AN INTRODUCTION TO STABILITY THEORY

DANIEL PALACÍN

These lecture notes are a slightly extended version of my course on stability theory given at the Münster Model Theory Month in May of 2016. The course is based on the first section of Pillay's book [3] and it covers fundamental notions of stability theory such as definable types, forking calculus and canonical bases, as well as, stable groups and homogeneous spaces. The approach followed here is originally due to Hrushovski and Pillay [2], who presented stability from a local point of view.

Through the notes some general knowledge of model theory is assumed. I recommend the book of Tent and Ziegler [4] as an introduction to model theory. Furthermore, the texts of Casanovas [1] and Wagner [5] may also be useful to the reader to obtain a different approach to stability theory.

I am in debt to all the participants of the Model Theory Month in Münster, as well as Martin Bays, for their comments and remarks during and after the course which help me to improve the presentation of these notes. Furthermore, this text was used in a reading group at the Hebrew University of Jerusalem in 2016. I would like to thank the participants, in particular Yatir Halevi and Itay Kaplan, for making numerous suggestions and remarks. Last but not least, I thank the anonymous reviewer for their work, comments and remarks.

1. Preliminaries

Throughout the text T is a complete first-order theory with infinite models in a language L. We shall be working inside a *monster model* $\mathbb{M}$ of the theory, i.e. a sufficiently saturated and homogeneous model. Thus tuples of elements and sets consist of elements from this model, and we assume that they have a small size compared to the monster model. We use the letters $a, b, c, \ldots$ to denote tuples (not necessarily finite) of elements and $A, B, C, \ldots$ for sets, while $x, y, z, \ldots$ are for tuples of variables. Types over small sets of parameters are denoted by $p, q, \ldots$ and global types (i.e. types over $\mathbb{M}$) are written in Fraktur $\mathfrak{p}, \mathfrak{q}, \ldots$

We shall recall Shelah's construction of imaginaries, which allow us to deal with equivalence classes. Given a model M of the theory, we construct M^{eq} as follows: We add a new sort M^n/E for every formula $E(x, y)$ in

Date: September 5, 2017.

the language defining an equivalence relation on a finite power M^n and additionally, we extend the language L to L^{eq} by adding an n-ary function symbol π_E for the projection map $\pi_E : M^n \to M^n/E$ that maps an n-tuple a onto its equivalence class $\pi_E(a) = [a]_E$. We identify M with $M/=$. Observe that all symbols of L are in the expanded language L^{eq}. The theory T^{eq} is the complete L^{eq}-theory of the model M^{eq}. It can be checked that T^{eq} does not depend on the choice of our initial model. In particular, the many-sorted structure $\mathbb{M}^{\mathrm{eq}}$ is indeed a monster model of T^{eq}, whose elements are called *imaginaries*. Furthermore, observe that every automorphism of $\mathbb{M}$ extends uniquely to an automorphism of $\mathbb{M}^{\mathrm{eq}}$.

The main difference while working with T^{eq} instead of T is that now variables, functions and relations must specify the sorts they live on.

Lemma 1.1. *For every formula $\psi(x, x_1^{E_1}, \ldots, x_n^{E_n})$ in L^{eq}, where x is a tuple from the home sort and each $x_i^{E_i}$ is of sort E_i, there is a formula $\varphi(x, y_1, \ldots, y_n)$, with y_i having the length of the arity of E_i, such that for all tuples $a, a_1, \ldots, a_n$ in $\mathbb{M}$ of the right length we have that:*

$$\mathbb{M}^{\mathrm{eq}} \models \psi(a, \pi_{E_1}(a_1), \ldots, \pi_{E_n}(a_n)) \;\Leftrightarrow\; \mathbb{M} \models \psi(a, a_1, \ldots, a_n).$$

Now, given a formula $\psi(x, a)$, consider the equivalence relation $E_\psi(y, z)$ given by $\forall x(\psi(x, y) \leftrightarrow \psi(x, z))$, and define the *canonical parameter* $\ulcorner\psi(x, a)\urcorner$ of $\psi(x, a)$ as the imaginary $[a]_{E_\psi}$. Notice that $E_\psi(y, z)$ is equivalent to say $\psi(\mathbb{M}, y) = \psi(\mathbb{M}, z)$. By a canonical parameter $\ulcorner X\urcorner$ of a definable set X we mean the canonical parameter of a formula defining X. Observe that any two canonical parameters are interdefinable. Thus we shall talk about the canonical parameter of a definable set.

Canonical parameters are useful when dealing with automorphisms:

Lemma 1.2. *Let X be a definable subset of $\mathbb{M}^n$. Then the following are equivalent:*

(1) The set X is definable over A.
(2) The set X is A-invariant, i.e. for any $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$, $\sigma(X) = X$.
(3) The canonical parameter of X belongs to $\mathrm{dcl}^{\mathrm{eq}}(A)$.

Proof. We first show (2) and (3) are equivalent. For this, let $\varphi(x, y)$ be a formula, let c be a tuple and let x be a tuple of variables of length n such that $\varphi(\mathbb{M}, c) = X$. Now, let $E(y, z)$ be the equivalence relation $\varphi(\mathbb{M}, y) = \varphi(\mathbb{M}, z)$. Thus $[c]_E = \ulcorner\varphi(x, c)\urcorner$. Now, we have

$$\sigma([c]_E) = [c]_E \;\Leftrightarrow\; E(c, \sigma(c)) \;\Leftrightarrow\; \varphi(\mathbb{M}, c) = \varphi(\mathbb{M}, \sigma(c)) \;\Leftrightarrow\; \sigma(X) = X$$

and so we obtain the equivalence.

To see that (1) implies (2). Suppose that X is defined by a formula $\psi(x, b)$ with b a tuple in A, and x a tuple of variables of length n, i.e. $X = \psi(\mathbb{M}, b)$.

Now, given an automorphism $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$ we have that $\sigma(b) = b$ and so:

$$a \in X \iff \phi(a, b) \text{ holds} \iff \phi(\sigma(a), b) \text{ holds} \iff \sigma(a) \in X.$$

Finally, to see that (2) yields (1), suppose that X is A-invariant and consider a formula $\phi(x, y)$, a tuple b in $\mathbb{M}$ and a tuple of variables x of length n such that $X = \phi(\mathbb{M}, b)$. Set $p(y) = \mathrm{tp}(b/A)$. By invariance notice that

$$p(y) \vdash \forall x\big(\phi(x, y) \leftrightarrow \phi(x, b)\big).$$

By compactness there exists a formula $\theta(y) \in p(y)$ witnessing this, and set $\chi(x)$ be the formula $\exists z(\theta(z) \wedge \phi(x, z))$. It is easy to see that $\chi(\mathbb{M}) = X$. □

Similarly, we have the following:

Lemma 1.3. *Let X be a definable subset of $\mathbb{M}^n$. Then the following are equivalent:*

(1) The set X has a finite (bounded) orbit under $\mathrm{Aut}(\mathbb{M}/A)$.
(2) The canonical parameter of X belongs to $\mathrm{acl}^{\mathrm{eq}}(A)$.

Proof. By the previous lemma we know that a set is unequivocally determined by its canonical parameter. Hence, a set X has a finite orbit if and only if so does $\ulcorner X \urcorner$. Moreover, by compactness an imaginary element has finite orbit if and only if it has a bounded orbit. This yields the statement. □

2. Forking, dividing and satisfiability

Definition 2.1. A collection $\mathcal{I}$ of definable sets in a fixed variable x is an *ideal* if it is closed under subsets and finite unions, and additionally $\emptyset \in \mathcal{I}$.

Lemma 2.2. *If a partial type $\pi(x)$ over A does not imply a formula from an ideal $\mathcal{I}$, then for any set $B \supseteq A$ there is a complete type $p(x)$ over B extending $\pi(x)$ which does not contain any formula from $\mathcal{I}$.*

Proof. Given a partial type $\pi(x)$ and a set B, it is enough to show the consistency of the following set of formulas

$$\pi(x) \cup \{\neg\varphi(x, b) : b \in B \text{ and } \varphi(x, b) \in \mathcal{I}\}.$$

If it is inconsistent, then by compactness there are finitely many formulas in $\mathcal{I}$ whose disjunction is implied by $\pi(x)$. As $\mathcal{I}$ is an ideal, we obtain a contradiction. Therefore, this set is consistent. Furthermore, if the formula $\varphi(x, b)$ is in $\mathcal{I}$, then its negation does not belong to $\mathcal{I}$, as neither does the formula $x = x$ by assumption. Hence, it follows that any completion of the set above satisfies the requirements. □

Definition 2.3. A partial type π is said to be finitely satisfiable in a set A if any finite conjunction of formulas from π is realized by a tuple in A.

As the collection of formulas which are not satisfied in a fixed set form an ideal, an easy application of Lemma 2.2 yields that any partial type $\pi(x)$ which is finitely satisfiable in A, has a complete extension over any set $B \supseteq A$ which is finitely satisfiable in A as well. In particular, any complete type over a model M has a global extension which is finitely satisfiable in M.

Lemma 2.4. *The following holds:*

(1) If $\mathfrak{p}$ is finitely satisfiable in A, then it is A-invariant.
(2) If $\mathfrak{p}$ is A-invariant, and $(a_i)_{i<\alpha}$ is a sequence such that a_i realizes $\mathfrak{p}_{|A\cup\{a_j\}_{j<i}}$, then $(a_i)_{i<\alpha}$ is A-indiscernible.

Proof. If (1) does not hold, then there are some tuples b and c having the same type over A and the formula $\phi(x,b) \wedge \neg\phi(x,c)$ belongs to $\mathfrak{p}$. However, as $\mathfrak{p}$ is finitely satisfiable we get a contradiction.

To prove (2), we show by induction on n that $a_0 \dots a_n \equiv_A a_{i_0} \dots a_{i_n}$ for $i_0 < \dots < i_n$. By induction, assume that there is an automorphism $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$ such that $\sigma(a_0 \dots a_{n-1}) = a_{i_0} \dots a_{i_{n-1}}$. Thus

$$\mathrm{tp}(a_{i_n}/A, (a_{i_j})_{j<n})^\sigma = (\mathfrak{p}_{|A\cup\{a_{i_j}\}_{j<n}})^\sigma = \mathfrak{p}_{|A\cup\{a_j\}_{j<n}} = \mathrm{tp}(a_n/A, (a_j)_{j<n})$$

and so

$$a_{i_0} \dots a_{i_n} \equiv_A \sigma(a_{i_0} \dots a_{i_n}) \equiv_A a_0 \dots a_n,$$

as desired. □

Definition 2.5. A formula $\psi(x,a)$ *divides* over a set A if there is an A-indiscernible sequence $(a_i)_{i<\omega}$ with each $a_i \equiv_A a$ such that $\{\psi(x,a_i)\}_{i<\omega}$ is inconsistent. We say that a formula *forks* over A if it implies a finite disjunction of formulas, each of which divides over A.

Observe that any inconsistent formula divides over any set, and that if a formula $\varphi(x,a)$ forks over A, then so does any formula $\psi(x,b)$ implying $\varphi(x,a)$. Therefore, forking over A means that the formula belongs to the ideal generated by the formulas that divide over A. In general there are formulas that fork but do not divide as is exhibited in the next example.

Example 2.6. Let T be the theory of the circle S^1 with a ternary relation $R(x,y,z)$ interpreted as "y lies on the arc between x and z, ordered clockwise". It is an exercise to see that

(1) This theory has quantifier elimination and so, there is a unique 2-type $p(x,y)$ without parameters consistent with the formula $x \neq y$.
(2) The formula $R(a,y,b)$ divides over $\emptyset$ for any elements a,b.

