.

We can again form the group of isomorphism classes of line bundles with the tensor product. The product of sections of two line bundles corresponds to a section of their tensor product. The transition functions of the tensor product are the product of the transition functions. We denote by $\mathcal{O}(n)$ the line bundle which is the n-fold tensor product of $\mathcal{O}(1)$.

If $X \subseteq \mathbb{P}^3$ is a subspace, we denote by $\mathcal{O}_X(n)$ the restriction of $\mathcal{O}(n)$ to X.

The global sections of a vector bundle $\mathcal{E}$ on a space X are denoted by $\mathrm{H}^0(X, \mathcal{E})$.

Example 57. The tautological line bundle on $\mathbb{P}^3$ is $\mathcal{O}(-1)$. To see this, one can write down explicit equations for the transition functions of the tautological line bundle and check that they agree with the transition functions of $\mathcal{O}(-1)$.

Back to our question, how many lines are there on a given cubic surface S?

To answer this question, we first need to study the cohomology ring of the Grassmannian $G(2, 4)$, which parametrizes lines in $\mathbb{P}^3$.

Lemma 58. Let $\mathbb{G} = G(2,4)$ be the Grassmannian of lines in $\mathbb{P}^3$. Fix a complete flag $\{p_0\} \subseteq L_0 \subseteq H_0 \subseteq \mathbb{P}^3$, which may be assumed to be given by $H_0 = \{x_3 = 0\}$, $L_0 = \{x_3 = x_2 = 0\}$ and $p_0 = \{x_1 = x_2 = x_3 = 0\}$. Define the *Schubert cycles* as follows

- $\sum_0 = \sum_{0,0} = \mathbb{G}$,
- $\sum_1 = \sum_{1,0} = \{L \in \mathbb{G} \mid L \cap L_0 \neq \varnothing\}$,
- $\sum_2 = \sum_{2,0} = \{L \in \mathbb{G} \mid p_0 \in L\}$,
- $\sum_{1,1} = \{L \in \mathbb{G} \mid L \subseteq H_0\}$,
- $\sum_{2,1} = \{L \in \mathbb{G} \mid p_0 \in L \subseteq H_0\}$,
- $\sum_{2,2} = \{L_0\}$.

We have inclusions

$$\begin{array}{ccccccc} & & & \sum_2 & & & \\ & & \nearrow & & \searrow & & \\ \{L_0\} & \hookrightarrow & \sum_{2,1} & & & \sum_1 & \hookrightarrow \mathbb{G} \\ & & \searrow & & \nearrow & & \\ & & & \sum_{1,1} & & & \end{array}$$

The lines contained in each Schubert cycle and not contained in the previous ones form euclidean disks of dimensions 0, 2, 4, 6 and 8 respectively (from left to right). These constitute the cells of a CW structure on $\mathbb{G}$, each of them freely generating the corresponding homology group by dimension arguments. Define the *Schubert classes* $\sigma_{i,j} \in \mathrm{H}^{2(i+j)}(\mathbb{G})$ to be the Poincaré dual of the corresponding homology class. Then we have relations

- $\sigma_1^2 = \sigma_{1,1} + \sigma_2$,
- $\sigma_1\sigma_{1,1} = \sigma_1\sigma_2 = \sigma_{2,1}$,
- $\sigma_1\sigma_{2,1} = \sigma_{2,2}$,
- $\sigma_{1,1}^2 = \sigma_2^2 = \sigma_{2,2}$,
- $\sigma_{1,1}\sigma_2 = 0$.

It follows that $\sigma_1^3 = 2\sigma_{2,1}$, $\sigma_1^4 = 2\sigma_{2,2}$ and $\sigma_1^2\sigma_{1,1} = \sigma_1^2\sigma_2 = \sigma_{2,2}$, and by dimension arguments any product with degree more than 8 vanishes. We get

$$\mathrm{H}^*(\mathbb{G}) = \mathbb{Z}[\sigma_1, \sigma_2]/(\sigma_1^3 - 2\sigma_1\sigma_2, \sigma_1^2\sigma_2 - \sigma_2^2)$$

Proof: See [EH16, Thm. 3.10.] for a proof. To compute cup products, use Poincaré duality and compute intersection products of the corresponding homology cycles, which correspond to intersections of linear spaces. □

Theorem 59. Every smooth cubic surface $S \subseteq \mathbb{P}^3$ contains exactly 27 distinct lines.

Proof: (Sketch)

Let $F \in \mathbb{C}[x_0, x_1, x_2, x_3]$ be the polynomial defining S. Note that this polynomial F corresopnds to a global section of $\mathcal{O}(3)$.

The Grassmannian $\mathbb{G} = G(2,4)$ parametrizes lines in $\mathbb{P}^3$. Let $L \in \mathbb{G}$. What does it mean for L to be contained in S? The surface S is given by the cubic form F. A line L is contained in S if this cubic form restricts to the zero cubic form on L. But cubic forms on $L \cong \mathbb{P}^1$ are a vector space $\mathrm{H}^0(L, \mathcal{O}_L(3))$ of dimension 4, and as L varies in $\mathbb{G}$ this 4 dimensional vector space varies forming a vector bundle $\mathcal{E}$ of rank 4 over $\mathbb{G}$. The cubic form $F \in \mathrm{H}^0(\mathbb{P}, \mathcal{O}(3))$ gives then a global section s_F whose zero locus is the set of lines $L \subseteq S$. As pointed out in corollary 19, this corresponds then to (the Poincaré dual of) the Euler class $e(\mathcal{E})$, which by corollary 33 corresponds to $c_4(\mathcal{E}) \in \mathrm{H}^8(\mathbb{G}) = \mathbb{Z}[\sigma_1, \sigma_2]/(\sigma_1^3 - 2\sigma_1\sigma_2, \sigma_1^2\sigma_2 - \sigma_2^2)$, where $\sigma_1 \in \mathrm{H}^2(\mathbb{G})$ and $\sigma_2 \in \mathrm{H}^4(\mathbb{G})$ are the Schubert classes of lemma 58.

