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**The Erweiterungssatz for the
Intersection Cohomology of Schubert Varieties**

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Introduzione

La teoria delle rappresentazioni è un settore della matematica che nell'ultimo secolo, anche grazie al suo interesse in fisica teorica, è stato fatto oggetto delle più estese investigazioni. Lo studio delle rappresentazioni si concentra innanzitutto sulla classificazione e lo studio delle rappresentazioni irriducibili. Uno dei primi risultati fondamentali in questo contesto, dimostrato da H. Weyl nel 1925 [Hum78, §23.3], fornisce una formula per calcolare i caratteri delle rappresentazioni irriducibili $L(\lambda)$ di dimensione finita di un'algebra di Lie riduttiva in termini di caratteri dei moduli di Verma $M(\mu)$:

$$\text{ch}(L(\lambda)) = \sum_{\omega \in W} (-1)^{l(\omega)} \text{ch}(M(\omega(\lambda + \rho) - \rho))$$

I moduli di Verma sono le rappresentazioni di peso più alto senza ulteriori relazioni: questi non sono sempre irriducibili ma sono, generalmente, di facile manipolazione.

La formula di Weyl è stata generalizzata a rappresentazioni irriducibili di dimensione infinita nel 1981 da Beilinson e Bernstein [BB81] e indipendentemente da Brylinski e Kashiwara [BK81].

$$\text{ch}(L(-\omega\rho - \rho)) = \sum_{\nu \leq \omega} (-1)^{l(\omega) - l(\nu)} P_{\nu, \omega}(1) \text{ch}(M(-\nu\rho - \rho)) \quad (1)$$

Entrambe le dimostrazioni di questa formula fanno ricorso a strumenti tecnici completamente nuovi per la teoria delle rappresentazioni dell'epoca: in particolare si stabilisce una corrispondenza tra le rappresentazioni considerate e una classe di oggetti geometrici, i "D-moduli", coi corrispondenti "fasci perversi" ad essi associati. Tali oggetti geometrici erano in quegli anni al centro di intense ricerche motivate da problemi matematici di diversa natura:

- lo studio della coomologia di intersezione, a opera di M. Goresky e R. MacPherson, che trae le sue motivazioni originali da questioni riguardanti la topologia delle varietà singolari;
- la cosiddetta "analisi algebrica", ovvero lo sviluppo della teoria algebrica delle equazioni lineari alle derivate parziali, soprattutto a opera di M. Kashiwara e, indipendentemente, dello stesso J. Bernstein;
- lo studio, ad opera principalmente di P. Deligne, della categoria derivata dei fasci l-adici costruibili su una varietà algebrica definita su un campo finito,

con le conseguenti nozioni di pesi dell'azione del morfismo di Frobenius e di purezza. Si tratta di una teoria estremamente profonda e potente, motivata principalmente dalle “congetture di Weil” sulle proprietà aritmetiche della varietà algebriche, che si avvale del potente arsenale coomologico messo a punto da A. Grothendieck e la sua scuola negli anni '60.

Per una stimolante ricostruzione storica di questi sviluppi si veda [Kle07]

Per la novità dei metodi usati, il risultato di Beilinson-Bernstein e Brlylinski-Kashiwara ha costituito un vero e proprio punto di svolta nello studio della materia, che ha posto le basi di una nuova area di ricerca, chiamata *teoria geometrica delle rappresentazioni*.

Questa tesi si pone in questo ambito e in essa ripercorreremo e approfondiremo alcuni degli strumenti e dei risultati tipici di questa teoria con l'obiettivo di avvicinarci ai risultati più recenti e ai settori in corrente sviluppo.

Nella formula (1) i termini $P_{\nu,\omega}$ sono i cosiddetti polinomi di Kazhdan-Lusztig. Questi appaiono nella definizione di una particolare base dell'algebra di Hecke, un oggetto algebrico ottenuto come deformazione dell'algebra di gruppo $\mathbb{Z}[W]$ di un gruppo di Coxeter W . Uno dei risultati fondamentali più sorprendenti in teoria geometrica delle rappresentazioni è proprio l'interpretazione geometrica di questi polinomi e dell'intera algebra di Hecke. Questa interpretazione coinvolge appunto la “Coomologia d'Intersezione”.

La coomologia d'intersezione fu definita negli anni '80 da Goresky e MacPherson e come detto costituisce uno strumento adatto allo studio topologico delle varietà singolari. Per la definizione si prende il fascio costante \mathbb{C}_U sulla parte nonsingolare U di una varietà X e si cerca un'estensione “minimale”, in un senso appropriato che sarebbe troppo lungo spiegare adesso, di questo fascio a tutto X : questa estensione conduce in realtà non ad un fascio ma ad un complesso di fasci di spazi vettoriali, vale a dire un oggetto della categorie derivata dei complessi a fasci di coomologia costruibili. Vari risultati classici riguardanti enunciati in termini di gruppi di coomologia di varietà lisce, quali la dualità di Poincarè o, nel caso di varietà singolari proiettive, il cosiddetto teorema di Lefschetz “difficile”, valgono per varietà singolari se, al posto della coomologia, si considera la coomologia di intersezione.

Preso un gruppo riduttivo G e un suo gruppo di Borel B , la varietà delle bandiere $X = G/B$ è una varietà liscia e proiettiva dotata di una stratificazione in B -orbite, $G = \bigsqcup B\omega B$, parametrizzate dagli elementi ω del gruppo di Weyl di G . Le chiusure di queste orbite, le varietà di Schubert X_ω , sono varietà singolari. I loro complessi di coomologia d'intersezione \mathcal{L}_ω forniscono appunto l'interpretazione geometrica dei polinomi di Kazhdan-Lusztig: i coefficienti dei polinomi risultano essere le dimensioni delle spighe dei fasci della coomologia d'intersezione nei vari strati della varietà di Schubert X_ω . In questo modo si può dimostrare il fatto, tutt'altro che evidente dalla definizione combinatoria dei polinomi, che, se W è il gruppo di Weyl di un gruppo algebrico riduttivo, i coefficienti dei polinomi $P_{\nu,\omega}$ sono interi positivi. A tutt'oggi non esiste una dimostrazione puramente combinatorica di questa positività.

Se si passa dalle proprietà locali del complesso di intersezione a quelle globali,

l'ipercoomologia $\mathbb{H}(\mathcal{L}_\omega)$ dei complessi \mathcal{L}_ω (la cosiddetta coomologia di intersezione $IH(X_\omega)$), è in modo naturale un modulo sull'anello (commutativo artinian) di coomologia $C = H^\bullet(X)$ della varietà delle bandiere. Il teorema di decomposizione, uno dei teoremi più profondi riguardanti la coomologia d'intersezione, dimostrato da Beilinson, Bernstein Deligne e Gabber in [BBD], afferma in questo contesto che $\mathbb{H}(\mathcal{L}_\omega)$, considerato come C -modulo, è un addendo diretto di $H^\bullet(\tilde{X}_\omega)$. Qui \tilde{X}_ω è una naturale risoluzione delle singolarità di X_ω , chiamata varietà di Bott-Samelson.

Nel 1990 Soergel, nell'importante lavoro [Soe90] ha dimostrato l'“Erweiterungssatz” (Teorema di Estensione): presi due elementi $\nu, \omega \in W$ si ha un isomorfismo di \mathbb{C} -spazi vettoriali graduati su

$$\mathrm{Hom}(\mathcal{L}_\nu, \mathcal{L}_\omega) \cong \mathrm{Hom}_{C\text{-Mod}}(\mathbb{H}(\mathcal{L}_\nu), \mathbb{H}(\mathcal{L}_\omega)),$$

dove gli omomorfismi a sinistra si intendono calcolati nella categoria derivata dei complessi a coomologia costruibile. In particolare questo teorema rende possibile la determinazione puramente algebrica dell'addendo $\mathbb{H}\mathcal{L}_\omega$ di $H^\bullet(\tilde{X}_\omega)$. Infatti $\mathbb{H}\mathcal{L}_\omega$ è isomorfo all'addendo contenente 1 in una qualunque decomposizione in indecomponibile di $H^\bullet(\tilde{X}_\omega)$.

Basandosi su questi risultati, B. Elias e G. Williamson [EW14a] hanno dimostrato nel 2012 una congettura di Kazhdan e Lusztig che resisteva da più di 30 anni: i polinomi $P_{\nu, \omega}$ hanno coefficienti positivi per un qualunque gruppo di Coxeter.

Veniamo ora alla struttura della tesi. Nel primo capitolo sono introdotte le varietà considerate nel seguito: la varietà della bandiera, le varietà di Schubert e le loro risoluzioni, le varietà di Bott-Samelson. Nel capitolo 2 si discute l'algebra di Hecke di un gruppo di Coxeter e in particolare la sua base di Kazhdan-Lusztig. Il capitolo 3 fornisce l'interpretazione geometrica dei polinomi di Kazhdan-Lusztig, costruendo una corrispondenza (o meglio una categorificazione) tra l'algebra di Hecke e un'algebra costruita a partire dai complessi di coomologia d'intersezione delle varietà di Schubert. Nel quarto capitolo si vede come questa corrispondenza può essere definita anche in termini di una particolare classe di C -bimoduli, detti bimoduli di Soergel. Questo risultato è una delle conseguenze dell'“Erweiterungssatz”. Nella seconda parte del capitolo è presente una dimostrazione di questo Teorema che si avvale di un teorema di localizzazione dovuto a V. Ginzburg [Gin91],

Le appendici contengono alcune parti tecniche, che, se introdotte nel corpo principale della tesi, ne avrebbero appesantito la lettura ed oscurato la linea argomentativa. In particolare: l'appendice A contiene un riepilogo della categoria derivata dei complessi di fasci a coomologia costruibile, e dei principali funtori naturalmente definiti su questa. Nell'appendice B sono introdotti, in modo abbastanza esteso, i fasci perversi e la coomologia d'intersezione. L'appendice C contiene una breve introduzione alla teoria dei moduli misti di Hodge, sviluppata da Saito nei primi anni '90. Questa teoria, assai complessa, individua un analogo per varietà complesse della teoria dei fasci l -adici e del conseguente formalismo dei pesi. Dalla teoria di Saito dipendono in modo cruciale alcuni risultati esposti nei capitoli 3 e 4.

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Chapter 1

Bott-Samelson Resolution

1.1 Reductive Groups and Weyl groups

In this first section we recall some fundamental properties of reductive linear algebraic groups. We refer to the book of Springer [Spr98] for the definitions and for a more detailed account. All the groups and varieties in this section are defined over \mathbb{C} .