Therefore, the formula $x = x$ forks over $\emptyset$ but it does not divide (notice that to divide parameters are essential).

Definition 2.7. A partial type divides (forks) over A if it implies a formula that does it.

Remark 2.8. The following holds:

(1) If $\pi(x)$ divides over A, then it divides over some model containing A and so over $\mathrm{acl}^{\mathrm{eq}}(A)$.
(2) If $\pi(x)$ forks (divides) over A, then it forks (divides) over any subset of A.
(3) If $\pi(x)$ forks (divides) over A, then so does some conjunction of formulas from π.
(4) If $\pi(x)$ does not fork over A, then it has an extension over any set of parameters which does not fork over A.
(5) If $\pi(x)$ is finitely satisfiable in A, then it does not fork over A.

Proof. Only the first point requires some checking. For this, we recall the following fact: An indiscernible sequence over a set A is indeed indiscernible over some model containing A. This fact is shown using Erdős-Rado. We refer to [1, Corollary 1.7] for a detailed proof. Using this, the statement is immediate. □

3. Local stability

We shall be working in the imaginary monster model of the theory.

Let $\phi(x, y)$ be a formula. By a ϕ-formula we mean a formula of the form $\phi(x, a)$ or $\neg\phi(x, a)$, and by a *complete ϕ-type* over a set of parameters A we mean a maximal consistent collection of ϕ-formulas with parameters over A. We denote the space of complete ϕ-types over A by $S_\phi(A)$.

Definition 3.1. Let $\phi(x, y)$ be a formula. A complete ϕ-type $p(x)$ over A is definable over a set B if there exists a formula $\psi(y)$ with parameters over B such that for any tuple a in A, we obtain

$$\phi(x, a) \in p \;\Leftrightarrow\; \psi(a) \text{ holds.}$$

We denote the formula $\psi(y)$ as $\mathrm{d}_p x\phi(x, y)$ and we say that p is definable if it is definable over its domain. Furthermore, if q is a complete type we denote by $q_{|\phi}$ its corresponding complete ϕ-type and by $\mathrm{d}_q x\phi(x, y)$ the definition of $q_{|\phi}$.

Lemma 3.2. *If $\mathfrak{p} \in S_\phi(\mathbb{M})$ is finitely satisfiable in A and definable, then it is definable over $\mathrm{dcl}^{\mathrm{eq}}(A)$.*

Proof. Let $\mathfrak{p}$ be a global complete ϕ-type which is finitely satisfiable in A. By Lemma 2.4(i), or its proof, we obtain that $\mathfrak{p}$ is A-invariant. Thus, by Lemma 1.2 we obtain that the canonical parameter of $\mathrm{d}_{\mathfrak{p}} x\phi(x, y)$ belongs to $\mathrm{dcl}^{\mathrm{eq}}(A)$ and hence $\mathfrak{p}$ is definable over $\mathrm{dcl}^{\mathrm{eq}}(A)$. □

Definition 3.3. A formula $\phi(x, y)$ is *stable* if there is no sequence $(a_i, b_i)_{i<\omega}$ such that $\phi(a_i, b_j)$ holds if and only if $i < j$.

Remark 3.4. Observe that by compactness we may replace $(\omega, <)$ by any infinite linear order. Consequently, the following holds:

(1) If $\phi(x, y)$ is stable, then so is $\neg\phi(x, y)$.
(2) If $\phi_1(x, y)$ and $\phi_2(x, z)$ are stable, then so is the formula $\chi(x, yz)$ given by $\phi_1(x, y) \vee \phi_2(x, z)$.
(3) If $\phi(x, y)$ is stable, then so is the formula $\phi^*(y, x)$ consisting of switching the roles of x and y.

In particular, it follows from (1) and (2) that a Boolean combination of stable formulas is stable.

Proof. To see (3) it is enough to consider the reverse order ω^* of ω. Similarly, and replacing b_i by $b'_i = b_{i+1}$ we obtain (1). Finally (2) follows by an application of Ramsey's theorem. If $\chi(x, yz)$ is not stable and this is witnessed by a sequence $(a_i, b_i^1 b_i^2)_{i<\omega}$, then set

$$X_k = \{(i, j) \in \mathbb{N} \times \mathbb{N} : i < j \text{ and } \phi_k(a_i, b_j^k)\} \quad \text{for} \quad k = 1, 2.$$

By Ramsey's theorem, there is an infinite set I of $\mathbb{N}$ such that all increasing pairs from I belong to X_1, in which case $\phi_1(x, y)$ is not stable, or all increasing pairs belong to X_2 showing that $\phi_2(x, y)$ is not stable, a contradiction. □

Lemma 3.5. *Let $\phi(x, y)$ be stable and let $p(x) \in S_\phi(A)$ and B a subset of A. Then:*

(1) For any global type $\mathfrak{q}$ containing p, there is a finite sequence $(c_i)_{i<n}$ with c_i realizing the type $\mathfrak{q}_{|B\cup\{c_j\}_{j<i}}$ such that p is defined by a positive Boolean combination of the formulas $\phi^*(y, c_i) = \phi(c_i, y)$.
(2) If p is finitely satisfiable in a subset B of A, then p is definable and the definition is given as a positive Boolean combination of the formulas $\phi^*(y, b) = \phi(b, y)$ with b in B.

Proof. We first show (1). Consider a complete ϕ-type $p(x) \in S_\phi(A)$ and let $\mathfrak{q}$ be a global completion of p. Suppose, towards a contradiction, that there is no finite sequence $(c_i)_{i<n}$ with c_i realizing $\mathfrak{q}_{|A\cup\{c_j\}_{j<i}}$ such that $p = (\mathfrak{q}_{|A})_{|\phi}$ is definable by a positive Boolean combination of the formulas $\phi(c_i, y)$. Following Erdős-Makkai, we construct inductively on n a sequence of parameters $(b_n, b'_n)_{n<\omega}$ in A and another sequence $(c_n)_{n<\omega}$ with c_n realizing $\mathfrak{q}_{|B\cup\{c_i\}_{i<n}}$ such that:

(i) $\phi(x, b_i)$ and $\neg\phi(x, b'_i)$ belong to p for every $i < \omega$,
(ii) $\phi(c_i, b_j) \to \phi(c_i, b'_j)$ holds for every $i < j$, and
(iii) $\phi(c_i, b_j)$ and $\neg\phi(c_i, b'_j)$ hold when $i \geq j$.

To do so, assume that we have already obtained $(b_i, b'_i)_{i<n}$ and $(c_i)_{i<n}$. As p is not definable by a positive Boolean combination of the formulas $\phi(c_i, y)$ for $i < n$, there are tuples b_n and b'_n with $\phi(x, b_n) \in p$ and $\phi(x, b'_n) \notin p$

such that if $\phi(c_i, b_n)$ holds, then so does $\phi(c_i, b'_n)$[1]. Thus, setting c_n to be a realization of $\mathfrak{q}_{|B\cup\{b_i,b'_i\}_{i\leq n}\cup\{c_i\}_{i<n}}$ we obtain the desired sequence.

Now, by Ramsey's theorem we may assume that either $\phi(c_i, b_j)$ holds for all $i < j$ or $\neg\phi(c_i, b_j)$ for all $i < j$. In the first case, the sequence $(c_i, b'_i)_{i<\omega}$ witnesses that $\phi(x, y)$ is not stable, and in the second case the sequence $(c_i, b_{i+1})_{i<\omega}$, a contradiction.

To prove (2) notice that the same construction works, but now taking c_n to be a tuple in B realizing the finite set $\{\phi(x, b_i)\}_{i\leq n} \cup \{\phi(x, b'_i)\}_{i\leq n}$, which is a subset of formulas from p. □

Corollary 3.6. *If $\phi(x, y)$ is stable, then any complete ϕ-type over a model M is definable by a positive Boolean combination of ϕ^*-formulas over M.*

It then follows that any complete ϕ-type over a model has a definable global extension. Definable extensions over arbitrary sets is given by the next lemma:

Lemma 3.7. *Let $\phi(x, y)$ be stable and let $p(x) \in S(A)$. Then there is some $\mathfrak{q}(x) \in S_\phi(\mathbb{M})$ such that $p(x) \cup \mathfrak{q}(x)$ is consistent and $\mathfrak{q}$ is definable over $\mathrm{acl}^{\mathrm{eq}}(A)$.*

Proof. Let X be the space of ϕ-types over $\mathbb{M}$ which are consistent with p, i.e.

$$X = \{\mathfrak{q}(x) \in S_\phi(\mathbb{M}) : p(x) \cup \mathfrak{q}(x) \text{ is consistent}\}.$$

Observe that X is the image under the restriction map $S_x(\mathbb{M}) \to S_\phi(\mathbb{M})$ of the closed space formed by global types in $S_x(\mathbb{M})$ extending $p(x)$. Since the restriction map is closed, the space X is closed in $S_\phi(\mathbb{M})$. Recall that the space $S_\phi(\mathbb{M})$ is endowed with a compact Hausdorff totally disconnected topology, where a basis of clopen sets is given by all sets of the form $[\psi] = \{\mathfrak{q} \in S_\phi(\mathbb{M}) : \mathfrak{q} \vdash \psi\}$ for any Boolean combination ψ of ϕ-formulas.

Now, set $X^{(0)} = X$ and for each natural number i set $X^{(i+1)}$ to be the set of accumulation points in $X^{(i)}$, i.e. the collection of global ϕ-types $\mathfrak{q}(x) \in X^{(i)}$ which are not isolated within $X^{(i)}$ by a Boolean combination of ϕ-formulas. That is, the set $X^{(i+1)}$ is formed by types $\mathfrak{q}(x) \in X^{(i)}$ such that for any Boolean combination $\psi(x)$ of formulas of the form $\phi(x, a)$ or $\neg\phi(x, a)$, if $\mathfrak{q} \vdash \psi$ then there is a distinct $\mathfrak{q}' \in X^{(i)}$ with $\mathfrak{q}' \vdash \psi$. Inductively, it is easy to see that each $X^{(i)}$ is closed in $S_\phi(\mathbb{M})$.

We claim that some $X^{(n+1)}$ is empty. Otherwise, for any natural number n, there is some ϕ-type $\mathfrak{p}$ in $X^{(n)}$ such that given a Boolean combination $\psi(x)$ of ϕ-formulas with $\mathfrak{p} \vdash \psi$, there is a distinct global ϕ-type $\mathfrak{p}'$ in $X^{(n)}$

[1]Let X be a proper subset of A. Then X is a positive Boolean combination of sets $X_0, \ldots, X_n$ if and only if for any elements x, y in A the following holds: If $x \in X$ and for any $i \leq n$ it holds that if $x \in X_i$ then $y \in X_i$, then $y \in X$. Apply this with X being $\{b \in A : \phi(x, b) \in p\}$ and $X_i = \{b \in A : \phi(c_i, b) \text{ holds}\}$.

implying ψ. Thus, we can find an instance $\phi(x,a)$ with $\phi(x,a) \in \mathfrak{p}$ and $\neg\phi(x,a) \in \mathfrak{p}'$. It then follows that $\psi(x) \wedge \phi(x,a)$ and $\psi(x) \wedge \neg\phi(x,a)$ are consistent and obviously contradictory. This argument yields the existence of a binary tree $\{\phi(x, a_{\eta|i})^{\eta(i)}\}_{\eta\in 2^n, i<n}$ such that each branch is consistent but the conjunction of any two branches is inconsistent, where $\phi^0 = \phi$ and $\phi^1 = \neg\phi$. Now, let μ be the smallest cardinal such that $2^\mu > |T|$ and note that $|2^{<\mu}| \leq |T|$. By compactness, we obtain a binary tree $\{\phi(x, b_{\eta|i})^{\eta(i)}\}_{\eta\in 2^\mu, i<\lambda}$ with each branch being consistent but the conjunction of any two branches is inconsistent, where as before $\phi^0 = \phi$ and $\phi^1 = \neg\phi$. As there are only $|2^{<\mu}|$ many $b_{\eta|i}$'s, we can find a model M of size $|T|$ containing all these parameters. Thus $|S_\phi(M)| = 2^\mu$ as each branch of the tree yields a complete ϕ-type. However, as $\phi(x,y)$ is stable, all ϕ-types over M are definable by Corollary 3.6 and so $|S_\phi(M)| \leq |M|$ since there are only $|M|$ many formulas with parameters over M, a contradiction.