To compute $c_4(\mathcal{E})$ we want to understand $\mathcal{E}$ in terms of a simpler bundle. This will be the dual of the tautological bundle $\mathcal{S} \to \mathbb{G}$. The fibre of $\mathcal{S}$ over a point $L \in \mathbb{G}$ is the 2-dimensional vector space $V \subset \mathbb{C}^4$ such that $L = \mathbb{P}(V)$. Its dual $\mathcal{S}^\vee$ is then a rank 2 bundle whose fibre over each line $L \in \mathbb{G}$ is the vector space of linear forms on the plane $V \subseteq \mathbb{C}^4$ such that $\mathbb{P}(V) = L$. This is precisely the vector space of global sections $\mathrm{H}^0(L, \mathcal{O}_L(1)) \cong \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(1))$. By the discussion in the previous paragraph, the bundle $\mathcal{E}$ is given over each fibre by $\mathrm{H}^0(L, \mathcal{O}_L(3))$, so we have

$$\mathcal{E} \cong S^3(\mathcal{S}^*)$$

Now we need to compute the Chern class of $\mathcal{S}^*$. Let f be a linear form on $\mathbb{C}^4$. Then f restricts to a linear form on each plane V representing a line $L \in \mathbb{G}$, giving a global section of $\mathcal{S}^*$. By example 46 we know that $c_1(\mathcal{S}^*)$ is the Poincaré dual of the locus over which two generic global sections f_1, f_2 become linearly dependent, which is precisely the set of lines $L = \mathbb{P}(V) \in \mathbb{G}$ such that

$$V \cap Z(f_1) \cap Z(f_2) \supsetneq \{0\}$$

because this means that there is a whole line in V along which f_1 and f_2 are zero, thus a non trivial linear combination of f_1 and f_2 over V which is zero. But $V \cap Z(f_1) \cap Z(f_2)$ containing a line means that $\mathbb{P}(V) \cap \mathbb{P}(Z(f_1) \cap Z(f_2)) \neq \varnothing$, so this zero locus is given by the Schubert cycle $\sum_1$. We get

$$c_1(\mathcal{S}^*) = \sigma_1$$

For $c_2(\mathcal{S}^*)$ we need to compute the zero locus of a section f. This is now the set of lines $L = \mathbb{P}(V) \in \mathbb{G}$ such that $f|_L = 0$, i.e. $L \subseteq Z(f)$ which is a hyperplane. So we get

$$c_2(\mathcal{S}^*) = \sigma_{1,1}$$

And therefore

$$c(\mathcal{S}^*) = 1 + \sigma_1 + \sigma_{1,1}$$

The claim now is that $c_4(\mathcal{E}) = c_4(S^3(\mathcal{S}^*)) = 27\sigma_{2,2}$. To prove this formula we may assume by proposition 32 that $\mathcal{S}^*$ splits as the sum of two line bundles $\mathcal{L} \oplus \mathcal{M}$. Let $\alpha = c_1(\mathcal{L})$ and $\beta = c_1(\mathcal{M})$. Then

$$c(\mathcal{S}^*) = 1 + \sigma_1 + \sigma_{1,1} = (1+\alpha)(1+\beta)$$

and therefore $\alpha + \beta = \sigma_1$ and $\alpha\beta = \sigma_{1,1}$. Now

$$\mathcal{S}^3(\mathcal{S}^*) = \mathcal{L}^3 \oplus (\mathcal{L}^2 \otimes \mathcal{M}) \oplus (\mathcal{L} \otimes \mathcal{M}^2) \oplus \mathcal{M}^3$$

We get from lemma 36 that

$$c(S^3(\mathcal{S}^*)) = (1+3\alpha)(1+2\alpha+\beta)(1+\alpha+2\beta)(1+3\beta)$$

We are only interested in the top degree of this polynomial expression. This is given by

$$3\alpha(2\alpha+\beta)(\alpha+2\beta)3\beta = 9\alpha\beta(2\alpha^2+5\alpha\beta+2\beta^2) = 9\alpha\beta(2(\alpha+\beta)^2+\alpha\beta)$$

Using the relations obtained from the splitting, we can express this again in terms of Chern classes of $\mathcal{S}^*$ as $9\sigma_{1,1}(2\sigma_1^2 + \sigma_{1,1})$. Using the relations in lemma 58 we get $9\sigma_{1,1}(2\sigma_{1,1} + 2\sigma_2 + \sigma_{1,1}) = 27\sigma_{1,1}^2 = 27\sigma_{2,2}$. Therefore

$$c_4(\mathcal{E}) = c_4(S^3(\mathcal{S}^*)) = 27\sigma_{2,2}$$

This corresponds in homology to 27 copies of the ground class $[\mathbb{G}] \in \mathrm{H}_0(\mathbb{G})$, each of them being a point (line in $\mathbb{P}^3$). So there are exactly 27 lines contained in a smooth cubic surface. □

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