Let G be a reductive linear algebraic group and T a maximal torus. T acts on G by conjugation and this determines a root decomposition in weight spaces of the Lie algebra \mathfrak{g} of G

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$$

Here R is a finite subset of the group of characters $X(T)$ of the torus T .

$$X(T) = \left\{ \phi : T \rightarrow \mathbb{C}^* \mid \begin{array}{l} \phi \text{ is a morphism of algebraic} \\ \text{varieties and a group homomorphism} \end{array} \right\}$$

In the decomposition above, \mathfrak{g}_0 is the tangent algebra of T while each \mathfrak{g}_α is unidimensional and $t.g = \alpha(t)g \forall t \in T, g \in \mathfrak{g}_\alpha$

For any $\alpha \in R$ there exists an unidimensional subgroup U_α of G , whose Lie algebra is \mathfrak{g}_α , and an isomorphism $u_\alpha : \mathbb{C} \rightarrow U_\alpha$ (where the additive group $(\mathbb{C}, +)$ is regarded as a unipotent linear algebraic group) such that

$$tu_\alpha(x)t^{-1} = u_\alpha(\alpha(t)x) \quad \forall t \in T \quad \forall x \in \mathbb{C}$$

The torus T and the groups U_α , $\alpha \in R$, generate G . R has several properties and it is called a *root system*

- R spans $X_{\mathbb{R}}(T) = X(T) \otimes \mathbb{R}$;
- there exists a suitable positive definite symmetric bilinear form (\cdot, \cdot) on $X_{\mathbb{R}}(T)$ such that for every root $\alpha \in R$, R is stable under the reflection

$$s_\alpha(x) = x - 2 \frac{(\alpha, x)}{(\alpha, \alpha)} \alpha$$

- For any two roots $\alpha, \beta \in R$, $2\frac{(\alpha, \beta)}{(\beta, \beta)}$ is an integer

There exists a subset S of R , whose elements are called *simple roots*, with the following properties:

1. The roots in S form a basis of $X_{\mathbb{R}}(T)$
2. Every root $\alpha \in R$ can be written as a positive (or negative) integral linear combination of simple roots, i.e. $\alpha = \sum_{\beta \in S} c_s \beta$, where the coefficients c_s are integers and they are all positive (or all negative).

The choice of S gives also a partition of R into *positive roots* R^+ (those with positive coefficients) and the *negative roots* $R^- = -R^+$.

We denote by $W = N_G(T)/T$ the Weyl group. W is also the group generated by the reflections of the root system R . Actually, simple reflections (i.e. given by the s_α with $\alpha \in S$) are enough to generate W . For a representative $\dot{\omega} \in N_G(T)$ of ω we have $\dot{\omega}^{-1}t\dot{\omega} = t' \in T$ and

$$t(\dot{\omega}u_\alpha(x)\dot{\omega}^{-1})t^{-1} = \dot{\omega}t'u_\alpha(x)(t')^{-1}\dot{\omega}^{-1} = \dot{\omega}u_\alpha(\alpha(t')x)\dot{\omega}^{-1} = \dot{\omega}u_\alpha(\omega(\alpha)(t))\dot{\omega}^{-1}$$

and this yields to the equality $\dot{\omega}U_\alpha\dot{\omega}^{-1} = U_{\omega(\alpha)}$.

Let B a Borel subgroup of G containing T . Restricting the T -action to B we get the decomposition

$$\text{Lie}(B) = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha$$

where R^+ is a positive system of roots, and $U_\alpha \subseteq B$ if and only if $\alpha \in R^+$. One can choose the isomorphisms u_α in such a way that $n_\alpha = u_\alpha(1)u_{-\alpha}(1)u_\alpha(1)$ represents the reflection s_α in W .

B is a solvable subgroup, hence there is a decomposition $B = TB_u = B_uT$, where $B_u = [B, B]$ is the unipotent part of B . After choosing any numbering of the roots $\{\alpha_1, \dots, \alpha_N\}$ in R^+ we have an isomorphism of algebraic varieties (but not of groups)

$$\phi : \mathbb{C}^N \rightarrow B_u \quad \phi(x_1, \dots, x_n) = u_{\alpha_1}(x_1) \cdot \dots \cdot u_{\alpha_N}(x_N) \quad (1.1)$$

For $\omega \in W$ we set $R(\omega) = \{\alpha \in R^+ \mid \omega(\alpha) \in -R^+\}$. The number of elements of $R(\omega)$ is equal to the length $l(\omega)$, the minimum integer l such that there exists a reduced expression $\omega = s_{\alpha_1} \cdot \dots \cdot s_{\alpha_l}$. Here the α_i are simple roots relative to R^+ . Given a reduced expression for ω one can describe explicitly the elements of $R(\omega)$

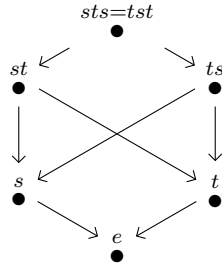
$$R(\omega) = \{\alpha_l, s_{\alpha_l}(\alpha_{l-1}), \dots, s_{\alpha_l} \cdot \dots \cdot s_{\alpha_2}(\alpha_1)\} \quad (1.2)$$

Let $S = \{s_1, \dots, s_n\}$ be the set of simple reflections for W . We say that an element in W is a reflection if it has a conjugate in S . Thus, the set of reflections is

$$Q = \bigcup_{\omega \in W} \omega S \omega^{-1}$$

There is a natural partial order $<$ on W , called the *Bruhat order*. It is generated by $xq < x$, if $q \in Q$ and $l(xq) < l(x)$. We notice that the definition does not change if we consider multiplication on the left by elements of Q , since $xq = (xqx^{-1})x$ and xqx is clearly a reflection. There is another equivalent description of the Bruhat order: we say that $x \geq y$ if there exists a reduced expression $(s_{i_1}, \dots, s_{i_k})$ of x containing a subsequence which is a reduced expression for y .

Example 1.1.1. The symmetric group S_3 has 6 elements and 2 generators $s = (1, 2)$ and $t = (2, 3)$. The Bruhat order can be drawn as



where $\bullet \xrightarrow{x} \bullet$ means $y \geq x$.

Let ω_0 be the longest element in W . It is unique and $R(\omega_0) = -R^+$, thus $l(\omega_0) = N = \frac{1}{2}|R|$. By definition we see that $R(\omega)$ and $R(\omega\omega_0)$ are disjoint subsets and their union is the whole R^+ . Then we can define

$$U_\omega = \prod_{\alpha \in R(\omega)} U_\alpha$$

This is well defined as the product does not depend on the order of the factors. Then the product morphism $U_\omega \times U_{\omega\omega_0} \rightarrow B_u$ is an isomorphism of variety by (1.1).

1.2 Bruhat Decomposition

We now consider the homogeneous space $X = G/B$, also known as *flag variety*.

Example 1.2.1. Let $G = SL_n(\mathbb{C})$. A Borel subgroup for G is given by the group of upper triangular matrices. The flag variety can be identified with the set of flags in \mathbb{C}^n

$$\text{Flag}(\mathbb{C}^n) = \{(V_i)_{0 \leq i \leq n} \mid V_i \text{ is a } i\text{-dimensional subspace of } \mathbb{C}^n \text{ and } V_i \subseteq V_{i+1}\}$$

G acts transitively on the set $\text{Flag}(\mathbb{C}^n)$ and the stabilizer of the "standard" flag

$$0 \subseteq \mathbb{C}e_1 \subseteq \mathbb{C}e_1 \oplus \mathbb{C}e_2 \subseteq \dots \subseteq \mathbb{C}e_1 \oplus \dots \oplus \mathbb{C}e_n = \mathbb{C}^n$$

is exactly the group of upper triangular matrices B .

This is a smooth projective variety and is supplied with a left G -action given by left multiplication. Let's restrict it to a B -action. We have the following:

Theorem 1.2.2 (Bruhat Decomposition). *The B -action on X decomposes the flag variety in a finite number of orbits, each of which is of the form $B\dot{\omega}B/B$, where $\dot{\omega} \in G$ is a representative for $\omega \in W$. Every orbit $B\dot{\omega}B/B$ is isomorphic, as a variety, to $\mathbb{C}^{l(\omega)}$ and is called a Schubert cell. The closure $X_\omega = \overline{B\dot{\omega}B/B}$ of a Schubert cell is called a Schubert variety and is a union of B -orbits. More precisely,*

$$X_\omega = \bigsqcup_{\nu \leq \omega} B\nu B/B$$

where \leq is the Bruhat order.

If there is no room for confusion, from now on we will denote simply by ω any element of G in the coset $\omega \in N_G(T)/T$.

For example the Schubert variety X_e of the identity element $e \in G$ is a single point. For a simple reflection $s_\alpha \in S$, the Schubert variety X_{s_α} is isomorphic to $\mathbb{P}_{\mathbb{C}}^1$. In general the Schubert varieties are singular projective varieties. The aim of this chapter is to define a natural resolution of singularities of these Schubert varieties, i.e. a projective smooth variety $\tilde{X}(\omega)$ along with a birational morphism $\pi : \tilde{X}(\omega) \rightarrow X_\omega$. The following lemma will be useful for this purpose:

Lemma 1.2.3. *Let $\omega = s_{\alpha_1} \cdot \dots \cdot s_{\alpha_l}$ be a reduced expression for $\omega \in W$. Then the morphism*

$$U_{\omega^{-1}} \times B \rightarrow B\omega B \quad (u, b) \mapsto u\omega b$$

defines an isomorphism of varieties.