As $X^{(n+1)}$ is empty, all types in $X^{(n)}$ are isolated and so by (topological) compactness we obtain that it is finite.

Now, let $\mathfrak{q}$ be a ϕ-type from $X^{(n)}$ and note that it is definable by Corollary 3.6. Thus, as $X^{(n)}$ is an A-invariant set, the orbit of $\mathfrak{q}$ under $\mathrm{Aut}(\mathbb{M}/A)$ is finite and hence, the canonical parameter of the definition of $\mathfrak{q}$ belongs to $\mathrm{acl}^{\mathrm{eq}}(A)$. □

Lemma 3.8 (Harrington)**.** *Let $\phi(x,y)$ be a stable formula and let $\mathfrak{p}(x)$ and $\mathfrak{q}(y)$ be complete global types. Then*

$$\mathrm{d}_{\mathfrak{p}} x\phi(x,y) \in \mathfrak{q}(y) \ \Leftrightarrow\ \mathrm{d}_{\mathfrak{q}} y\phi(x,y) \in \mathfrak{p}(x).$$

Proof. Assume that $\mathrm{d}_{\mathfrak{q}} y\phi(x,y)$ and $\mathrm{d}_{\mathfrak{p}} x\phi(x,y)$ are both over A. We construct recursively on n a sequence $(a_n, b_n)_{n<\omega}$ such that b_n realizes $\mathfrak{q}_{|A\cup\{a_j\}_{j<n}}$ and then we take a_n realizing $\mathfrak{p}_{|A\cup\{b_j\}_{j\leq n}}$. Thus, for $i \geq j$ we obtain that

$$\phi(a_i, b_j) \ \Leftrightarrow\ \phi(x, b_j) \in \mathfrak{p} \ \Leftrightarrow\ \mathrm{d}_{\mathfrak{p}} x\phi(x,y) \in \mathfrak{q}$$

and similarly for $i < j$ we get

$$\phi(a_i, b_j) \ \Leftrightarrow\ \phi(a_i, y) \in \mathfrak{q} \ \Leftrightarrow\ \mathrm{d}_{\mathfrak{q}} y\phi(x,y) \in \mathfrak{p}.$$

As $\phi(x,y)$ is stable, we obtain the result. □

Next, we shall prove uniqueness of definable extension over algebraically closed sets. To do so, we need to consider the following more robust notion of ϕ-type. A *generalized ϕ-type* over a set A is maximal consistent collection of formulas which are equivalent to a Boolean combination of ϕ-formulas, possibly with parameters not in A. Notice that over a model M a generalized ϕ-type is equivalent to a ϕ-type over M. Hence, we do not make any distinction over models between ordinary and generalized ϕ-types.

Corollary 3.9. *Let $\phi(x,y)$ be stable. A complete generalized ϕ-type over an $\mathrm{acl}^{\mathrm{eq}}$-closed set has a unique global ϕ-type extension.*

Proof. Let $A = \mathrm{acl}^{\mathrm{eq}}(A)$ and let p be a complete generalized ϕ-type over A. Existence follows from Lemma 3.7 applied to any completion of p over A. To show uniqueness, consider two global complete ϕ-types $\mathfrak{p}_1$ and $\mathfrak{p}_2$ extending p and assume that both are definable over A. Let $\phi(x,b)$ be a formula and let $q(y) = \mathrm{tp}(b/A)$. By Lemma 3.7 there is a global complete ϕ^*-type $\mathfrak{q}(y)$ which is definable over A and such that $q(y) \cup \mathfrak{q}(y)$ is consistent. Let $\mathfrak{q}'$ be a completion of $q(y) \cup \mathfrak{q}(y)$; note that $\mathfrak{q}'_{|\phi^*} = \mathfrak{q}$. As the ϕ^*-type of $\mathfrak{q}'$, and the ϕ-types of $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are definable over A, by Harrington's lemma we obtain:

$$\phi(x,b) \in \mathfrak{p}_i \;\Leftrightarrow\; \mathrm{d}_{\mathfrak{p}_i} x\phi(x,y) \in \mathfrak{q}' \;\Leftrightarrow\; \mathrm{d}_{\mathfrak{q}'} y\phi(x,y) \in \mathfrak{p}'_i,$$

where $\mathfrak{p}'_1$ and $\mathfrak{p}'_2$ are arbitrary completions of $\mathfrak{p}_1$ and $\mathfrak{p}_2$ respectively. On the other hand, by Lemma 3.5 the definition $\mathrm{d}_{\mathfrak{q}'} y\phi(x,y)$ of $\mathfrak{q}'_{|\phi^*}$ is equivalent to a positive Boolean combination of ϕ-formulas. Thus, since both $\mathfrak{p}'_1$ and $\mathfrak{p}'_2$ extend the generalized ϕ-type p, we have that

$$\mathrm{d}_{\mathfrak{q}'} y\phi(x,y) \in \mathfrak{p}'_i \;\Leftrightarrow\; \mathrm{d}_{\mathfrak{q}'} y\phi(x,y) \in p.$$

Hence, putting everything together we obtain that the condition $\phi(x,b) \in \mathfrak{p}_i$ does not depend on i and as $\phi(x,b)$ was arbitrary, the ϕ-types of $\mathfrak{p}_1$ and $\mathfrak{p}_2$ coincide. □

Lemma 3.10. *Let $\phi(x,y)$ be a stable formula. If $\mathfrak{p}$ is a global complete ϕ-type which is definable over a model M and consistent with a partial type $\pi(x)$ over M, then $\pi(x) \cup \mathfrak{p}(x)$ is finitely satisfiable in M.*

Proof. (Due to I. Kaplan) Let $\mathfrak{p}$ and π be given, and note that $\pi(x) \cup \mathfrak{p}_{|M}(x)$ is finitely satisfiable in M. Thus, there is some global type $\mathfrak{q}$ extending π and $\mathfrak{p}_{|M}$ which is also finitely satisfiable in M and hence, its restriction $\mathfrak{q}_{|\phi}$ is definable over M by Lemma 3.2 and Corollary 3.6.

To conclude, it suffices to see that $\mathfrak{q}_{|\phi}$ is precisely $\mathfrak{p}$. To do so, assume that $\phi(x,c)$ belongs to $\mathfrak{p}$ but it does not belong to $\mathfrak{q}_{|\phi}$. Thus, the formulas $\mathrm{d}_{\mathfrak{p}} x\phi(x,c)$ and $\neg\mathrm{d}_{\mathfrak{q}_{|\phi}} x\phi(x,c)$ hold, and so the sentence

$$\exists y\big(\mathrm{d}_{\mathfrak{p}} x\phi(x,y) \wedge \neg\mathrm{d}_{\mathfrak{q}_{|\phi}} x\phi(x,y)\big)$$

is true of the ambient theory. However, this yields the existence of some element d in M such that $\neg\phi(x,d)$ belongs to $\mathfrak{q}_{|\phi}$ and $\phi(x,d)$ belongs to $\mathfrak{p}$, a contradiction since $\mathfrak{q}_{|\phi}$ and $\mathfrak{p}$ agree over M by construction. □

Proposition 3.11. *Let $\phi(x,y)$ be a stable formula. Then the following are equivalent for a formula $\phi(x,a)$:*

1. *it is satisfiable in every model containing A.*
2. *it does not fork over any model containing A.*
3. *it does not divide over A.*
4. *there is a positive Boolean combination of A-conjugates of $\phi(x,a)$ which is equivalent to a consistent formula with parameters over A.*

(5) there is a global complete ϕ-type $\mathfrak{p}$ containing $\phi(x,a)$ which is definable over $\mathrm{acl}^{\mathrm{eq}}(A)$.

Proof. We already know that (1) implies (2) and it is clear that (2) yields (3) since a formula divides over a set if and only if it divides over a model containing such a set.

Now, we show that (3) implies (4). Suppose that $\phi(x,a)$ does not divide over A, set $q(y) = \mathrm{tp}(a/A)$ and let $\mathfrak{q}'(y)$ be a global ϕ^*-type such that $q(y) \cup \mathfrak{q}'(y)$ is consistent and $\mathfrak{q}'$ is definable over $\mathrm{acl}^{\mathrm{eq}}(A)$. In particular, it is definable over a model M and so $q(y) \cup \mathfrak{q}'(y)$ is finitely satisfiable in M by Lemma 3.10. Thus, there is a global type $\mathfrak{q}(y)$ extending $q(y) \cup \mathfrak{q}'(y)$ which is finitely satisfiable in M; note that $\mathfrak{q}_{|\phi^*} = \mathfrak{q}'$. On the other hand, by Lemma 3.5, there is a sequence $(c_i)_{i<n}$ with c_i realizing $\mathfrak{q}_{|M\cup\{c_j\}_{j<i}}$ such that $\mathfrak{q}_{|\phi^*} = \mathfrak{q}'$ is definable by a positive Boolean combination $\psi(x)$ of the formulas $\phi(x,c_i)$. Note that we can extend such a sequence to a sequence $(c_i)_{i<\omega}$ of realizations of $\mathfrak{q}_{|M}$ such that each c_i realizes $\mathfrak{q}_{|M\cup\{c_j\}_{j<i}}$, and so $(c_i)_{i<\omega}$ is M-indiscernible. In particular, it is A-indiscernible. Moreover, as $\mathfrak{q}(y)$ extends $q(y) = \mathrm{tp}(a/A)$, each $c_i \equiv_A a$ and so $\psi(x)$ is consistent since $\phi(x,a)$ does not divide over A. On the other hand, as $\mathfrak{q}_{|\phi^*} = \mathfrak{q}'$ we obtain that $\psi(x)$ is equivalent to a formula $\varphi(x)$ with parameters over $\mathrm{acl}^{\mathrm{eq}}(A)$ since any two definitions of $\mathfrak{q}_{|\phi^*}$ must be equivalent. Now, let $\chi(x)$ be the disjunction of all the finitely many A-conjugates of such a formula. Hence $\chi(x)$ is equivalent to a finite positive Boolean combination of A-conjugates of $\phi(x,a)$.

Now, we show that (4) yields (5). Let $\chi(x)$ be a consistent formula with parameters over A which is equivalent to a positive Boolean combination of A-conjugates of $\phi(x,a)$. By Lemma 3.7, there is a global ϕ-type $\mathfrak{q}(x)$ consistent with $\chi(x)$ (in fact, with any completion of it) which in addition is definable over $\mathrm{acl}^{\mathrm{eq}}(A)$. Thus, some A-conjugates of $\phi(x,a)$ belongs to $\mathfrak{q}$ and hence, the formula $\phi(x,a)$ belongs to an A-conjugate of $\mathfrak{q}$. Therefore, the formula $\phi(x,a)$ belongs to global ϕ-type which is definable over $\mathrm{acl}^{\mathrm{eq}}(A)$.

Finally, observe that (1) follows from (5) since a type $\mathfrak{p}$ given by (5) is definable over any model containing A and so it is finitely satisfiable in any model containing A by Lemma 3.10. This finishes the proof. $\square$

To conclude this section we consider the following more general situation. Let Δ be a finite set of formulas $\{\phi_i(x,y_i)\}_{i\leq n}$. By a Δ-formula over A we mean a formula of the form $\phi_i(x,a)$ or $\neg\phi_i(x,a)$ with a in A and $i \leq n$, and a complete Δ-type over A is nothing other than a maximal consistent set of Δ-formulas over A.

The following lemma allows us to code Δ-types in terms of local ψ-types for a suitable formula.

Lemma 3.12. *There is a formula $\psi_\Delta(x, y_0 \dots y_n z z_0 \dots z_{2n})$ such that*

(1) *If A has at least two elements, then each Δ-formula over A is equivalent to a positive ψ_Δ-formula over A.*
(2) *Any consistent positive ψ_Δ-formula over A is equivalent to a Δ-formula over A.*

Furthermore, if all formulas in Δ are stable, then so is ψ.

Proof. Set $\psi_\Delta(x, y_0 \dots y_n z z_0 \dots z_{2n})$ as

$$\bigwedge_{i\leq n} \left(\left(z = z_i \to \phi_i(x, y_i)\right) \wedge \left(z = z_{n+i} \to \neg\phi_i(x, y_i)\right) \wedge \bigvee_{i\leq 2n} z = z_i \wedge \bigwedge_{i<j} z_i \neq z_j.\right.$$

It is easy to see that this formula satisfies the requirements. □

As a consequence observe that global complete Δ-types are equivalent to complete ψ_Δ-types.

Corollary 3.13. *Let $\phi(x, y)$ and $\varphi(x, z)$ be stable formulas and suppose that $\phi(x, a)$ and $\varphi(x, b)$ divide over A. Then so does $\phi(x, a) \vee \varphi(x, b)$.*

Proof. Assume, as we may, that $A = \mathrm{acl}^{\mathrm{eq}}(A)$. Set $\chi(x, yz)$ be the stable formula $\phi(x, y) \vee \varphi(x, z)$ and set Δ to be $\{\phi(x, y), \varphi(x, z), \chi(x, yz)\}$. Now, let ψ_Δ be the stable formula given by Lemma 3.12. If the formula $\chi(x, ab)$ does not divide over A, then by Proposition 3.11 there exists a global χ-type $\mathfrak{p}$ containing $\chi(x, ab)$ which is definable over $\mathrm{acl}^{\mathrm{eq}}(A)$. Recall that a ϕ-type over a model is equivalent to a generalized ϕ-type. Seeing $\mathfrak{p}$ as a generalized ϕ-type, consider its restriction $\mathfrak{p}_{|A}$ over A, a generalized ϕ-type over A. By Lemma 3.7 applied to (a completion of) $\mathfrak{p}_{|A}$, we find some global ψ_Δ-type $\mathfrak{q}$ which is consistent with $\mathfrak{p}_{|A}$ and is definable over A. Note that $\mathfrak{q}$ is equivalent to a Δ-type $\mathfrak{q}'$ which of course is A-invariant, as $\mathfrak{q}$ is. Thus, its restriction $\mathfrak{q}'_{|\chi}$ is also A-invariant and so it is definable over A by Lemma 3.2 and Corollary 3.6. Again, seeing $\mathfrak{q}'_{|\chi}$ as a generalized ϕ-type, note that $\mathfrak{q}'_{|\chi}$ and $\mathfrak{p}$ extend p. Hence uniqueness of definable extensions over algebraically closed sets (i.e. Corollary 3.9) yields that $\mathfrak{q}'_{|\chi} = \mathfrak{p}$ and so $\chi(x, ab) \in \mathfrak{q}'$. Thus, we have that either $\phi(x, a) \in \mathfrak{q}'$ or $\varphi(x, b) \in \mathfrak{q}'$. Hence, one of these formulas does not divide over A by Proposition 3.11, a contradiction. □

As a consequence we obtain:

Corollary 3.14. *Let $\phi(x, y)$ be a stable formula. Then for any tuple a and any set A, either $\phi(x, a)$ or $\neg\phi(x, a)$ does not divide over A.*

4. Stable theories

In this section we present the general theory of (global) stability theory.

Definition 4.1. A theory is *stable* if all formulas are stable.

Remark 4.2. The imaginary expansion of a stable theory is again stable.

Remark 4.3. In a stable theory, a formula $\phi(x,a)$ forks over A if and only if it divides over A.

Proof. It suffices to show that if $\phi(x,a)$ forks over A, then it divides over A. Suppose that $\phi(x,a)$ forks over A and this is exemplified by $\varphi_i(x,b_i)$ with $i<n$. By Corollary 3.13, the disjunction of all $\varphi_i(x,b_i)$ divides over A and so does $\phi(x,a)$. □

From now on, we say that a type q is a *non-forking extension* of p if $p\subseteq q$ and q does not fork over the parameters over p. Therefore, the previous remark yields that a (complete) type over a set A does not fork over A and so, any type has non-forking extensions. As a consequence we obtain:

Proposition 4.4. *Assume that the theory is stable, let $p\in S(A)$ and let B be a subset of A. The following are equivalent:*

(1) *The type p does not fork over B.*
(2) *There is a global type extending p which is invariant over* $\mathrm{acl}^{\mathrm{eq}}(B)$
(3) *There is a global type extending p which is definable over* $\mathrm{acl}^{\mathrm{eq}}(B)$.

Proof. The equivalence between (2) and (3) is given by Lemma 1.2 and the fact that any complete ϕ-type is definable. Now, suppose that p does not fork over B and let $\mathfrak{p}$ be a global type extending p which does not fork over B. Thus, by Proposition 3.11 this type is finitely satisfiable in any model containing B. As for any formula $\phi(x,y)$, the ϕ-type $\mathfrak{p}_{|\phi}$ is definable, Lemma 3.2 yields that $\mathfrak{p}$ is definable over $\mathrm{acl}^{\mathrm{eq}}(B)$. For the converse, consider a global type $\mathfrak{p}$ extending p which is $\mathrm{acl}^{\mathrm{eq}}(B)$-invariant. Thus, since a formula divides over B if and only if it divides over $\mathrm{acl}^{\mathrm{eq}}(B)$, we obtain the result. □

Corollary 4.5 (Local Character)**.** *In a stable theory, any type $p\in S(A)$ does not fork over a subset B of A with $|B|\leq|T|$.*

Proof. As forking equals dividing, the type p does not fork over A and so it has a global extension $\mathfrak{p}$ which does not fork over A. Hence, it is definable over $\mathrm{acl}^{\mathrm{eq}}(A)$ by Proposition 4.4. Thus, there is a subset B of A of size at most $|T|$ which contain all canonical parameters of the definition of $\mathfrak{p}$, so $\mathfrak{p}$ is definable over $\mathrm{acl}^{\mathrm{eq}}(B)$ and therefore p does not fork over B by Proposition 4.4. □

Now we aim to analyze those types with a unique global non-forking extension.

Definition 4.6. A type is stationary if it has a unique global non-forking extension.

By a *strong type* we mean a type with parameters over an imaginary algebraically closed set. We denote the strong type $\mathrm{tp}(a/\mathrm{acl}^{\mathrm{eq}}(A))$ as $\mathrm{stp}(a/A)$.

Lemma 4.7. *In a stable theory, any strong type is stationary. In particular, any type over a model is stationary.*

Proof. The existence of non-forking extensions is clear since any type does not fork over its set of parameters. To show uniqueness, observe that if p is a complete type over $\mathrm{acl}^{\mathrm{eq}}(A)$, then for any two non-forking extensions $\mathfrak{p}_1$ and $\mathfrak{p}_2$ of p we have by Corollary 3.9 that their ϕ-types agree for any formula $\phi(x, y)$. Thus, they must coincide. □

Lemma 4.8. *In a stable theory, any two non-forking global extensions of a common type over A are A-conjugate.*

Proof. Let p be a complete type over A and consider two global non-forking extensions $\mathfrak{p}$ and $\mathfrak{q}$. Set $B = \mathrm{acl}^{\mathrm{eq}}(A)$ and observe that there is an automorphism $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$ mapping a realization of $\mathfrak{p}_{|B}$ to a realization of $\mathfrak{q}_{|B}$. Namely, fix a realization a of $\mathfrak{p}_{|B}$, a realization b of $\mathfrak{q}_{|B}$ and take $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$ with $\sigma(a) = b$. Thus, as σ fixes B setwise we get $\mathfrak{p}^{\sigma}_{|B} = (\mathfrak{p}_{|B})^{\sigma} = \mathfrak{q}_{|B}$. By Lemma 4.7, the type $\mathfrak{q}_{|B}$ is stationary and so its non-forking extensions $\mathfrak{q}$ and $\mathfrak{p}^{\sigma}$ coincide. This finishes the proof. □

Corollary 4.9 (Transitivity)**.** *Assume the theory is stable and let $A \subseteq B$. Then a global type $\mathfrak{p}$ does not fork over A if and only if it does not fork over B and also $\mathfrak{p}_{|B}$ does not fork over A.*

Proof. Observe that right to left is immediate from the properties of forking. For the other direction, let $\mathfrak{q}$ be a global extension of $\mathfrak{p}_{|B}$ which does not fork over A, and note that it also does not fork over B. Thus $\mathfrak{p}$ is a B-conjugate of $\mathfrak{q}$ by Lemma 4.8 and so it does not fork over A. □

Definition 4.10. Let $\mathfrak{p}$ be a definable global type. The *canonical base* of $\mathfrak{p}$, denoted by $\mathrm{Cb}(\mathfrak{p})$, is the definable closure of the collection of canonical parameters for the definitions $\mathrm{d}_{\mathfrak{p}} x\phi(x, y)$.

Observe that an automorphism of $\mathrm{Aut}(\mathbb{M})$ fixes the canonical parameter of $\mathrm{d}_{\mathfrak{p}} x\phi(x, y)$ if and only if it permutes the set $\mathfrak{p}_{|\phi}$. Hence, it is immediate to see the following using Lemma 1.2.

Remark 4.11. Let $\mathfrak{p}$ be a definable global type. The type $\mathfrak{p}$ is A-invariant if and only if $\mathrm{Cb}(\mathfrak{p})$ is contained in $\mathrm{dcl}^{\mathrm{eq}}(A)$. In particular, the canonical base $\mathrm{Cb}(\mathfrak{p})$ is the smallest definably closed set over which $\mathfrak{p}$ is invariant (or equivalently, definable).

Lemma 4.12. *Assume that the theory is stable and let $\mathfrak{p}$ be a global type. The following holds:*

1. *(1) The type $\mathfrak{p}$ does not fork over A if and only if $\mathrm{Cb}(\mathfrak{p})$ is contained in $\mathrm{acl}^{\mathrm{eq}}(A)$.*
2. *(2) The type $\mathfrak{p}$ does not fork over A and $\mathfrak{p}_{|A}$ is stationary if and only if $\mathrm{Cb}(\mathfrak{p})$ is contained in $\mathrm{dcl}^{\mathrm{eq}}(A)$.*

Proof. As we pointed out in Remark 4.11, the canonical base $\mathrm{Cb}(\mathfrak{p})$ is contained in a set B if and only if $\mathfrak{p}$ is invariant over B. Thus, (1) is an immediate consequence of Proposition 4.4. For (2), observe that $\mathfrak{p}$ does not fork over A and $\mathfrak{p}_{|A}$ is stationary if and only if $\mathfrak{p}$ is the unique non-forking extension of $\mathfrak{p}_{|A}$, and the latter is equivalent to saying that $\mathfrak{p}$ is A-invariant by Lemma 4.8. Hence, by Remark 4.11 we obtain the result. □

Definition 4.13. The set A is *independent* from B over C if for every finite tuple a in A, the type $\mathrm{tp}(a/BC)$ does not fork over C. We write $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ for this.