Proof. We have

$$B\omega B = B_u T \omega B = U_{\omega^{-1}} U_{\omega_0 \omega^{-1}} \omega B$$

Furthermore,

$$\omega^{-1} (U_{\omega_0 \omega^{-1}}) \omega = \omega^{-1} \left(\prod_{\alpha \in R(\omega_0 \omega^{-1})} U_\alpha \right) \omega = \prod_{\alpha \in R(\omega_0 \omega^{-1})} U_{\omega^{-1}(\alpha)} \subseteq B$$

since, by definition, if $\alpha \in R(\omega_0 \omega^{-1})$ then $\omega^{-1}(\alpha) \in R^+$ and $U_{\omega^{-1}(\alpha)} \subseteq B$. The statement is thus equivalent to the fact that the map $(u, b) \mapsto \omega^{-1} u \omega b$ defines an isomorphism of $U_{\omega^{-1}} \times B$ onto $\omega^{-1} B \omega B$. Since this map is bijective, and, regarding both these spaces as homogeneous $U_{\omega^{-1}} \times B$ -spaces, equivariant, we conclude from the general fact stated in the next theorem. \square

Theorem 1.2.4. [Spr98, 5.3.2.(iii)] *Let G be a complex algebraic group and let $\phi : X \rightarrow Y$ be an equivariant homomorphism of G -homogeneous spaces. Then ϕ is an isomorphism if and only if it is bijective.*

Later we will also need:

Lemma 1.2.5. *Let $\omega \in W$, $s \in S$. We have*

$$\begin{aligned} BsB \cdot B\omega B &= B(s\omega)B & \text{if } l(s\omega) &= l(\omega) + 1 \\ BsB \cdot B\omega B &= B(s\omega)B \sqcup B(\omega)B & \text{if } l(s\omega) &= l(\omega) - 1 \end{aligned}$$

Proof. Let $s = s_\alpha$. Then, as in the proof of Lemma 1.2.3, we have $BsB = U_\alpha sB$. Thus, $BsB \cdot B\omega B = U_\alpha sB\omega B$. For $l(s\omega) = l(\omega) + 1$, by (1.2) we see that $R(\omega^{-1}s_\alpha) = \{\alpha\} \sqcup s_\alpha(R(\omega^{-1}))$ and

$$sU_{\omega^{-1}s} = \prod_{\beta \in R(\omega^{-1})} U_{s_\alpha(\beta)} = \prod_{\beta \in s_\alpha(R(\omega^{-1}))} U_\beta$$

Hence

$$U_\alpha sB\omega B = U_\alpha (sU_{\omega^{-1}s})s\omega B = U_{\omega^{-1}s}s\omega B = Bs\omega B$$

For $l(s\omega) < l(\omega)$ we have

$$BsB \cdot B\omega B = BsB \cdot BsB \cdot B\omega B$$

It remains to show that $BsB \cdot BsB = BsB \sqcup B = P_s$ is the minimal parabolic subgroup of G containing s . After taking the quotient by the radical of P_{s_α} , the statement is reduced to the case in which the Weyl group has only 2 elements, namely $G = SL_2(\mathbb{C})$ or $G = PSL_2(\mathbb{C})$. In these cases we need only to show that the existence of elements $x, y \in BsB$ such that $xy \in BsB$. For example, if B is the subgroup of the upper triangular matrices and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is a representative for s , we can choose

$$x = y = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{for } G = SL_2(\mathbb{C})$$

(or, similarly, $x = y = \left[\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right]$ for $G = PSL_2(\mathbb{C})$. □

1.3 The main construction

Let X and Y be two smooth varieties with, respectively, a right and a left action of an algebraic group G . Then we denote by $X \times_G Y$ the quotient of $X \times Y$ by the right G -action

$$(x, y) \cdot g = (x \cdot g, g^{-1} \cdot y) \quad \forall g \in G$$

This quotient is not an algebraic variety in general. However in our situation we can make several regularity assumptions on the action. If:

- the action on X is free,
- the quotient X/G exists and is smooth,
- the projection $X \rightarrow X/G$ is a locally trivial fibration (i.e. locally on X/G it is $U \times G \rightarrow U$),

then $X \times_G Y$ exists and it is a smooth variety because locally $(U \times G) \times_G Y \cong U \times Y$.

All the cases we are interested in will satisfy these hypotheses. As a significant example, take $X = P$ any parabolic subgroup of G containing B (the action is the B -right multiplication). In fact, cfr. [Spr98, 8.4], there exists an element $\omega_P \in W$

such that the Schubert cell $B\omega_P B$ is an open dense subvariety of P , and this yields the morphism

$$U_{\omega_P^{-1}} \times B \rightarrow P \quad (u, b) \longmapsto u\omega_P b$$

to be an isomorphism with an open dense subvariety of P . This gives the fibration structure in a neighborhood of $e \in P$. We can easily translate it to a neighborhood of any $p \in P$ by multiplying on the left by p .

Let $\omega = s_{\alpha_1} \cdot \dots \cdot s_{\alpha_l}$ be a reduced expression for ω . For any α_i there exists a minimal parabolic subgroup P_{α_i} (in which B has codimension 1) and for such a parabolic a subgroup we have $\omega_{P_{\alpha_i}} = s_{\alpha_i}$. We can iterate the above procedure to obtain:

Definition 1.3.1. Let $\tilde{X}(\alpha_1, \dots, \alpha_l) = P_{\alpha_1} \times_B P_{\alpha_2} \times_B \dots \times_B P_{\alpha_l} / B$. Here the final quotient by B is taken with respect to the B -right multiplication action. This quotient is called the *Bott-Samelson variety* relative to (the reduced expression of) ω and it is a smooth and projective variety (projectivity will follow from Lemma 1.4.3). The morphism

$$\pi : \tilde{X}(\alpha_1, \dots, \alpha_l) \rightarrow G/B \quad (p_1, \dots, p_l) \longmapsto p_1 \cdot \dots \cdot p_l B$$

is called the *Bott-Samelson resolution*. The image of π is exactly the Schubert variety X_ω .

We remark that we can also obtain $\tilde{X}(\alpha_1, \dots, \alpha_l)$ at once quotienting P^l by the right B^l -action

$$(p_1, p_2, \dots, p_l)(b_1, b_2, \dots, b_l) = (p_1 b_1, b_1^{-1} p_2 b_2, \dots, b_{l-1}^{-1} p_l b_l)$$

Theorem 1.3.2. *The Bott-Samelson resolution is a resolution of singularities of the Schubert variety X_ω , that is, $\tilde{X}(\alpha_1, \dots, \alpha_l)$ is a smooth variety and*

$$\pi : \tilde{X}(\alpha_1, \dots, \alpha_l) \rightarrow X_\omega$$

is a birational morphism.

Proof. The smoothness of $\tilde{X}(\alpha_1, \dots, \alpha_l)$ follows from the above discussion. We regard $U_{-\alpha_1} \times \dots \times U_{-\alpha_l}$ as a subvariety of P^l . Then, when restricted to this subvariety, the morphism $P^l \rightarrow P^l/B^l$ becomes injective. In fact, if $(u_1, \dots, u_l), (u'_1, \dots, u'_l) \in U_{-\alpha_1} \times \dots \times U_{-\alpha_l}$ are in the same B^l -orbit, then there exists, $b_1, \dots, b_l \in B$ such that $u_1 b_1 = u'_1, b_1^{-1} u_2 b_2 = u'_2, \dots, b_{l-1}^{-1} u_l b_l = u'_l$. But $b_1 = u_1^{-1} u'_1 \in U_{-\alpha_1} \cap B \implies b_1 = 1$ and continuing by induction all the b_i must be 1. We call V the image of $U_{-\alpha_1} \times \dots \times U_{-\alpha_l}$.

In view of Theorem 1.2.4, the morphism

$$\pi|_{U_{-\alpha_1} \times \dots \times U_{-\alpha_l}} : U_{-\alpha_1} \times \dots \times U_{-\alpha_l} \rightarrow V$$

is an isomorphism since it is bijective and it is an equivariant morphism between $U_{-\alpha_1} \times \dots \times U_{-\alpha_l}$ -homogeneous spaces. Also, the subvariety V has the same dimension of $\tilde{X}(\alpha_1, \dots, \alpha_l)$, hence it is an open dense subvariety. Then we conclude by Lemma 1.2.3 \square

1.4 G -orbits on $X \times X$

Let's consider the diagonal G -action on $X \times X$. Then an analogue of the Bruhat decomposition holds:

Proposition 1.4.1. *i) Every G -orbit on $X \times X$ contains an element of the form $(B, \omega B)$, $\omega \in W$. Let \mathcal{O}_ω be the orbit containing $(B, \omega B)$. Then its closure $\overline{\mathcal{O}_\omega}$ is a variety of dimension $l(\omega) + \dim X$ and it is a union of orbits:*

$$\overline{\mathcal{O}_\omega} = \bigsqcup_{\nu \leq \omega} \mathcal{O}_\nu$$

ii) The first projection $p_1 : \overline{\mathcal{O}_\omega} \rightarrow X$ is a locally trivial fibration with fibers isomorphic to X_ω

Proof. Let $\mathcal{O} \subseteq X \times X$ a G -orbit. Then $\mathcal{O} \cap (\{B\} \times X)$ is non-empty and it is a B -orbit in the second component. Now i) is an immediate consequence of the Bruhat decomposition for X .

We define $U^- = U_{\omega_0} \omega_0$. Then U^- is also the unipotent radical of the opposite Borel subgroup to B , the one corresponding to the positive root system R^- . So $U^- B / B \cong U^-$ is an open subset of X and we have the morphism

$$U^- \times X_\omega \rightarrow p_1^{-1}(U^- B / B) \quad (u, xB) \mapsto (uB, uxB)$$

which is an isomorphism since $(uB, vB) \mapsto (u, u^{-1}vB)$ is the inverse. Hence p_1 is a fibration in a neighborhood of $eB \in X$. By letting G act we can get a local trivialization in a neighborhood of any point. \square

In general, the $\overline{\mathcal{O}_\omega}$'s are singular projective varieties: more precisely, by ii) of the Proposition above, $\overline{\mathcal{O}_\omega}$ is singular if and only if X_ω is singular. As we have already done for X_ω , we will now look for a resolution of singularities for $\overline{\mathcal{O}_\omega}$.

We fix a reduced expression $\omega = s_{\alpha_1} \cdot \dots \cdot s_{\alpha_l}$. Then we define

$$\tilde{\mathcal{O}}(s_{\alpha_1}, \dots, s_{\alpha_l}) = \{(x_0, \dots, x_l) \in X^{l+1} \mid (x_{i-1}, x_i) \in \overline{\mathcal{O}_{s_{\alpha_i}}} \forall i, 1 \leq i \leq l\}$$

We have a sequence of projection

$$\begin{aligned} \tilde{\mathcal{O}}(s_{\alpha_1}, \dots, s_{\alpha_l}) &\rightarrow \tilde{\mathcal{O}}(s_{\alpha_1}, \dots, s_{\alpha_{l-1}}) \rightarrow \dots \rightarrow \tilde{\mathcal{O}}(s_{\alpha_1}) = \overline{\mathcal{O}_{s_{\alpha_1}}} \rightarrow X \\ (x_0, \dots, x_l) &\mapsto (x_1, \dots, x_{l-1}) \mapsto \dots \mapsto (x_0, x_1) \mapsto x_0 \end{aligned}$$

is a sequence of locally trivial fibrations and for each of them the fibers are isomorphic to $X_{s_{\alpha_i}} \cong \mathbb{P}_{\mathbb{C}}^1$. This immediately shows that $\tilde{\mathcal{O}}(s_{\alpha_1}, \dots, s_{\alpha_l})$ is nonsingular. Now we consider \tilde{p}_1 , the first projection on $\tilde{\mathcal{O}}(s_{\alpha_1}, \dots, s_{\alpha_l})$:

Proposition 1.4.2. *$\tilde{p}_1 : \tilde{\mathcal{O}}(s_{\alpha_1}, \dots, s_{\alpha_l}) \rightarrow X$ is a locally trivial fibration whose fibers are isomorphic to $\tilde{X}(\alpha_1, \dots, \alpha_l)$*

Proof. To prove this proposition, we need an alternative and equivalent definition of the Bott-Samelson variety, this time as a subvariety rather than as a quotient. Let $p_{\alpha_i} : G/B \rightarrow G/P_{\alpha_i}$ the projection. Then we define the variety

$$Y(\alpha_1, \dots, \alpha_l) = \{(x_1, \dots, x_l) \in X^l \mid x_0 = e \text{ and } p_{\alpha_i}(x_i) = p_{\alpha_i}(x_{i-1})\}$$

Lemma 1.4.3. $Y(\alpha_1, \dots, \alpha_l)$ is a variety isomorphic to $\tilde{X}(\alpha_1, \dots, \alpha_l)$

Proof of Lemma 1.4.3. We define the morphism

$$\phi : \tilde{X}(\alpha_1, \dots, \alpha_l) \rightarrow Y(\alpha_1, \dots, \alpha_l) \quad [(p_1, p_2, \dots, p_l)] \mapsto (p_1B, p_1p_2B, \dots, p_1 \dots p_lB)$$

$p_1 \dots p_i P_{\alpha_i} = p_1 \dots p_{i-1} P_{\alpha_i}$ since $p_i \in P_{\alpha_i}$ and the morphism is well defined. To show that ϕ is an isomorphism we have simply to exhibit an inverse

$$\psi : Y(\alpha_1, \dots, \alpha_l) \rightarrow \tilde{X}(\alpha_1, \dots, \alpha_l) \quad (g_1B, g_2B, \dots, g_lB) \mapsto [(g_1, g_1^{-1}g_2, \dots, g_{l-1}^{-1}g_l)]$$

ψ is well defined and the lemma is proven. \square

We return to the proof of the proposition. The inverse image of U^-B/B is isomorphic to $U^- \times Y(\alpha_1, \dots, \alpha_l)$ through the morphism

$$U^- \times Y(\alpha_1, \dots, \alpha_l) \rightarrow \tilde{p}_1^{-1}(U^-B/B) \quad (u, (x_1, \dots, x_l)) \mapsto (u, ux_1, \dots, ux_l)$$

This describes the fibration in a neighborhood of eB . As before, we can multiply by elements to get the thesis. \square

Let $\pi : \tilde{\mathcal{O}}(s_{\alpha_1}, \dots, s_{\alpha_l}) \rightarrow \overline{\mathcal{O}_\omega}$ the morphism defined by $\pi(x_1, \dots, x_{l+1}) \rightarrow (x_1, x_{l+1})$.

Theorem 1.4.4. π is a resolution of singularities of $\overline{\mathcal{O}_\omega}$.

Proof. We have the following chain of morphisms

$$\tilde{\mathcal{O}}(s_{\alpha_1}, \dots, s_{\alpha_l}) \xrightarrow{\pi} \overline{\mathcal{O}_\omega} \xrightarrow{p_1} X \quad \pi \circ p_1 = \tilde{p}_1$$

and locally on a suitable open set $U \subseteq X$ this can be written as

$$U \times \tilde{X}(\alpha_1, \dots, \alpha_l) \xrightarrow{\text{Id}_U \times \pi} U \times X_\omega \xrightarrow{p_1} U$$

Since $\tilde{X}(\alpha_1, \dots, \alpha_l)$ is a resolution of singularities of X_ω , then clearly $\tilde{\mathcal{O}}(s_{\alpha_1}, \dots, s_{\alpha_l})$ is a resolution of $\overline{\mathcal{O}_\omega}$. \square

Moreover, let $\nu < \omega$ and $\pi : Y(\alpha_1, \dots, \alpha_l) \cong \tilde{X}(\alpha_1, \dots, \alpha_l) \rightarrow X(\omega)$. We have that $U_{\nu^{-1}} \cong B\nu B/B \subseteq X_\omega$ and

$$U_{\nu^{-1}} \times \pi^{-1}(\nu B) \cong \pi^{-1}(B\nu B/B)$$

where the isomorphism is given by $(u, (y_1, \dots, y_l)) \rightarrow (uy_1, \dots, uy_l)$. Thus $\pi : \pi^{-1}(B\nu B) \rightarrow B\nu B$ is a trivial fibration.

Arguing like in Theorem 1.4.4, we can get the analogous result for $\mathcal{O}_\nu \subseteq \overline{\mathcal{O}_\omega}$ and $\pi : \tilde{\mathcal{O}}(\alpha_1, \dots, \alpha_l) \rightarrow \overline{\mathcal{O}_\omega}$, i.e. $\pi : \pi^{-1}(\mathcal{O}_\nu) \rightarrow \mathcal{O}_\nu$ is a locally trivial fibration.

1.5 The dual cell decomposition

We conclude this chapter with a discussion of the dual Bruhat decomposition, which will be needed in Chapter 4.

The flag variety $X = G/B$ can be seen also as the variety parameterizing the Borel subgroups of G through the map

$$f : G/B \rightarrow \text{Bor}(G) \quad gB \longmapsto gBg^{-1}$$

Of course, there is nothing special about B so we can replace B with any other Borel subgroup. For example we can consider the opposite Borel subgroup $\tilde{B} = \omega_0 B \omega_0^{-1}$ (the one corresponding to negative roots) and we can define analogously $\tilde{f} : G/\tilde{B} \rightarrow \text{Bor}(G)$. Thus we obtain the isomorphism $\theta = f^{-1} \circ \tilde{f} : G/\tilde{B} \rightarrow G/B$, coming from the isomorphism $x \rightarrow x\omega_0$ on G .

The family of locally closed subsets $\tilde{B}\omega\tilde{B}/\tilde{B}$, for $\omega \in W$ defines a cell decomposition related to the usual one through $\theta(\tilde{B}\omega\tilde{B}/\tilde{B}) = \omega_0 B \omega_0 \omega \omega_0 B/B$. So, if we define, $Y_\omega = \theta(\tilde{B}\omega_0\omega\tilde{B}/\tilde{B})$ we have $Y_\omega = \omega_0 X_{\omega\omega_0}$.

Lemma 1.5.1. *Let $\omega, \nu \in W$ with $l(\omega) \leq l(\nu)$ and $\omega \neq \nu$. Then*

- i) *The intersection $X_\omega \cap \omega_0 X_{\omega_0\nu}$ is empty.*
- ii) *$X_\omega \cap \omega_0 X_{\omega_0\omega}$ is the singleton $\{\omega B\}$.*

Proof. Let's assume that $X_\omega \cap \omega_0 X_{\omega_0\nu} \neq \emptyset$ and let A be an irreducible component of this intersection. A is stable with respect to the action of T and it is a proper variety so it must contain a fixed point [Spr98, 6.2.6] of the form μB , $\mu \in W$. Therefore $\mu B \in X_\omega$ and $\omega_0 \mu B \in X_{\omega_0\nu}$ but this, in particular, means that $l(\mu) \leq l(\omega)$ and $l(\omega_0\mu) \leq l(\omega_0\nu) \implies l(\mu) \geq l(\nu)$. From the hypothesis $l(\omega) \leq l(\nu)$ we get $l(\omega) = l(\mu) = l(\nu)$. But then $\mu B \in X_\omega$ if and only if $\mu = \omega \omega_0 \mu B \in X_{\omega\nu B}$ if and only if $\omega_0\mu = \omega_0\nu$ thus $\omega = \mu = \nu$ and we reach a contradiction.

For the statement ii), by the same argument we obtain that every irreducible component of $X_\omega \cap \omega_0 X_{\omega_0\omega}$ must contain the point ωB . To conclude it suffices to show that for a suitable neighborhood V of ωB we have $X_\omega \cap \omega_0 X_{\omega_0\omega} \cap V = \{\omega B\}$ (this, indeed, forces $\{\omega B\}$ to be the whole irreducible component and $X_\omega \cap \omega_0 X_{\omega_0\omega} = \{\omega B\}$). Clearly, by shrinking it if necessary, we can limit ourselves to consider $B\omega B/B \cap \omega_0 B\omega_0\omega B/B \cap V$.

Let $\tilde{B}_\omega = \omega\tilde{B}\omega^{-1}$ and let $U_\omega^- = (\tilde{B}_\omega, \tilde{B}_\omega)$ be its unipotent part. Analogously we define $B_\omega = \omega B \omega^{-1}$ and U_ω . The morphism

$$\phi : U_\omega^- \rightarrow X \quad u \longmapsto u\omega B$$

is an open embedding and the image is a neighborhood of ωB . ϕ sends $U \cap U_\omega^-$ onto $B\omega B/B$: it is a bijection since $U \cap U_\omega^- \cong U/(U \cap U_\omega \rightarrow B\omega B)$ is clearly surjective. Similarly ϕ induces a bijection between $U^- \cap U_\omega^-$ and $\omega_0 B \omega_0 \omega B/B$. We get

$$\phi^{-1}(B\omega B \cap \omega_0 B \omega_0 \omega B/B) = \{e\} \implies B\omega B \cap \omega_0 B \omega_0 \omega B/B \cap \phi(U_\omega^-) = \{\omega B\}$$

□

Chapter 2

Hecke algebras

In this chapter we will define and describe the Hecke algebra of a general Coxeter group W . The Hecke algebra may be thought as a deformation of the group algebra $\mathbb{C}[W]$ depending on a parameter q . Hecke algebras play an important role in many, often apparently unrelated, important problems in representation theory.

2.1 Coxeter Groups

Definition 2.1.1. A Coxeter Group W is a group which has finitely many generators s_1, \dots, s_n subject to the relations $s_i^2 = 1, \forall i \in \{1, \dots, n\}$, and $(s_i s_j)^{m_{ij}} = 1$, where $m_{ij} = m_{ji} \in \{2, 3, \dots, \infty\}$ ($m_{ij} = \infty$ means that there is no relation).

The relation $(s_i s_j)^{m_{ij}}$ can be written also as

$$\underbrace{s_i s_j \cdot \dots \cdot s_i}_{m_{ij} \text{ times}} = \underbrace{s_j s_i \cdot \dots \cdot s_j}_{m_{ij} \text{ times}}$$

and it is called *braid relation*

Example 2.1.2. The symmetric group S_n is a Coxeter Group. In fact, it is generated by the transpositions $s_i = (i, i + 1)$. The coefficient m_{ij} is 2 if $|i - j| > 1$ (this means that s_i and s_j commute) while $m_{i, i+1} = 3$

Coxeter groups form a very interesting class of groups playing an important role in different areas of mathematics, such as Lie theory and finite group theory. Every Coxeter group can be described using reflections. In fact, every Coxeter group has a faithful linear representation in which it acts as a group generated by reflections.