Theorem 4.14. *Assume that the theory is stable. Then the ternary relation* $\mathop{\smile\!\!\!\!\!\!\mid}$ *satisfies the following properties:*

1. *Invariance: If* $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ *and* $f \in \mathrm{Aut}(\mathbb{M})$*, then* $f(A) \mathop{\smile\!\!\!\!\!\!\mid}_{f(C)} f(B)$*.*
2. *Finite character:* $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ *if and only if* $A_0 \mathop{\smile\!\!\!\!\!\!\mid}_C B_0$ *for any finite subset* A_0 *of* A *and* B_0 *of* B*.*
3. *Extension: If* $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$*, then for any set* D *there is an automorphism* $f \in \mathrm{Aut}(\mathbb{M}/CB)$ *such that* $f(A) \mathop{\smile\!\!\!\!\!\!\mid}_C BD$*.*
4. *Local character: If* A *is finite and* B *is any set, there is a subset* C *of* B *with* $|C| \leq |T|$ *such that* $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$*.*
5. *Transitivity:* $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ *and* $A \mathop{\smile\!\!\!\!\!\!\mid}_{CB} D$ *if and only if* $A \mathop{\smile\!\!\!\!\!\!\mid}_C BD$*.*
6. *Symmetry: If* $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$*, then* $B \mathop{\smile\!\!\!\!\!\!\mid}_C A$*.*
7. *Algebraicity: If* $A \mathop{\smile\!\!\!\!\!\!\mid}_C A$*, then* $A \subseteq \mathrm{acl}^{\mathrm{eq}}(C)$*.*
8. *Stationarity: If* a *and* b *have the same strong type over* A *with* $a \mathop{\smile\!\!\!\!\!\!\mid}_A B$ *and* $b \mathop{\smile\!\!\!\!\!\!\mid}_A B$ *, then* $\mathrm{stp}(a/AB) = \mathrm{stp}(b/AB)$*.*

Proof. (1) and (2) follow from the definition and (3) by basic properties of forking noticing that the argument given in Lemma 2.2 works also when x an infinite tuple of variables. Observe that (4) is nothing else than Lemma 4.5. For (7), use the fact that the algebraic formula $x = a$ with a an element in A divides over any set C unless $a \in \mathrm{acl}^{\mathrm{eq}}(C)$. This can be seen using indiscernible sequences or for instance Proposition 4.4. Moreover, property (8) is a mere translation of uniqueness of non-forking extensions for strong types.

To prove (5), by finite character we may assume that A is a finite set enumerated by the tuple a. Then the statement follows easily applying twice Corollary 4.9. Namely, let $\mathfrak{p}$ be a global non-forking extension of $\mathrm{tp}(a/BCD)$. Then $\mathfrak{p}$ does not fork over C if and only if $\mathfrak{p}$ does not fork over BC and $\mathfrak{p}_{|BC}$ does not fork over C which, since $\mathfrak{p}$ does not fork over BCD, is equivalent to say that $\mathfrak{p}_{|BCD}$ does not fork over BC and $\mathfrak{p}_{|BC}$ does not fork over C.

For (6), suppose that $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ and assume first that $C = \mathrm{acl}^{\mathrm{eq}}(C)$. Observe that by definition we may assume that $C \subseteq B$. Moreover, if for a formula $\varphi(x, y, z)$ we have that $\varphi(x, b, z)$ divides over C and $c \in C$, then $\varphi(x, b, c)$ also divides over C. Thus $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ implies $AC \mathop{\smile\!\!\!\!\!\!\mid}_C B$ and so we may assume that $C \subseteq A$. Now, suppose towards a contradiction that there is a formula $\phi(x, y)$ such that $\phi(a, b)$ holds for some finite tuples $a \in A$ and

$b \in B$ and $\phi(a, y)$ forks over C. Now, as $\mathrm{tp}(a/C, b)$ does not fork over C, there exists a global type $\mathfrak{p}(x)$ extending $\mathrm{tp}(a/C, b)$ which is definable over C by Proposition 4.4. Let $\mathfrak{q}(y)$ be a non-forking extension of $\mathrm{tp}(b/C)$ which of course is also definable over C by Proposition 4.4. Hence, we have that

$$\phi(a, b) \text{ holds } \Leftrightarrow \phi(x, b) \in \mathfrak{p} \Leftrightarrow \mathrm{d}_{\mathfrak{p}} x\phi(x, y) \in \mathfrak{q},$$

and so $\mathrm{d}_{\mathfrak{q}} y\phi(x, y) \in \mathfrak{p}$ by Harrington's lemma. As $\mathrm{d}_{\mathfrak{q}} y\phi(x, y)$ has parameters over C, it belongs to $\mathrm{tp}(a/C)$ and thus $\phi(a, y) \in \mathfrak{q}$. However, this implies that $\mathfrak{q}$ forks over C, a contradiction. For the general case, observe that for any finite tuple a in A, the strong type $\mathrm{stp}(a/BC)$ does not fork over C; this can be easily seen using extension: take $a' \equiv_{BC} a$ with $a' \mathop{\smile\!\!\!\!\!\!\mid}_C \mathrm{acl}^{\mathrm{eq}}(BC)$ and then apply invariance. Thus, transitivity and finite character yield that $A \mathop{\smile\!\!\!\!\!\!\mid}_{\mathrm{acl}^{\mathrm{eq}}(C)} B$. ☐

Remark 4.15. In fact, the result above characterises stable theories. More precisely, a theory is stable if and only if there is a ternary relation defined among imaginary sets satisfying the properties (1) – (8) from Theorem 4.14 is stable. See [4, Theorem 36.10] or [1, Chapter 12] for a proof.

Definition 4.16. Let $p(x) \in S(A)$ be a stationary type in a stable theory. The canonical base of p is the canonical base of its unique global non-forking extension. We denote it by $\mathrm{Cb}(p)$. If $p = \mathrm{stp}(a/A)$ we simply write $\mathrm{Cb}(a/A)$.

Observe that $\mathrm{Cb}(a/A)$ is always contained in $\mathrm{acl}^{\mathrm{eq}}(A)$ by Lemma 4.12 since the unique non-forking extension of $\mathrm{stp}(a/A)$ does not fork over $\mathrm{acl}^{\mathrm{eq}}(A)$ nor over A.

Lemma 4.17. *For a tuple a and set $A \subseteq B$, the following are equivalent:*

(1) $a \mathop{\smile\!\!\!\!\!\!\mid}_A B$.
(2) $\mathrm{Cb}(a/B) \subseteq \mathrm{acl}^{\mathrm{eq}}(A)$.
(3) $\mathrm{Cb}(a/A) = \mathrm{Cb}(a/B)$.

Proof. Observe that (1) is equivalent to the statement that $\mathrm{stp}(a/B)$ does not fork over A. Hence (1) holds if and only if $\mathrm{stp}(a/A)$ and $\mathrm{stp}(a/B)$ have the same non-forking extension, which by definition implies (3). Moreover, (3) implies (2) by the remarks above and finally (1) follows from (2) by Lemma 4.12. ☐

Definition 4.18. A sequence $(a_i)_{i\in I}$ is said to be a Morley sequence in $p(x) \in S(A)$ if the sequence is indiscernible over A with a_i realizing p and $a_i \mathop{\smile\!\!\!\!\!\!\mid}_A (a_j)_{j<i}$ for any $i \in I$.

Lemma 4.19. *Given a type $p(x) \in S(A)$ and an ordinal α, there exists a Morley sequence $(a_i)_{i<\alpha}$ in p. Moreover, if p is stationary then any two Morley sequences in p are A-conjugated.*

Proof. Set $B = \mathrm{acl}^{\mathrm{eq}}(A)$. As p does not fork over A, there is a global type $\mathfrak{p}$ extending p which is B-invariant by Proposition 4.4. Then take $(a_i)_{i<\alpha}$ in a way that a_i realizes $\mathfrak{p}_{|B\cup\{a_j\}_{j<i}}$ and note that this sequence is indiscernible over A by Lemma 2.4. Moreover, as $\mathfrak{p}$ does not fork over A (by Proposition 4.4), neither does $\mathfrak{p}_{|B\cup\{a_j\}_{j<i}}$ and so $a_i \mathop{\smile\!\!\!\!\!\!|}_A (a_j)_{j<i}$.

For the second part, consider two Morley sequences $(a_i)_{i<\alpha}$ and $(b_i)_{i<\alpha}$ in p. We show inductively on β that $(a_i)_{i<\beta} \equiv_A (b_i)_{i<\beta}$; the initial case is obvious and the limit case is clear by induction. Let $(a_i)_{i<\beta}$ be an A-conjugate of $(b_i)_{i<\beta}$, i.e. there is some $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$ mapping $b_i \mapsto a_i$. Let $c = \sigma(b_\beta)$ and note that $c \mathop{\smile\!\!\!\!\!\!|}_A (a_i)_{i<\beta}$ by invariance since $b_\beta \mathop{\smile\!\!\!\!\!\!|}_A (b_i)_{i<\beta}$. As p has a unique non-forking extension over $A\cup\{a_i\}_{i<\beta}$ we obtain that a_β and c are realizations of such a unique non-forking extension of p and so they have the same type over $A \cup \{a_i\}_{i<\beta}$. Thus

$$(a_i)_{i<\beta}a_\beta \equiv_A (a_i)_{i<\beta}c \equiv_A (b_i)_{i<\beta}b_\beta,$$

as desired. □

Proposition 4.20. *Let p be a stationary type and let $(a_i)_{i<\omega}$ be a Morley sequence in p. Then, the canonical base of p is contained in $\mathrm{dcl}^{\mathrm{eq}}((a_i)_{i<\omega})$.*

Proof. Let p be a complete type over A and let $(a_i)_{i\leq\omega}$ be a Morley sequence in p. Thus $a_\omega \mathop{\smile\!\!\!\!\!\!|}_A (a_i)_{i<\omega}$ and hence there is a global type $\mathfrak{p}$ containing $\mathrm{tp}(a_\omega/A,(a_i)_{i<\omega})$ which does not fork over A. Hence $\mathfrak{p}$ is a non-forking extension of p by Corollary 4.9. On the other hand, the type $\mathrm{tp}(a_\omega/A,(a_i)_{i<\omega})$ is finitely satisfiable in $\{a_i\}_{i<\omega}$, so there exists a global type $\mathfrak{q}$ extending $\mathrm{tp}(a_\omega/A,(a_i)_{i<\omega})$ which is finitely satisfiable in $\{a_i\}_{i<\omega}$ and hence it is $\{a_i\}_{i<\omega}$-invariant. Thus $\mathfrak{q}$ does not fork over $\{a_i\}_{i<\omega}$ and so it is a non-forking extension of $\mathrm{tp}(a_\omega/A,(a_i)_{i<\omega})$. By Corollary 4.9 we obtain that $\mathfrak{q}$ is a non-forking extension of $p = \mathrm{tp}(a_\omega/A)$ and so $\mathfrak{p} = \mathfrak{q}$ since p is stationary. Therefore, as $\mathfrak{q}$ is $\{a_i\}_{i<\omega}$-invariant, we get $\mathrm{Cb}(p) = \mathrm{Cb}(\mathfrak{p}) \subseteq \mathrm{dcl}^{\mathrm{eq}}((a_i)_{i<\omega})$ by Remark 4.11. □

Lemma 4.21. *If $(a_i)_{i<\alpha}$ is independent over A, i.e. for each $i < \alpha$ we have that $a_i \mathop{\smile\!\!\!\!\!\!|}_A (a_j)_{j<i}$ then for any two disjoint subsets of subindexes I and J we have that $(a_i)_{i\in I} \mathop{\smile\!\!\!\!\!\!|}_A (a_i)_{i\in J}$.*

Proof. Left to the reader. □

Corollary 4.22. *Let p be a stationary type and let $(a_i)_{i<\omega+\omega}$ be a Morley sequence in p. Then*

$$\mathrm{acl}^{\mathrm{eq}}(\mathrm{Cb}(p)) = \mathrm{acl}^{\mathrm{eq}}((a_i)_{i<\omega}) \cap \mathrm{acl}^{\mathrm{eq}}((a_{\omega+i})_{i<\omega}).$$

Proof. Let $p(x) \in S(A)$ be a stationary type; thus its unique global non-forking extension is A-invariant and so $\mathrm{Cb}(p) \subseteq \mathrm{dcl}^{\mathrm{eq}}(A)$. Hence, for each $i < \omega + \omega$ we have that $a_i \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{Cb}(p)} A$ and so $a_i \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{Cb}(p)} (a_j)_{j<i}$ by transitivity. Whence the sequence $(a_i)_{i<\omega+\omega}$ is independent over $\mathrm{Cb}(p)$ and so $(a_i)_{i<\omega} \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{Cb}(p)} (a_{\omega+i})_{i<\omega}$. Therefore

$$\mathrm{acl}^{\mathrm{eq}}((a_i)_{i<\omega}) \cap \mathrm{acl}^{\mathrm{eq}}((a_{\omega+i})_{i<\omega}) \subseteq \mathrm{acl}^{\mathrm{eq}}(\mathrm{Cb}(p)).$$

The other direction follows from Proposition 4.20. □

5. Superstable theories

Definition 5.1. A theory is *superstable* if it is stable and for any finite tuple a and any set B, there is some finite subset C of B such that $a \mathop{\smile\!\!\!\!\!\!|}_C B$.

Definition 5.2. We define $\mathrm{U}(p) \geq \alpha$ for a type p by recursion on α:

- $\mathrm{U}(p) \geq 0$
- $\mathrm{U}(p) \geq \alpha + 1$ if there is a forking extension q of p with $\mathrm{U}(q) \geq \alpha$.
- $\mathrm{U}(p) \geq \gamma$ for a limit ordinal γ if $\mathrm{U}(p) \geq \alpha$ for all $\alpha < \gamma$.

We define the U-rank of p (also called Lascar rank) as the maximal α such that $\mathrm{U}(p) \geq \alpha$. If there is no maximum we set $\mathrm{U}(p) = \infty$. To ease notation, we write $\mathrm{U}(a/A)$ for $\mathrm{U}(\mathrm{tp}(a/A))$.

Remark 5.3. A type p has U-rank 0 if and only if it is algebraic.

Lemma 5.4. *Assume the theory is stable and let $q \in S(B)$ be an extension of $p \in S(A)$. Then:*

1. *If q is a non-forking extension of p, then $\mathrm{U}(p) = \mathrm{U}(q)$.*
2. *If $\mathrm{U}(p) = \mathrm{U}(q) < \infty$, then q is a non-forking extension of p.*

Proof. For (1) observe first that $\mathrm{U}(p) \geq \mathrm{U}(q)$ by definition since a forking extension of q is also a forking extension of p. For the other inequality, we proceed by induction on α. We show that if $\mathrm{U}(p) \geq \alpha$ then $\mathrm{U}(q) \geq \alpha$. Observe that the initial and the limit cases are clear. Suppose now that $\mathrm{U}(p) \geq \alpha + 1$ and let p' be a forking extension of p with $\mathrm{U}(p') \geq \alpha$. As $p'_{|A} = q_{|A}$, there is an automorphism $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$ mapping a realization of p' to a realization of q and so p'^{σ} and q have a common realization. Let b be a realization of $p'^{\sigma} \cup q$ and suppose that p'^{σ} and q are complete types over C' and B respectively. Observe by invariance that p'^{σ} is also a forking extension of p. Thus, in terms of forking calculus we have that $b \mathop{\smile\!\!\!\!\!\!|}_A B$ and $b \not\mathop{\smile\!\!\!\!\!\!|}_A C'$. By extension, there is some $C'' \equiv_{Ab} C'$ with $C'' \mathop{\smile\!\!\!\!\!\!|}_{Ab} B$; notice that $\mathrm{tp}(b/C'')$ forks over A by invariance and $\mathrm{U}(b/C'') = \mathrm{U}(p') \geq \alpha$. Thus $C''b \mathop{\smile\!\!\!\!\!\!|}_A B$ and so $b \mathop{\smile\!\!\!\!\!\!|}_{C''} B$ by transitivity and invariance. Therefore, as $\mathrm{tp}(b/BC'')$ is a non-forking extension of $\mathrm{tp}(b/C'')$ and the latter has U-rank at least α, the induction hypothesis yields that $\mathrm{U}(b/BC'') \geq \alpha$. On the other hand, observe that $b \not\mathop{\smile\!\!\!\!\!\!|}_B C''$ as otherwise we would have that $b \mathop{\smile\!\!\!\!\!\!|}_A C''$ by transitivity and finite character, contradicting the fact that $\mathrm{tp}(b/C'')$ forks over A. Hence $\mathrm{U}(b/B) \geq \alpha + 1$, as desired.

To conclude, note that (2) follows from the definition. □

Remark 5.5. Since every type does not fork over a subset of cardinality at most $|T|$, there are at most $2^{|T|}$ different U-ranks. As they form an initial segment of ordinals, all ordinal ranks are smaller than $(2^{|T|})^+$. Hence, there is an ordinal α such that $\mathrm{U}(p) \geq \alpha$ implies $\mathrm{U}(p) = \infty$.

Proposition 5.6. *A stable theory is superstable if and only if the* U*-rank of any type is ordinal-valued.*

Proof. If there is a type p with $\mathrm{U}(p) = \infty$, then in particular $\mathrm{U}(p) \geq \alpha + 1$ with α given by the remark. Thus there is a forking extension q of p with $\mathrm{U}(q) = \infty$. Iterating this process, we obtain an infinite sequence $(p_i(x))_{i<\omega}$ of types such that their union is consistent (by compactness) and each p_{i+1} is a forking extension of p_i. However, by construction $\bigcup_{i<\omega} p_i$ forks over any finite subset of its domain, a contradiction.

For the other direction, given a complete type $\mathrm{tp}(a/A)$ let B be a finite subset of A such that $\mathrm{U}(a/B)$ is minimal. It then follows that $\mathrm{U}(a/BA') = \mathrm{U}(a/B)$ for any finite subset A' of A and so $a \mathop{\smile\!\!\!\!\!\!|}_B A'$ by Proposition 5.4. Thus by finite character we get $a \mathop{\smile\!\!\!\!\!\!|}_B A$. □

Recall that every ordinal α can be written in the Cantor normal form as a finite sum $\omega^{\alpha_1} \cdot n_1 + \ldots + \omega^{\alpha_k} \cdot n_k$ for ordinals $\alpha_1 > \ldots > \alpha_k$ and natural numbers $n_1, \ldots, n_k$. Moreover, this sum is unique if we require all summands to be non-zero. If $\beta = \omega^{\alpha_1} \cdot m_1 + \ldots + \omega^{\alpha_k} \cdot m_k$, then $\alpha \oplus \beta$ is defined to be $\omega^{\alpha_1} \cdot (n_1 + m_1) + \ldots + \omega^{\alpha_k} \cdot (n_k + m_k)$. In fact, the function $\oplus$ is the smallest symmetric strictly increasing function f among pairs of ordinals such that $f(\alpha, \beta + 1) = f(\alpha, \beta) + 1$.

Theorem 5.7 (Lascar Inequalities)**.**

$$\mathrm{U}(a/Ab) + \mathrm{U}(b/A) \leq \mathrm{U}(ab/A) \leq \mathrm{U}(a/Ab) \oplus \mathrm{U}(b/A).$$

Proof. We first prove by induction on α that $\mathrm{U}(b/A) \geq \alpha$ implies that $\mathrm{U}(ab/A) \geq \mathrm{U}(a/Ab) + \alpha$. The case $\alpha = 0$ is left to the reader and the limit case is obvious. Thus, assume that the inequality holds for α and suppose that $\mathrm{U}(b/A) \geq \alpha + 1$. Thus, we can find some set B extending A such that $b \not\mathop{\smile\!\!\!\!\!\!|}_A B$ and $\mathrm{U}(b/B) \geq \alpha$. By extension there is some $B' \equiv_{Ab} B$ with $B' \mathop{\smile\!\!\!\!\!\!|}_{Ab} a$ and so $\mathrm{U}(a/Ab) = \mathrm{U}(a/B'b)$. Moreover, note by invariance that $\mathrm{U}(b/B') \geq \alpha$ and so, the inductive hypothesis yields that

$$\mathrm{U}(ab/B') \geq \mathrm{U}(a/B'b) + \alpha = \mathrm{U}(a/Ab) + \alpha.$$

On the other hand, as $B' \not\mathop{\smile\!\!\!\!\!\!|}_A ab$ by transitivity we obtain that $\mathrm{U}(ab/A) \geq \mathrm{U}(ab/B') + 1$ by Lemma 5.4 and hence $\mathrm{U}(ab/A) \geq \mathrm{U}(a/Ab) + \alpha + 1$, as desired.

Now, to show the second part we prove by induction on α that $\mathrm{U}(ab/A) \geq \alpha$ implies that $\mathrm{U}(a/Ab) \oplus \mathrm{U}(b/A) \geq \alpha$. This is clear for $\alpha = 0$ and the limit case, so assume that it holds for α and suppose that $\mathrm{U}(ab/A) \geq \alpha + 1$. Thus, we can find some superset B of A such that $\mathrm{U}(ab/B) \geq \alpha$ and $ab \not\mathop{\smile\!\!\!\!\!\!|}_A B$. By induction $\mathrm{U}(a/Bb) \oplus \mathrm{U}(b/B) \geq \alpha$. Now, if $b \not\mathop{\smile\!\!\!\!\!\!|}_A B$ then using Lemma 5.4

we obtain that

$$\mathrm{U}(a/Ab) \oplus \mathrm{U}(b/A) \geq \mathrm{U}(a/Bb) \oplus \mathrm{U}(b/B) + 1 \geq \alpha + 1.$$

Otherwise $b \mathop{\smile\!\!\!\!\!\!\!|}_A B$ and so $a \not\mathop{\smile\!\!\!\!\!\!\!|}_{Ab} B$ by transitivity. Thus, we get $\mathrm{U}(a/Ab) \geq \mathrm{U}(a/Bb) + 1$ and so $\mathrm{U}(a/Ab) \oplus \mathrm{U}(b/A) \geq \alpha + 1$. □

6. Stable groups and homogeneous spaces

A set is *type-definable* if it is given as the intersection of definable sets, and it is *relatively* definable in a superset X if it is the intersection of a definable set with X. A type-definable group $(G, \cdot)$ is a type-definable set G together with a relatively definable subset of $G \times G \times G$ which is the graph of the group operation. A subgroup H of a group G is relatively definable if the underlying set H is relatively definable in G. Note that the quotient of a type-definable group by a relatively definable normal subgroup is still type-definable.

We assume the ambient theory to be stable.