In this sense, Coxeter groups form a generalization of Weyl groups. On the other hand the Weyl Groups form a significant classes of examples for finite Coxeter Groups. In this case the generators are the simple reflections for a given choice of the positive roots.

The notions of length and of Bruhat order introduced in the previous chapter are easily generalized to an arbitrary Coxeter group.

Definition 2.1.3. A reduced expression for $\omega \in W$ is a sequence $(s_{i_1}, \dots, s_{i_k})$, $s_{i_j} \in S$ such that $s_{i_1} \cdot \dots \cdot s_{i_k} = \omega$ and that k is minimal. In this case we call $l(\omega) = k$ the length of ω .

We say that $x \geq y$ in the *Bruhat order* if there exists a reduced expression $(s_{i_1}, \dots, s_{i_k})$ containing a subsequence which is a reduced expression for y .

2.2 Definition of the Hecke Algebra

Let W be a Coxeter Group and S its set of generators. The Hecke algebra is a deformation of the group algebra $\mathbb{Z}[W]$ where the relations $e_s^2 = 1$ are replaced by a different quadratic relations involving a parameter q

Definition 2.2.1. The Hecke Algebra $\mathfrak{H}(W, S)$ of a Coxeter group is the free algebra (with unity) over $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ with basis $\{T_\omega\}_{\omega \in W}$. The multiplication is given by

$$T_s^2 = (q - 1)T_s + q \quad \text{if } s \in S$$

$$T_s T_\omega = T_{s\omega} \quad \text{if } l(s\omega) > l(s)$$

So, if $(s_{i_1}, \dots, s_{i_k})$ is a reduced expression for ω , we have

$$T_\omega = T_{s_{i_1}} \cdot \dots \cdot T_{s_{i_k}}.$$

Since $T_s \cdot (T_s - q + 1)q^{-1} = 1$, every T_s , for $s \in S$, is invertible. Hence all T_ω , $\omega \in W$ are invertible.

2.3 The Hecke Algebra of a Chevalley group

We will now provide another construction of the Hecke Algebra, which works for a Weyl Groups W . This provides also a natural way in which Hecke algebra arises.

Let \mathbb{F}_q be the finite field with q elements, and let $G = G(\mathbb{F}_q)$ be the Chevalley group corresponding to W . A general Chevalley group contains a subgroup $T \subseteq G$, analogous to the maximal torus and $B = TU$, analogous to the Borel subgroup, and there is a Bruhat decomposition $G = \sqcup B\omega B$ [Car93, §8]. However, as this section has only a motivational purpose and will not have any direct consequence on the rest of this work, the reader may just keep in mind the case $G = SL_n(\mathbb{F}_q)$.

Lemma 2.3.1. *For any $\omega \in W$ the order of $B\omega B/B$ is $q^{l(\omega)}$*

Proof. As in the complex case, there exists a bijection $U_{\omega^{-1}}^- \times B \rightarrow B\omega B$ [Car93, Cor 8.4.4.]. Here U_ω is a subgroup of G in which each element can be written in a unique way as an element $\prod_{\alpha \in R(\omega)} U_\alpha$ (after choosing any order of the factors) and the U_α , $\alpha \in R$ are one parameter subgroups isomorphic to \mathbb{F}_q . \square

We keep G and q fixed. We define the \mathbb{C} -algebra $\tilde{\mathfrak{H}}$ of B -bi-invariant \mathbb{C} -valued functions of G , that is, functions on G that are constants on the cells $B\omega B$ for $\omega \in B$.

A basis of $\tilde{\mathfrak{H}}$ is formed by the characteristic functions χ_ω of the double coset $B\omega B$

$$\chi_\omega(x) = \begin{cases} 1 & \text{if } x \in B\omega B \\ 0 & \text{if } x \notin B\omega B \end{cases}$$

The convolution product $*$, defined as

$$(f_1 * f_2)(g) = \frac{1}{|B|} \sum_{x \in G} f_1(x) f_2(x^{-1}g) = \frac{1}{|B|} \sum_{x \in G} f_1(gx) f_2(x^{-1}),$$

gives $\tilde{\mathfrak{H}}$ a \mathbb{C} -algebra structure.

The convolution is normalized, dividing by $|B|$, in such a way that χ_e , the characteristic function of B , is the identity element in $\tilde{\mathfrak{H}}$.

It is easy to verify that the product of two B -biinvariant functions is again a B -biinvariant function. In fact, $\forall b \in B, \forall g \in G$:

$$f_1 * f_2(bg) = \sum_{x \in G} f_1(bgx) f_2(x^{-1}) = \sum_{x \in G} f_1(gx) f_2(x^{-1}) = f_1 * f_2(g)$$

$$f_1 * f_2(gb) = \sum_{x \in G} f_1(x) f_2(x^{-1}gb) = \sum_{x \in G} f_1(x) f_2(x^{-1}g) = f_1 * f_2(g)$$

Apart from the normalization, we can notice that $\tilde{\mathfrak{H}}$ is the subalgebra of $\mathbb{C}[G]$ formed by B -biinvariant functions. From this, we can deduce immediately the associativity of the product.

We can define the so-called *augmentation map* $\epsilon : \tilde{\mathfrak{H}} \rightarrow \mathbb{C}$

$$\epsilon(f) = \frac{1}{|B|} \sum_{x \in G} f(x)$$

Lemma 2.3.1 implies $\epsilon(\chi_\omega) = q^{l(\omega)}$. Furthermore, ϵ is a \mathbb{C} -algebra homomorphism, that is $\epsilon(f_1 * f_2) = \epsilon(f_1)\epsilon(f_2)$. We now prove that $\tilde{\mathfrak{H}}$ is another realization of the Hecke Algebra.

For the proof we rely on the fact that a result completely analogous to Lemma 1.2.5 holds for Chevalley groups, see [Car93, 8.1.5.], namely, for a reflection s :

$$B\omega B \cdot BsB = B\omega sB \text{ if } l(\omega s) = l(\omega) + 1$$

and

$$B\omega B \cdot BsB = BsB \sqcup B \text{ if } l(\omega s) = l(\omega) - 1$$

Lemma 2.3.2. *Let $\omega, \omega' \in W$ such that $l(\omega\omega') = l(\omega) + l(\omega')$. Then*

$$\chi_\omega * \chi_{\omega'} = \chi_{\omega\omega'}$$

Proof. It suffices to show this when $\omega' = s$ is a simple reflection. By the extension of Lemma 1.2.5 to Chevalley groups we have $B\omega B \cdot BsB = B\omega sB$.

This means that $\chi_\omega * \chi_s$ is supported on $B\omega sB$ and thus, by the B -bi-invariance, it should be a constant multiple of $\chi_{\omega\omega'}$. So $\chi_\omega * \chi'_\omega = c\chi_{\omega\omega'}$

Applying the augmentation map we immediately obtain that the constant c must be 1, since $q^{l(\omega)}q^{l(\omega')} = q^{l(\omega\omega')}$. \square

Lemma 2.3.3. *Let s be a simple reflection in W . Then*

$$\chi_s * \chi_s = q\chi_e + (q - 1)\chi_s$$

Proof. By the extension of Lemma 1.2.5 to Chevalley groups we have $B\omega B \cdot BsB = BsB \sqcup B$. This means that $\chi_s * \chi_s$ is supported on $BsB \sqcup B$ and thus $\chi_s * \chi_s = c_1\chi_e + c_2\chi_s$ for some $c_1, c_2 \in \mathbb{C}$. By evaluating both sides at the identity e we get

$$\chi_s * \chi_s(e) = \frac{1}{|B|} \sum_{x \in BsB} \chi_s(x)\chi_s(x^{-1}) = \frac{|BsB|}{|B|} = q$$

and it follows that $c_1 = q$. At this point, applying the augmentation map, we get $c_2 = \frac{q-q}{q} = q - 1$. \square

Remark 2.3.4. The parameter q in the definition of the Hecke Algebra \mathfrak{H} can be given a specific value, for example we can set q to be any nonzero complex number $z \in \mathbb{C}$. More formally we are considering

$$\mathfrak{H}_z = \mathfrak{H} \otimes_{\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]} \mathbb{C}$$

where \mathbb{C} is regarded as a $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ -algebra through the morphism $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}] \rightarrow \mathbb{C}$ which sends q to z . This process is called *specialization* at $q = z$.

If we specialize q to 1 the quadratic relation for T_s becomes $T_s^2 = 1$ and we recover the group algebra $\mathfrak{H}_1 \cong \mathbb{C}[W]$

Theorem 2.3.5 (Iwahori). *Let $q = p^n$, p prime. Then the algebra $\tilde{\mathfrak{H}}_q$ is isomorphic to the Hecke algebra \mathfrak{H}_q*

Proof. The characteristic functions χ_ω in $\tilde{\mathfrak{H}}_q$ satisfy the defining relations of the algebra \mathfrak{H}_q and thus we get an homomorphism $\phi : \tilde{\mathfrak{H}}_q \rightarrow \mathfrak{H}_q$ such that $\phi(T_\omega) = \chi_\omega$. Also, since ϕ sends the basis $\{T_\omega\}_{\omega \in W}$ to the basis $\{\chi_\omega\}_{\omega \in W}$, ϕ is an isomorphism. \square

We can give an equivalent definition of the convolution product, only in terms of pullback and push-forward functors. As we will see in the chapter 3, this has an analogue in a quite different context.

Firstly we notice that B -bi-invariant functions on G are in correspondence with B -left invariant functions on $X = G/B$. These are in turn in bijection with G -invariant map on $X \times X$ (i.e. functions h on $X \times X$ such that $h(g \cdot x, g \cdot y) = h(x, y) \forall x, y \in X, g \in G$) Thus we have established the bijection

$$\left\{ \begin{array}{l} B\text{-bi-invariant} \\ \text{functions on } G \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} B\text{-left invariant} \\ \text{functions on } X \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} G\text{-left invariant} \\ \text{functions on } X \times X \end{array} \right\}$$

and the composite map is

$$\begin{aligned} f &\longmapsto \sigma(f) \\ \tau(h) &\longleftarrow h \end{aligned}$$

where $\sigma(f)(xB, yB) = f(x^{-1}yB)$ and $\tau(h)(g) = h(eB, gB)$, $\forall x, y, g \in G$.