Let G be a type-definable group acting transitively and definably on a type-definable set S. Assume that everything is defined without parameters in a stable theory. We denote by $G(x)$ and $S(x)$ the partial types type-defining G and S respectively. Observe that the group G acts naturally on the collection of partial types $\pi(x)$ extending $S(x)$. Namely, if $\pi(x)$ is a partial type extending $S(x)$ we write

$$g \cdot \pi(x) = \{\varphi(g^{-1} \cdot x) : \varphi(x) \in \pi(x)\}.$$

Hence, for an element $a \in S$ we have that a realizes $\pi(x)$ if and only if $g \cdot a$ realizes $g \cdot \pi(x)$. Formally, observe that the formula $\varphi(g^{-1} \cdot x)$ corresponds to a formula $\psi(x, g)$ with $\psi(x, y) = \varphi(y^{-1} \cdot x) = \exists z (z = y^{-1} \cdot x \wedge \varphi(z))$.

6.1. Generic sets and types.

Definition 6.1. A relatively definable subset X of S is *generic* if there are elements $g_1, \ldots, g_n$ of G such that S is contained in the union of all $g_i X$. We say that a partial type $\pi(x)$ extending $S(x)$ is generic if any relatively definable subset of S containing the realizations of π is generic.

Lemma 6.2. *Let X be a relatively definable subset of S given by a formula $\varphi(x, b)$. Then the following holds:*

(1) Either X or $S \setminus X$ is generic.
(2) The set X is generic if and only if for any element $g \in G$ the set $S(x) \cup \{g \cdot \varphi(x, b)\}$ does not fork over $\emptyset$.

Proof. We first introduce an auxiliary two-sorted structure $\mathbb{M}_0 = (S, G, R)$ where $R(x, y)$ is a binary relation interpreted in $\mathbb{M}_0$ as follows: $R(x, y)$ if $x \in S$, $y \in G$ and $y^{-1} \cdot x \in X$. Let T_0 be the theory of $\mathbb{M}_0$ and observe that $R(x, y)$ is a stable formula in T_0; otherwise by saturation of $\mathbb{M}$ there is a sequence $(a_i, g_i)_{i<\omega}$ with $a_i \in S$, $g_i \in G$ and in the original theory $\varphi(g_j^{-1} \cdot a_i, b)$ holds if and only if $i < j$. However, the sequence $(a_i, bg_i)_{i<\omega}$ would witness that the formula $\psi(x, yz) = \varphi(z^{-1} \cdot x, y)$ is not stable, a contradiction.

Given an element $g \in G$ let σ_g be the map taking $s \in S \mapsto g \cdot s$ and $h \in G \mapsto gh$. It is easy to see that $\sigma_g \in \mathrm{Aut}(\mathbb{M}_0)$ and so, since G acts on S transitively there is a unique type without parameters in T_0 implying $S(x)$. Now, let 1 be the identity element of G. Note that $R(x, 1)$ defines X in the structure $\mathbb{M}_0$ and that $\neg R(x, 1)$ defines $S \setminus X$. Moreover, any $\emptyset$-conjugate of $R(x, 1)$ and $\neg R(x, 1)$ under the action of $\mathrm{Aut}(\mathbb{M}_0)$ has the form $R(x, g)$ and $\neg R(x, g)$ respectively.

Now, by Corollary 3.14 either $R(x, 1)$ or $\neg R(x, 1)$ does not divide over $\emptyset$. Assume first that $R(x, 1)$ does not divide and so by Proposition 3.11, some positive Boolean combination of $\emptyset$-conjugates (with respect to the automorphism group of some elementary extension $\mathbb{M}_0'$ of $\mathbb{M}_0$) of $R(x, 1)$ is consistent and $\emptyset$-definable. Consequently, since $S(x)$ determines the unique type without parameters in T_0, we have that $S(x)$ is equivalent to a positive Boolean combination of formulas of the form $R(x, g)$ with g in $G(\mathbb{M}_0')$ and so we can find elements $g_1, \ldots, g_n$ in G such that $S(x)$ implies the disjunction $\bigvee_{i \le n} R(x, g_i)$. Therefore, the set X is generic. Similarly, we can get that $S \setminus X$ is generic whenever $\neg R(x, 1)$ does not divide over $\emptyset$ and hence, we deduce that either X or $S \setminus X$ is generic. This shows (1).

For (2), assume first that X is not generic but for any $g \in G$ the partial type $S(x) \cup \{g \cdot \varphi(x, b)\}$ does not fork over $\emptyset$. Thus, in T_0 the formula $R(x, 1)$ divides over $\emptyset$ by the paragraph above and so, there is an infinite $\emptyset$-indiscernible sequence $(g_i')_{i<\omega}$ in some elementary extension of $\mathbb{M}_0$ such that $\{R(x, g_i')\}_{i<\omega}$ is inconsistent. Thus, there is some natural number k for which this set of formulas is k-inconsistent. Hence, by compactness it is consistent with the ambient theory that there is a sequence $(g_i)_{i<\kappa}$ with $\kappa > 2^{|T|}$ such that $S(x) \cup \{g_i \cdot \varphi(x, b)\}$ is k-inconsistent. As our model $\mathbb{M}$ is saturated, we may take such a sequence inside $\mathbb{M}$. By assumption each of the sets $S(x) \cup \{g_i \cdot \varphi(x, b)\}$ does not fork over $\emptyset$ and so, each of them is contained in a global type $\mathfrak{p}_i$ which does not fork over $\emptyset$. By Lemma 4.7, each $\mathfrak{p}_i$ is determined by its restriction to $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$. However, as $S(x) \cup \{g_i \cdot \varphi(x, b)\}_{i<\kappa}$ is k-inconsistent we obtain that there are κ many types $\mathfrak{p}_i$. However, there are only $2^{|T|}$ many types over $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$, a contradiction.

To show the other direction in point (2), we introduce another auxiliary structure $\mathbb{M}_1$ which we define as follows: first let Γ be the collection of sets $S \cap g \cdot \varphi(\mathbb{M}, a)$ where a is a tuple of parameters and $g \in G$, and $\epsilon(x, u)$ is a binary relation on $S \times \Gamma$ given by the membership relation. Let $\mathbb{M}_1$ be the two-sorted structure (S, Γ, ϵ). Let T_1 be its theory and observe that $\epsilon(x, u)$ is stable as otherwise there is a sequence $(s_i, Y_i)_{i<\omega}$ with $s_i \in S$ and $Y_i = S \cap g_i \cdot \varphi(\mathbb{M}, a_i)$ such that $s_i \in S \cap g_j \cdot \varphi(\mathbb{M}, a_j)$ if and only if $i < j$. However, the sequence $(s_i, g_i a_i)_{i<\omega}$ witnesses that the formula $\psi(x, yz) = \varphi(z^{-1} \cdot x, y)$ is not stable, yielding a contradiction. Moreover, for each $g \in G$ the map τ_g which takes $s \in S \mapsto g \cdot s$ and $Y \in \Gamma \mapsto g \cdot Y$ is an automorphism of $\mathbb{M}_1$.

Now, let $\mathfrak{p}_1 \in S_\epsilon(\mathbb{M}_1)$ be a ϵ-type in T_1 extending $S(x)$ which is definable over $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$ in the sense of T_1. Observe that such a type exists by Lemma 3.7. Moreover note that a type $\mathfrak{q}_1 \in S_\epsilon(\mathbb{M}_1)$ can be identified with the type $\mathfrak{q} \in S_\psi(\mathbb{M})$ that implies $S(x) \cup \{g \cdot \varphi(x, a)\}$ whenever $x \epsilon S \cap g \cdot \varphi(\mathbb{M}, a)$ belongs to $\mathfrak{q}_1$. Let $\mathfrak{p}$ be the ψ-type associated to $\mathfrak{p}_1$ and note that since $\mathfrak{p}_1$ is definable over $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$ in T_1, then so is $\mathfrak{p}$ in the ambient theory. To see the latter, note that any automorphism from $\mathrm{Aut}(\mathbb{M})$ induces an automorphism from $\mathrm{Aut}(\mathbb{M}_1)$; thus as $\mathfrak{p}$ is definable by stability, the canonical parameter of its definition has a finite orbit under the action of $\mathrm{Aut}(\mathbb{M})$ as so does the canonical parameter of the definition of $\mathfrak{p}_1$ under the action of $\mathrm{Aut}(\mathbb{M}_1)$. Now, suppose that X is generic, so there is some translate $h \cdot \varphi(x, b)$ that belongs to $\mathfrak{p}$ and so the formula $x \epsilon S \cap h \cdot \varphi(\mathbb{M}, b)$ belongs to $\mathfrak{p}_1$. Thus $x \epsilon S \cap \varphi(\mathbb{M}, b)$ belongs to $\mathfrak{p}_1^{\tau_{h^{-1}}}$, which is definable over $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$ by invariance in T_1. Hence, we obtain that $h^{-1} \cdot \mathfrak{p}$ is definable over $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$ and so $S(x) \cup \{\varphi(x, b)\}$ does not divide over $\emptyset$. As for any $g \in G$, the set $g \cdot X$ is generic, the argument above yields the result. □

Lemma 6.3. *The set of relatively definable subsets of S which are non-generic form an ideal.*

Proof. It is clear that such a set contains the empty-set and it is closed under subsets. To check that it is closed under unions, consider two non-generic subsets X and Y of S. If $X \cup Y$ is generic, then there are elements $g_0, \ldots, g_n$ in G such that

$$S = \bigcup_{i \leq n} g_i(X \cup Y) = \bigcup_{i \leq n} g_i X \cup \bigcup_{i \leq n} g_i Y.$$

As X is non-generic, the union $\bigcup_{i \leq n} g_i X$ cannot be generic and so, its complement $S \setminus \bigcup_{i \leq n} g_i X$ is generic by Lemma 6.2. However, the latter is clearly contained in $\bigcup_{i \leq n} g_i Y$. It then follows that $\bigcup_{i \leq n} g_i Y$ and so Y are generic, a contradiction. □

Consequently, by Lemma 2.2 we obtain:

Corollary 6.4. *Let $\pi(x)$ be a partial type extending $S(x)$. If π is generic then there is a generic type extending π. In particular, there are generic global types.*

Proposition 6.5. *Let $p(x)$ be a complete type over A extending $S(x)$. The following are equivalent:*

(1) The complete type p is generic.
(2) For any element g in G we have that $g \cdot p(x)$ does not fork over $\emptyset$.

(3) For any element g in G we have that $g \cdot p(x)$ does not fork over A.
(4) For any element $g \in G$ and a realizing p, if $a \mathop{\smile\!\!\!\!\!\!|}_A g$ then $g \cdot a \mathop{\smile\!\!\!\!\!\!|} A, g$.

Proof. The equivalence between (1) and (2) is given by Lemma 6.2. Moreover, the equivalence between (1) and (3) follows from the equivalence between (1) and (2) after naming A as constants. Additionally, it is clear that (4) implies (2): as $p(x)$ does not fork over A, there is some realization a of p with $a \mathop{\smile\!\!\!\!\!\!|}_A g$. Thus $g \cdot a \mathop{\smile\!\!\!\!\!\!|} A, g$ by assumption and hence $g \cdot p$ does not fork over $\emptyset$ since $g \cdot a$ realizes it. Finally, to see that (1) yields (4), let $\mathfrak{q}$ be a generic global type extending p and note that it does not fork over A by point (3). If $a \mathop{\smile\!\!\!\!\!\!|}_A g$, then $\mathrm{tp}(a/A, g)$ has a global extension $\mathfrak{p}$ which does not fork over A and hence $\mathfrak{p}$ and $\mathfrak{q}$ must be A-conjugates by Lemma 4.8. In particular, it follows that $\mathfrak{p}$ is generic and so is $\mathrm{tp}(a/A, g)$. Hence, by the second point we get that $\mathrm{tp}(g \cdot a/A, g)$ does not fork over $\emptyset$ which means $g \cdot a \mathop{\smile\!\!\!\!\!\!|} A, g$. □

We say that an element a is *generic* in S over A if $a \in S$ and $\mathrm{tp}(a/A)$ is a generic type (for the action of G on S). Similarly, we say that g is *generic* in G over A if $g \in G$ and $\mathrm{tp}(g/A)$ is a (left or right) generic type of G.