We can easily transport the definition of the convolution product in this new situation

$$h_1 * h_2(x, y) = \sum_{z \in X} h_1(x, z)h_2(z, y) = \frac{1}{|B|} \sum_{z \in g} h_1(x, zB)h_2(zB, y)$$

In fact, we have $\sigma(f_1) * \sigma(f_2) = \sigma(f_1 * f_2)$ since

$$\begin{aligned} \sigma(f_1) * \sigma(f_2)(xB, yB) &= \frac{1}{|B|} \sum_{z \in G} f(x^{-1}zB)g(z^{-1}yB) = \\ &= (f * g)(x^{-1}y) = \sigma(f * g)(xB, yB) \end{aligned}$$

If $\phi : A \rightarrow B$ is a map between finite sets, we can define the pullback and push-forward of \mathbb{C} -functions

$$\begin{aligned} \phi^*(f) &= f \circ \phi \quad \forall f : B \rightarrow \mathbb{C} \\ \phi_*(f)(b) &= \sum_{a \in \phi^{-1}(b)} f(a) \quad \forall b \in B \text{ and } \forall f : A \rightarrow \mathbb{C} \end{aligned}$$

Now let us consider the following diagram.

$$\begin{array}{ccccc} & & & \Delta & \\ & & & \longleftarrow & \\ X \times X \times X \times X & & & & X \times X \times X \\ & \swarrow p_{12} & & \searrow p_{34} & \downarrow r \\ X \times X & & & X \times X & X \times X \end{array}$$

$$\Delta(x, y, z) = (x, y, y, z) \quad r(x, y, z) = (x, z)$$

At this point we can define $f * g = r_*\Delta^*(p_{12}^*(f) \otimes p_{34}^*(g)) = r_*\Delta^*(f \boxtimes g)$ where \otimes denotes simply the product (i.e. $h_1 \otimes h_2(x) = h_1(x)h_2(x)$) and $f \boxtimes g(x, y, z, w) = f(x, y)g(z, w)$, $\forall x, y, z, w \in X$). We just check that this definition agrees with the one previously given.

$$r_*\Delta^*(f \boxtimes g)(x, y) = \sum_{z \in X} \Delta^*(f \boxtimes g)(x, z, y) = \sum_{z \in X} f(x, z)g(z, y) = (f * g)(x, y)$$

2.4 Kazhdan-Lusztig basis

The ring $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ has a natural involution $\overline{(\cdot)}$ given by $\overline{f(q)} = f(q^{-1})$. We can extend this to an involution of the whole algebra \mathfrak{H} .

$$\overline{\sum_{\omega \in W} c_{\omega}(q)T_{\omega}} = \sum_{\omega \in W} c_{\omega}(q^{-1})T_{\omega^{-1}}$$

We now want to construct a basis $\{C_{\omega}\}_{\omega \in W}$ whose elements are self-dual, i.e. they are fixed by $\overline{(\cdot)}$, and such that the change of basis matrix from the basis $\{T_{\omega}\}_{\omega \in W}$ is upper triangular.

Theorem 2.4.1. *For any $\omega \in W$, there exists a unique element $C_{\omega} \in \mathfrak{H}$ such that $\overline{C_{\omega}} = C_{\omega}$ and*

$$C_{\omega} = q^{-\frac{l(\omega)}{2}}T_{\omega} + \sum_{\nu < \omega} q^{-\frac{l(\nu)+1}{2}}h_{\nu,\omega}(q)T_{\nu}$$

where $h_{\nu,\omega}(q) \in \mathbb{Z}[q^{-\frac{1}{2}}]$

Proof. We start by proving the existence of C_{ω} : we proceed by induction on the length of ω . Of course, $C_e = T_e$ satisfies the requirements for $\omega = e$. If $\omega = s$ is a simple reflection, then $C_s = q^{-\frac{1}{2}}T_s + q^{-\frac{1}{2}}$ works since $\overline{C_s} = q^{\frac{1}{2}}(T_s - q + 1)q^{-1} + q^{\frac{1}{2}} = C_s$. For the inductive step we need the following formula concerning the multiplication on the right by C_s .

$$T_{\omega}C_s = \begin{cases} q^{-\frac{1}{2}}T_{\omega s} + q^{-\frac{1}{2}}T_{\omega} & \text{if } \omega s > \omega \\ q^{\frac{1}{2}}T_{\omega s} + q^{\frac{1}{2}}T_{\omega} & \text{if } \omega s < \omega \end{cases}$$

This follows from

$$T_{\omega}T_s = \begin{cases} T_{\omega s} & \text{if } \omega s > \omega \\ qT_{\omega s} + (q-1)T_{\omega} & \text{if } \omega s < \omega \end{cases}$$

Now we fix $\omega \in W$. By induction we can assume that C_{ν} is already defined for all the ν such that $\nu < \omega$. We can always choose a simple reflection s such that $x s < x$ (for instance, we can pick $s = s_k$ if $\omega = s_1 \cdot \dots \cdot s_k$ is a reduced expression for ω).

$$\begin{aligned} C_{\omega s} \cdot C_s &= \left(q^{-\frac{l(\omega s)}{2}}T_{\omega s} + \sum_{\nu < \omega s} h_{\nu,\omega s}(q)q^{-\frac{l(\nu)+1}{2}}T_{\nu} \right) \cdot C_s = \\ &= q^{-\frac{l(\omega)}{2}}T_{\omega} + q^{-\frac{l(\omega)}{2}}T_{\omega s} + \sum_{\substack{\nu < \omega s \\ \nu s < \nu}} h_{\nu,\omega s}(q)q^{-\frac{l(\nu)}{2}}(T_{\nu s} + T_{\nu}) + \sum_{\substack{\nu < \omega s \\ \nu s > \nu}} h_{\nu,\omega s}(q)q^{-\frac{l(\nu)+2}{2}}(T_{\nu s} + T_{\nu}) = \\ &= q^{-\frac{l(\omega)}{2}}T_{\omega} + \sum_{\nu < \omega} g_{\nu}(q)q^{-\frac{l(\nu)}{2}}T_{\nu} \end{aligned} \tag{2.1}$$

and since $h_{\nu,\omega s}(q)$ is a polynomial in $\mathbb{Z}[q^{-\frac{1}{2}}]$, we have that also $g_{\nu}(q)$ is in $\mathbb{Z}[q^{-\frac{1}{2}}]$.

We define C_ω as $C_\omega = C_{\omega s} C_s - \sum_{\nu < \omega} g_\nu(0) C_\nu$ (here, by $g_\nu(0)$, we just mean the constant term of $g_\nu(q)$). This is clearly self-dual with respect to $(\bar{\cdot})$. Writing the just defined C_ω in the $\{T_\omega\}$ basis, we get

$$C_\omega = q^{-\frac{l(\omega)}{2}} T_\omega + \sum_{\nu < \omega} \tilde{h}_{\nu, \omega}(q) q^{-\frac{l(\nu)}{2}} T_\nu$$

Now we focus our attention on a single coefficient $h_{\nu, \omega}(q)$ of a certain T_ν .

$$\tilde{h}_{\nu, \omega}(q) = g_\nu(q) - g_\nu(0) + (\text{polynomials in } \mathbb{Z}[q^{-\frac{1}{2}}] \text{ without constant terms})$$

Thus, we can write $\tilde{h}_{\nu, \omega}(q) = q^{-\frac{1}{2}} h_{\nu, \omega}(q)$ and C_ω satisfies both the required conditions.

Also an element in \mathfrak{H} with these properties is unique. In fact, suppose that the defining condition hold both for C_ω and C'_ω . Then we denote by $d = C_\omega - C'_\omega$ their difference.

$$d = \sum_{\nu < \omega} r_\nu(q) q^{-\frac{l(\nu)+1}{2}} T_\nu \quad \text{for some } r_\nu(q) \in \mathbb{Z}[q^{-\frac{1}{2}}]$$

If $d \neq 0$ we can take a maximal $z \in W$ such that $r_z(q) \neq 0$. But, using the self-duality of d , we obtain

$$r_z(q) q^{-\frac{l(z)+1}{2}} T_z + \sum_{\nu < z} r_\nu(q) q^{-\frac{l(\nu)+1}{2}} T_\nu = r_z(q^{-1}) q^{\frac{l(z)+1}{2}} T_{z^{-1}}^{-1} + \sum_{\nu < z} r_\nu(q^{-1}) q^{\frac{l(\nu)+1}{2}} T_{\nu^{-1}}^{-1} \quad (2.2)$$

If $\nu = s_1 \cdot \dots \cdot s_k$ is a reduced expression we have that

$$T_{\nu^{-1}}^{-1} = (T_{s_k} \cdot \dots \cdot T_{s_1})^{-1} = q^{-l(\nu)} (T_{s_1} - q + 1) \cdot \dots \cdot (T_{s_k} - q + 1)$$

and if we expand this expression in the $\{T_\omega\}$ -basis, this is in the span of $\{T_\mu\}_{\mu \leq \nu}$. In particular,

$$T_{\nu^{-1}}^{-1} \in q^{-l(\nu)} T_\nu + \text{span}\langle T_\mu \mid \mu < \nu \rangle$$

By (2.2) we get

$$r_z(q) q^{-\frac{l(z)+1}{2}} T_z = r_z(q^{-1}) q^{\frac{l(z)+1}{2}} q^{-l(z)} T_z \implies r_z(q) q^{-\frac{1}{2}} = r_z(q^{-1}) q^{\frac{1}{2}}$$

But, since $r_z(q) \in \mathbb{Z}[q^{-\frac{1}{2}}]$, this would imply $r_z(q) = 0$ and we get a contradiction. Hence $d = 0$ and the proof is complete \square

Remark 2.4.2. This basis is denoted by C'_ω in the original paper [KL79] where it was introduced. The polynomials $h_{\nu, \omega}$ are related to the standard Kazhdan-Lusztig polynomials by the formula:

$$P_{\nu, \omega}(q) = q^{\frac{l(\omega)-l(\nu)-1}{2}} h_{\nu, \omega}(q)$$

Using this polynomials the basis $\{C_\omega\}$ takes the form

$$C_\omega = q^{-\frac{l(\omega)}{2}} \left(T_\omega + \sum_{\nu < \omega} P_{\nu, \omega}(q) T_\nu \right)$$

Theorem 2.4.3. *The Kazhdan-Lusztig polynomials $P_{\nu,\omega}$ are polynomials in $\mathbb{Z}[q]$ of degree at most $\frac{1}{2}(l(\omega) - l(\nu) - 1)$*

Proof. We just need to show that $P_{\nu,\omega} \in \mathbb{Z}[q]$, $\forall \nu, \omega \in W$. The degree condition will then automatically follow since $P_{\nu,\omega}(q) = q^{\frac{l(\omega)-l(\nu)-1}{2}} h_{\nu,\omega}(q)$ and $h_{\nu,\omega}(q) \in \mathbb{Z}[q^{-\frac{1}{2}}]$.

This can be done by induction on the length of ω . For $\omega = s \in S$, $C_s = q^{-\frac{1}{2}}T_s + q^{-\frac{1}{2}}T_e$ and $P_{e,s} = 1$.

Let now ω be any element in W . As in the proof of 2.4.1 let $s \in S$ such that $\omega s < \omega$. By induction we can assume $P_{\nu,\omega s}(q) \in \mathbb{Z}[q]$ for any $\nu < \omega s$.

$$\begin{aligned} C_{\omega s} \cdot C_s &= q^{-\frac{l(\omega s)}{2}} \left(T_{\omega s} + \sum_{\nu < \omega s} P_{\nu,\omega s}(q) T_\nu \right) \cdot C_s = \\ &= q^{-\frac{l(\omega)}{2}} (T_\omega + T_{\omega s}) + \sum_{\substack{\nu < \omega s \\ \nu s < \nu}} P_{\nu,\omega s}(q) q^{-\frac{l(\omega)}{2}} q (T_{\nu s} + T_\nu) + \sum_{\substack{\nu < \omega s \\ \nu s > \nu}} P_{\nu,\omega s}(q) q^{-\frac{l(\omega)}{2}} (T_{\nu s} + T_\nu) \end{aligned}$$

In this equation all the coefficients of the T_ν s are in $q^{-\frac{l(\omega)}{2}}\mathbb{Z}[q]$. Furthermore, from this we can see that for the g_ν defined in (2.1), $g_\nu(0)$ must be 0 if $l(\omega) - l(\nu)$ is odd.

This means that also in

$$\sum_{\nu < \omega} g_\nu(0) C_\nu = q^{-\frac{l(\omega)}{2}} \sum_{\nu < \omega} g_\nu(0) q^{\frac{l(\omega)-l(\nu)}{2}} \left(T_\nu + \sum_{\mu < \nu} P_{\mu,\nu}(q) T_\mu \right)$$

the coefficients of the T_ν are in $q^{-\frac{l(\omega)}{2}}\mathbb{Z}[q]$. So this must hold also for $C_\omega = C_{\omega s} C_s - \sum_{\nu < \omega} g_\nu(0) C_\nu$ \square

Remark 2.4.4. From the proof of the previous theorem we see that $P_{\nu,\omega}(0)$, for $\nu < \omega s$ is exactly $P_{\nu,\omega s}(0)$ if $\nu s > \nu$, while it is $P_{\nu s,\omega s}$ if $\nu s < \nu$. Then we can easily see by induction on the length of ω that the constant term of every polynomial $P_{\nu,\omega}$ is 1.

Example 2.4.5. The above gives also an algorithm to compute the Kazhdan-Lusztig polynomials. Firstly we notice that $P_{\nu,\omega} = 1$ whenever $l(\omega) - l(\nu) \leq 1$. We can see that this implies that $C_s C_t = C_{st}$ if $s, t \in S$ and $s \neq t$.

Let $W = S_3$. Then the unique unknown polynomial is $P_{e,sts}$.

$$C_{st} C_s = q^{-1}(T_{st} + T_s + T_t + 1) C_s = q^{-\frac{3}{2}}(T_{sts} + T_{st} + T_{ts} + T_t + (1+q)T_s + (1+q)T_e)$$

We see that $g_\nu(0) = 0$ for any $\nu \neq s$, while $g_s(0) = 1$. So $C_{sts} = C_{st} C_s - C_s = q^{-\frac{3}{2}}(T_{sts} + T_{st} + T_{ts} + T_t + T_s + T_e)$ and all the Kazhdan-Lusztig are trivial.

However we can find the first nontrivial polynomial already for $W = S_4$. Let $S = \{s, t, u\}$ and we can check that $P_{\nu,tsu} = 1 \forall \nu < tsu$.

$$C_{tsu} C_t = \left(q^{-\frac{3}{2}} \sum_{\nu < tsu} T_\nu \right) C_s = q^{-2} \sum_{\nu < tsut} T_\nu + q^{-1}(T_t + T_e)$$

Here $g_\nu(0) = 0$ for any ν , so we have $C_{tsut} = C_{tsu} C_t$ and $P_{e,tsut}(q) = P_{t,tsut}(q) = q+1$.

Chapter 3

Geometric Construction of the Hecke Algebra

3.1 Convolution of sheaves

For any pair of complex algebraic varieties (X, Y) we consider the derived category $\mathcal{D}_c^b(X \times Y)$ of bounded complexes of $\mathbb{C}_{X \times Y}$ -sheaves whose cohomology sheaves are constructible, as defined in §B.2. Since we will always consider the bounded derived category, from now on we will omit the b .

In analogy with what we have already done in §2.2 in the finite Chevalley groups setting we can define a convolution between complexes of sheaves. IF X, Y and Z are three algebraic varieties, we have the bifunctor:

$$\mathcal{D}_c(X \times Y) \times \mathcal{D}_c(Y \times Z) \rightarrow \mathcal{D}_c(X \times Z) \quad (\mathcal{F}, \mathcal{G}) \mapsto \mathcal{F} * \mathcal{G}$$

$\mathcal{F} * \mathcal{G} = Rr_! \Delta^*(\mathcal{F} \boxtimes \mathcal{G}) = Rr_! \Delta^*(p_{12}^* \mathcal{F} \otimes p_{34}^* \mathcal{G})$ where the maps Δ and r are:

$$\begin{array}{ccccc}
 & & X \times Y \times Y \times Z & \xleftarrow{\Delta} & X \times Y \times Z \\
 & \swarrow p_{12} & & & \downarrow r \\
 X \times Y & & & & X \times Z \\
 & \searrow p_{34} & & & \\
 & & Y \times Z & &
 \end{array}$$

$$\Delta(x, y, z) = (x, y, y, z) \quad r(x, y, z) = (x, z)$$

Proposition 3.1.1. *The convolution product is canonically associative, that is, there is a canonical isomorphism*

$$(\mathcal{E} * \mathcal{F}) * \mathcal{G} \cong \mathcal{E} * (\mathcal{F} * \mathcal{G})$$

for any $\mathcal{E} \in \mathcal{D}_c(X \times Y)$, $\mathcal{F} \in \mathcal{D}_c(Y \times Z)$ and $\mathcal{G} \in \mathcal{D}_c(Z \times W)$.

Proof. We define the maps

$$v : X \times Y \times Z \times W \rightarrow X \times Y^2 \times Z^2 \times W \quad (x, y, z, w) \mapsto (x, y, y, z, z, w)$$

$$\pi : X \times Y \times Z \times W \rightarrow X \times W \quad (x, y, z, w) \mapsto (x, w)$$

We want to show that $(\mathcal{E} * \mathcal{F}) * \mathcal{G} \cong R\pi_! v^*(\mathcal{E} \boxtimes \mathcal{F} \boxtimes G)$. Then, in a symmetric way, one can show that also $\mathcal{E} * (\mathcal{F} * \mathcal{G}) \cong R\pi_! v^*(\mathcal{E} \boxtimes \mathcal{F} \boxtimes G)$.

Let's consider the following commutative diagram:

$$\begin{array}{ccccc} X \times Y \times Z \times W & \xrightarrow{\text{Id}_X \times \Delta} & X \times Y \times Z^2 \times W & \xrightarrow{\Delta \times \text{Id}_{Z \times W}} & X \times Y^2 \times Z^2 \times W \\ \downarrow u & & \downarrow r \times \text{Id}_{Z \times W} & & \\ X \times Z \times W & \xrightarrow{\Delta} & X \times Z^2 \times W & & \\ \downarrow r & & & & \\ X \times W & & & & \end{array}$$

where $u(x, y, z, w) = (x, z, w)$. Since the upper left square is cartesian, we can apply the Proper Base Change Theorem (A.2.4)

$$\begin{aligned} R\pi_! v^*(\mathcal{E} \boxtimes \mathcal{F} \boxtimes G) &= Rr_! Ru_!(\text{Id}_X \times \Delta)^*(\Delta \times \text{Id}_{Z \times W})^*(\mathcal{E} \boxtimes \mathcal{F} \boxtimes G) \cong \\ &\cong Rr_! \Delta^* R(r \times \text{Id}_{Z \times W})_!(\Delta \times \text{Id}_{Z \times W})^*(\mathcal{E} \boxtimes \mathcal{F} \boxtimes G) \cong Rr_! \Delta^*((\mathcal{E} * \mathcal{F}) \boxtimes \mathcal{G}) \end{aligned}$$

and the last term is, by definition, $(\mathcal{E} * \mathcal{F}) * \mathcal{G}$ □

3.2 Convolution on $X \times X$

Let now X be the flag variety of a reductive group G . We define $\mathcal{D}_G(X \times X)$ as the full subcategory of $\mathcal{D}_c(X \times X)$ consisting of objects whose cohomology sheaves are constructible with respect to the stratification by G -orbits.

In view of the above proposition we will always omit parentheses. Then we can define a map from this category into the Hecke algebra $\mathfrak{H} = \mathfrak{H}(W, S)$ of the Weyl group W of G .

$$h : \mathcal{D}_G(X \times X) \rightarrow \mathfrak{H} \quad \mathcal{F} \mapsto \sum_{\substack{i \in \mathbb{Z} \\ \omega \in W}} h^i(\mathcal{F})_\omega q^{\frac{i}{2}} T_\omega$$

where $h^i(\mathcal{F})_\omega$ is the dimension of the stalk $\mathcal{H}^i(\mathcal{F})_x$ at any point x of \mathcal{O}_ω : this is well-defined since $\mathcal{H}^i(\mathcal{F})$ is a locally constant sheaf when restricted to a single G -orbit due to the constructibility condition.

Let $N = \frac{1}{2}|R| = l(\omega_0)$ the dimension of $X = G/B$. We adopt the notation

$$\mathcal{J}_\omega = IC(\overline{\mathcal{O}_\omega})[-N] \quad \text{and} \quad \mathcal{L}_\omega = IC(X_\omega)$$

Here IC stands for the Intersection Cohomology Complex, defined in §B.5. Clearly $\mathcal{J}_\omega[i] \in \mathcal{D}_G(X \times X)$ for any $i \in \mathbb{Z}$.

We recall from Proposition 1.4.1 that $p_1 : \overline{\mathcal{O}_\omega} \rightarrow X$ is a locally trivial fibration with fibers isomorphic to X_ω . This means that locally $\overline{\mathcal{O}_\omega}$ is isomorphic to $U \times X_\omega$ where U is a smooth open subvariety of X .