Remark 6.6. Using Proposition 6.5 one can easily see that the following properties hold for any element a of S:

(1) For any B: a is generic over AB iff a is generic over A and $a \mathop{\smile\!\!\!\!\!\!|}_A B$.
(2) If $g \in G$ and a is generic over A, g, then $g \cdot a$ is generic over A, g as well.

Observe that all previous results apply when $S = G$ and G acts on itself regularly by left (or right) translation. A priori we shall distinguish between the notion of being left generic (as defined before) and right generic, i.e. finitely many right translates cover the group. Nevertheless, it turns out that both notions agree.

Lemma 6.7. *The following holds for any element g of G and any set of parameters A:*

(1) If g is left generic over A, then so is g^{-1}.
(2) The element g is left generic over A iff it is right generic over A.

Proof. We use Proposition 6.5(4) to characterise genericity.

(1) Suppose that g is left generic over A, and let h be a realization of $\mathrm{tp}(g/A)$ independent from g over A. Then $gh \mathop{\smile\!\!\!\!\!\!|} A, g$ since g is generic over A and so $(gh)^{-1} \mathop{\smile\!\!\!\!\!\!|}_A g$. Now, using the previous remark, we see that g is left generic over $A, (gh)^{-1}$, and so is $h^{-1} = (gh)^{-1}g$. Thus h^{-1} is generic over A and so is g^{-1} by invariance.

(2) Assume that g is left generic over A. Let h be such that $g \mathop{\smile\!\!\!\!\!\!|}_A h$; thus $g^{-1} \mathop{\smile\!\!\!\!\!\!|}_A h^{-1}$. By (1), we have that g^{-1} is left generic over A and so

$h^{-1}g^{-1} \mathop{\smile\!\!\!\!\!\!|}\, A, h^{-1}$. Hence $gh \mathop{\smile\!\!\!\!\!\!|}\, A, h$, yielding that g is right generic over A. The other direction is similar. □

Lemma 6.8. *Let a be an element in S and let $g \in G$ be generic in G over A, a then $g \cdot a$ is generic in S over A.*

Proof. To ease notation, assume that $A = \emptyset$. Let $h \in G$ be such that $h \mathop{\smile\!\!\!\!\!\!|}\, g \cdot a$ and by extension we can find some $h' \equiv_{g \cdot a} h$ with $h' \mathop{\smile\!\!\!\!\!\!|}\, g, a$. Thus $h' \mathop{\smile\!\!\!\!\!\!|}_a g$ and so $g \mathop{\smile\!\!\!\!\!\!|}_a h'$. Hence, genericity of g over a yields that $h' \cdot g \mathop{\smile\!\!\!\!\!\!|}\, a, h'$. But $h' \mathop{\smile\!\!\!\!\!\!|}\, a$ so $h' \cdot g, a \mathop{\smile\!\!\!\!\!\!|}\, h'$ by transitivity and so $h' \cdot g \cdot a \mathop{\smile\!\!\!\!\!\!|}\, h'$. Therefore, we obtain the result by invariance. □

6.2. **Stabilizers and connected components.**

Proposition 6.9. *There is a minimal type-definable subgroup of G which has bounded index in G.*

Proof. Set $S = G$ and let H be a type-definable subgroup of bounded index in G. By compactness, any relatively definable set containing H is generic in G and so H is generic as a partial type. Thus, there is global generic type $\mathfrak{p}$ extending the partial type $H(x)$. Observe that $g \cdot \mathfrak{p}$ extends the partial type defining the coset gH and so the index of H in G is bounded by the number of translates of $\mathfrak{p}$. As generic types do not fork over $\emptyset$, there are at most $2^{|T|}$ many of them (namely, at most as many as there are types over $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$) and thus we have an absolute bound on the index of H in G. Thus, the intersection of all type-definable subgroups (over any set of parameters) is indeed a small intersection (of size at most $2^{|T|}$) and so it is type-definable. □

Let G^0 denote the intersection of all relatively definable finite index subgroups of G, which we call the *connected component* of G. We say that the group is connected when it is equal to its connected component. Observe that G^0 has bounded index in G and it is generic as a partial type.

Definition 6.10. Let $\mathfrak{p}$ be a global type extending $S(x)$. By the stabilizer of $\mathfrak{p}$ we mean the group

$$\mathrm{Stab}(\mathfrak{p}) = \{g \in G : g \cdot \mathfrak{p} = \mathfrak{p}\}.$$

Remark 6.11. Let $\mathfrak{p}$ be a global type extending $S(x)$ definable over A. Then $g \in \mathrm{Stab}(\mathfrak{p})$ iff whenever a realizes $\mathfrak{p}_{|A,g}$ then $g \cdot a$ realizes $\mathfrak{p}_{|A,g}$.

Remark 6.12. The stabilizer of a complete type is an intersection of relatively definable subgroups. Namely, let the ϕ-stabilizer of $\mathfrak{p}$ be defined as

$$\mathrm{Stab}(\mathfrak{p}, \phi) = \{g \in G : \forall y \big(\mathrm{d}_{\mathfrak{p}} x \phi(x, y) \leftrightarrow \mathrm{d}_{\mathfrak{p}} x \phi(g^{-1} \cdot x, y)\big)\},$$

which is a relatively definable subgroup. Therefore, we obtain that

$$\mathrm{Stab}(\mathfrak{p}) = \bigcap_{\phi(x,y) \in L} \mathrm{Stab}(\mathfrak{p}, \phi)$$

and so the stabilizer of $\mathfrak{p}$ is equal to the intersection of relatively definable subgroups.

Proposition 6.13. *The following holds:*

1. *(1) The connected component G^0 of G acts trivially on the set of generic types (for the action of G on S). That is, for any global generic type $\mathfrak{p}$ we have that G^0 is contained in* $\mathrm{Stab}(\mathfrak{p})$.
2. *(2) The group G acts transitively on the set of generic global types.*
3. *(3) Suppose that G acts regularly on S and let $\mathfrak{p}$ be a global type extending $S(x)$. Then, the type $\mathfrak{p}$ is generic if and only if $G^0 = \mathrm{Stab}(\mathfrak{p})$.*

Proof. Through the proof we use without notice properties of generic elements and types.

(1) Let F be the intersection of all stabilizers of global generic types. Thus G/F acts faithfully on the set of generic global types, and since there is only a bounded number many of them, the group G/F has bounded cardinality. As each stabilizer is equal to an intersection of relatively definable subgroups, and each of these relatively definable subgroups has finite index by compactness, the subgroup G^0 is contained in any stabilizer. Thus, it acts trivially on the set of generic global types.

(2) Let $\mathfrak{p}$ and $\mathfrak{q}$ be two generic global types and let a and b be two realizations of their restriction to some model M, respectively. By transitivity of the action, choose some element g of G such that $g \cdot a = b$. Pick some generic element h of G^0 for the action of G on itself by right translation, which in addition is independent from M, a, b, g. Thus $hg \mathop{\smile\!\!\!\!\!\!\!\;|}_M a$ (using the fact that h is generic over M, a and $h \mathop{\smile\!\!\!\!\!\!\!\;|}_{M,a} g^{-1}$), so a realizes $\mathfrak{p}_{|M,hg}$ and hence $hg \cdot a$ realizes $hg \cdot (\mathfrak{p}_{|M,hg}) = (hg \cdot \mathfrak{p})_{|M,hg}$. Now, note that $h \in \mathrm{Stab}(\mathfrak{q})$ by (1). Thus, as b realizes $\mathfrak{q}_{|M,h}$ since $h \mathop{\smile\!\!\!\!\!\!\!\;|}_M b$, the element $hg \cdot a = h \cdot b$ realizes $\mathfrak{q}_{|M}$ by Remark 6.11. Consequently, we have that $(hg \cdot \mathfrak{p})_{|M} = \mathfrak{q}_{|M}$ and so $hg \cdot \mathfrak{p} = \mathfrak{q}$ by stationarity, as desired. In fact, note also that $g \cdot \mathfrak{p} = \mathfrak{q}$ since $h \in \mathrm{Stab}(g \cdot \mathfrak{p})$ by (1).

(3) Suppose first that $\mathfrak{p}$ is generic. By (1) it remains to show that $\mathrm{Stab}(\mathfrak{p})$ is contained in G^0. To do so, note that for each relatively definable subgroup H of finite index in G, the generic type $\mathfrak{p}$ must imply that $x \in H \cdot a$ for some element a in S since H has only finitely many orbits. A compactness argument yields the existence of some element b in S such that $\mathfrak{p}$ implies $x \in G^0 \cdot b$. Thus, the type $g \cdot \mathfrak{p}$ implies $x \in g \cdot (G^0 \cdot b)$. Normality of G^0 yields that $g \cdot (G^0 \cdot b) = G^0 \cdot (g \cdot b)$. Hence, assuming that g belongs to $\mathrm{Stab}(\mathfrak{p})$, we obtain that b and $g \cdot b$ belongs to the same orbit. Whence, we can find some h_1, h_2 in G^0 such that $(h_1 g) \cdot b = h_2 \cdot b$ and so g belongs to G^0 by regularity. This shows that $\mathrm{Stab}(\mathfrak{p}) = G^0$.

To prove the other direction, let M be a small model such that $\mathfrak{p}$ does not fork over M. By Remark 6.6 it suffices to show that $\mathfrak{p}_{|M}$ is generic. Let a be a realization of $\mathfrak{p}_{|M}$ and let g be a generic element of G over M, a which in addition belongs to G^0; note that this element exists since G^0 is generic as partial type. The assumption implies that $(g \cdot \mathfrak{p})_{|M,g} = \mathfrak{p}_{|M,g}$. Now, as g is generic over M, a, we have that $g \mathop{\smile\!\!\!\!\!\!\mid}_M a$, and so a realizes $(g \cdot \mathfrak{p})_{|M,g} = \mathfrak{p}_{|M,g}$. In particular, it satisfies $g \cdot \mathfrak{p}_{|M}$ and so $g^{-1} \cdot a$ realizes $\mathfrak{p}_{|M}$. Therefore, we obtain the result since $\mathfrak{p}_{|M} = \mathrm{tp}(g^{-1} \cdot a/M)$ is generic by Lemma 6.8. $\square$

Corollary 6.14. *There is a unique generic in every coset of G^0. In particular, the group G is connected if and only if it has a unique generic type.*

References

[1] Enrique Casanovas. *Simple theories and hyperimaginaries*, Lecture Notes in Logic, Cambridge University Press, 2011.

[2] Ehud Hrusvhoski and Anand Pillay. *Groups definable in local fields and pseudo-finite fields*, Israel Journal of Mathematics, Vol. 85 (1994), pp. 203–262.

[3] Anand Pillay. *Geometric stability theory.* Oxford Logic Guides 32. Oxford University Press, Oxford, GB, 1996.

[4] Katrin Tent and Martin Ziegler. *A course in model theory.* Lecture Notes in Logic, 40. Association for Symbolic Logic, Cambridge University Press, Cambridge, 2012.

[5] Frank O. Wagner. *Simple Theories.* Mathematics and Its Applications 503. Kluwer Academic Publishers, Dordrecht, NL, 2000.

Einstein Institute of Mathematics, Hebrew University of Jerusalem, Givat Ram, 9190401 Jerusalem, Israel

E-mail address: daniel.palacin@mail.huji.ac.il