The Intersection Cohomology complexes can be computed locally on Zariski dense open sets (B.4.4) and we have $\mathcal{J}_\omega|_{U \times X_\omega} = p_2^*(\mathcal{L}_\omega) = \mathbb{C} \boxtimes \mathcal{L}_\omega$. This implies that $\mathcal{H}^i(\mathcal{J}_\omega)_\nu = \mathcal{H}^i(\mathcal{L}_\omega)_\nu$, where $\mathcal{H}^i(\mathcal{L}_\omega)_\nu$ is the stalk of $\mathcal{H}^i(\mathcal{L}_\omega)$ at any point of the orbit $B\nu B/B$.

From the support conditions on Intersection Cohomology B.5.3 we obtain:

$$\text{if } \nu < \omega \text{ then } \mathcal{H}^{i-l(\omega)}(\mathcal{L}_\omega)_\nu = 0 \text{ for } i - l(\omega) \geq -l(\nu).$$

We define $\tilde{P}_{\nu,\omega}(q) = \sum_i \dim \mathcal{H}^{i-l(\omega)}(\mathcal{L}_\omega)_\nu q^{\frac{i}{2}}$. So far we only know that $\tilde{P}_{\nu,\omega}(q) \in \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ and that no power of q bigger than $\frac{1}{2}(l(\omega) - l(\nu) - 1)$ appears. We can rewrite $h(\mathcal{J}_\omega)$ as

$$h(\mathcal{J}_\omega) = q^{-\frac{l(\omega)}{2}} \sum_{\nu < \omega} \tilde{P}_{\nu,\omega}(q) T_\nu$$

The main goal of this section is to show that the convolution product between complexes of sheaves is the geometric counterpart of the product in the Hecke Algebra. To achieve this result we need to restrict our domain as the category $\mathcal{D}_G(X \times X)$ is too large.

Definition 3.2.1. We denote by \mathcal{K} the full subcategory of $\mathcal{D}_G(X \times X)$ formed by all objects in $\mathcal{D}_c(X \times X)$ that are direct sum of \mathcal{J}_ω , $\omega \in W$ and of their shifts $\mathcal{J}_\omega, i \in \mathbb{Z}$.

We start by dealing with a simple reflection s . In this case, $\overline{\mathcal{O}_s}$ is a smooth variety. \mathcal{J}_s is merely $\mathbb{C}_{\overline{\mathcal{O}_s}}[1]$ and $h(\mathcal{J}_s) = q^{-\frac{1}{2}}(T_e + T_s) = C_s$. The following Lemma will provide the fundamental step.

Lemma 3.2.2. *Let $A \in \mathcal{K}$ and suppose further that $\mathcal{H}^i(A) = 0$ for all odd i (or for all even i). Then*

$$i) \mathcal{J}_s * A \in \mathcal{D}_G(X \times X)$$

$$ii) \mathcal{H}^i(\mathcal{J}_s * A) = 0 \text{ for all even } i \text{ (or for all odd } i).$$

$$iii) h(\mathcal{J}_s * A) = C_s h(A)$$

Proof. i) holds because both Δ and r are G -equivariant morphisms. We need to compute $h^i(\mathcal{J}_s * A)_\omega$. We pick $x = (B, \omega B) \in \mathcal{O}_\omega$. Let $p_{ij} : X^4 \rightarrow X^2$ be the projection on the i -th and j -th factors. We have:

$$p_{12}^* \mathcal{J}_s = \mathbb{C}_{\overline{\mathcal{O}_s} \times X \times X}[1]$$

Then

$$\mathcal{J}_s * A = Rr_*(\Delta^*((p_{34}^* A)|_{\overline{\mathcal{O}_s} \times X \times X}))[1]$$

The following diagram is a Cartesian square:

$$\begin{array}{ccc}
r^{-1}(x) & \xrightarrow{i_{r^{-1}(x)}} & X \times X \times X \\
\downarrow r|_{r^{-1}(x)} & & \downarrow r \\
\{x\} & \xrightarrow{i_{\{x\}}} & X \times X
\end{array}$$

Thus by Proper Base Change we obtain

$$\mathcal{H}^i(\mathcal{J}_s * A)_x = i_{\{x\}}^*(R^i r_*)(\Delta^*((p_{34}^* A)|_{\overline{\mathcal{O}_s \times X \times X}})[1]) = H^{i+1}(\Delta^*((p_{34}^* A)|_{\overline{\mathcal{O}_s \times X \times X}})|_{r^{-1}(x)})$$

But the composition

$$r^{-1}(x) \cap \Delta^{-1}(\overline{\mathcal{O}_s} \times X \times X) \hookrightarrow \Delta^{-1}(\overline{\mathcal{O}_s} \times X \times X) \xrightarrow{\Delta} \overline{\mathcal{O}_s} \times X \times X \hookrightarrow X^4 \xrightarrow{p_{34}} X^2$$

is a closed embedding and

$$r^{-1}(x) \cap \Delta^{-1}(\overline{\mathcal{O}_s} \times X \times X) = \{(B, y, \omega B) \mid y \in X_s\} \cong \mathbb{P}^1$$

Setting $Y = X_s \times \{\omega B\} \subseteq X^2$ we have $\mathcal{H}^i(\mathcal{J}_s * A)_x = H^{i+1}(Y, A|_Y)$. Using the constructibility of the complex A we will prove that there exists an open subset $U \cong \mathbb{C}$, $U \subseteq Y$, such that $A|_U$ has locally constant cohomology sheaves. For this we divide into two cases:

- If $l(s\omega) > l(\omega)$, from Lemma 1.2.5, we have $BsB\omega B = B(s\omega)B$. The set $(BsB, \omega B)$ is contained in the G -orbit $G \cdot (B, sB\omega B) = G \cdot (B, BsB\omega B) = \mathcal{O}_{s\omega}$. So we can take $U = \mathcal{O}_{s\omega} \cap Y$ and $Y \setminus U = \{(B, \omega B)\} \in \mathcal{O}_\omega$
- If $l(s\omega) < l(\omega)$ then $BsB\omega B = B\omega B \sqcup B(s\omega)B$. Let $s = s_\alpha$. The set $(BsB, \omega B)$ is contained in $G \cdot (B, sB\omega B) = \mathcal{O}_{s\omega} \sqcup \mathcal{O}_\omega$.

Furthermore, we have $BsB = U_\alpha sB$ and the element $(B, su_\alpha(x)\omega B)$, or equivalently $(u_\alpha(-x)sB, \omega B)$, belongs to $\mathcal{O}_{s\omega}$ if and only $x = 0$. This is a consequence of the proof of Lemma 1.2.5. Thus we can take $U = \mathcal{O}_\omega \cap Y$ and $Y \setminus U = (sB, \omega B) \in \mathcal{O}_{s\omega}$

Up to a quasi-isomorphism (i.e. an isomorphism in $\mathcal{D}(U)$) we can assume that $A|_U = \tau^{\leq k} A|_U$ and $\mathcal{H}^k(A|_U) \neq 0$. We have the distinguished triangle

$$\longrightarrow \tau^{\leq k-1} A|_U \longrightarrow A|_U \longrightarrow \mathcal{H}^k(A|_U)[-k] \xrightarrow{+1}$$

Furthermore we notice that since $\mathcal{H}^{k-1}(A|_U) = 0$ we have $\tau^{\leq k-1} A|_U \cong \tau^{\leq k-2} A|_U$. We can apply the cohomological functor $R\Gamma_c$ to this triangle and we obtain the long exact sequence

$$\dots \rightarrow H_c^i(\tau^{\leq k-2} A|_U) \rightarrow H_c^i(A|_U) \rightarrow H_c^i(\mathcal{H}^k(A|_U)[-k]) \rightarrow H_c^{i+1}(\tau^{\leq k-2} A|_U) \rightarrow \dots$$

$\mathcal{H}^k(A|_U)$ is a constant sheaf on $U \cong \mathbb{A}_\mathbb{C}^1$ which is contractible, thus it is isomorphic to $\mathbb{C}_U^{n_k}$ for some $n_k \in \mathbb{N}$. If we apply the Poincaré Duality to $\mathcal{H}^k(A|_U)$ we obtain

$$H_c^i(\mathcal{H}^k(A|_U)[-k]) = H_c^{i-k}(\mathcal{H}^k(A|_U)) \cong H^{2-i+k}(\mathbb{C}_U^{n_k}) = \begin{cases} \mathbb{C}^{n_k} & \text{if } i = k + 2 \\ 0 & \text{if } i \neq k + 2 \end{cases}$$

We recall that for a sheaf \mathcal{F} on \mathbb{C} we have $H_c^i(\mathbb{C}, \mathcal{F}) = 0$ for any $i > 2$. Hence, $H_c^i(\tau^{\leq k-2} A|_U) = 0$ for any $i > k$. Thus, for any i we have

$$\dim H_c^i(A|_U) = \dim H_c^i(\tau^{\leq k-2} A|_U) + H_c^i(\mathcal{H}^k(A|_U)[-k])$$

. By induction on the cohomological length of $A|_U$ (i.e. the number of nonzero sheaves in the complex) we get

$$\begin{aligned} \dim H_c^i(A|_U) &= \sum_{k \in \mathbb{Z}} \dim H_c^i(\mathcal{H}^k(A|_U)[-k]) = \dim H_c^i(\mathcal{H}^{i-2}(A|_U)[-i+2]) = \\ &= H^{i-2}(\mathcal{H}^{i-2}(A|_U)[-i+2]) = \dim H^0(\mathcal{H}^{i-2}(A|_U)) \end{aligned}$$

But, due of the locally constancy, we can just pick any point $u \in U$ and

$$\dim H_c^i(A|_U) = \dim H^0(\mathcal{H}^{i-2}(A|_U)) = \dim \mathcal{H}^{i-2}(A|_U)_u = \dim \mathcal{H}^{i-2}(A)_u$$

The following triangle is distinguished

$$j_! j^! A \longrightarrow A \longrightarrow i_* i^* A \xrightarrow{+1}$$

where j and i are respectively the open and closed embeddings

$$j : U \hookrightarrow Y \quad i : Y \setminus U = \{u_0\} \rightarrow Y$$

Then we get the long exact sequence taking the cohomology

$$\dots \rightarrow \mathcal{H}^{i-1}(A)_{u_0} \rightarrow H_c^i(U, A|_U) \rightarrow H^i(Y, A|_Y) \rightarrow \mathcal{H}^i(A)_{u_0} \rightarrow$$

Then, by the hypothesis on the cohomology sheaves of A , $H_c^i(A|_U) = \mathcal{H}^{i-2}(A)_u$ and $\mathcal{H}^i(A)_{u_0}$ vanish for all the odd i , hence the long exact sequence splits into short exact sequences

$$0 \rightarrow H_c^i(U, A|_U) \rightarrow H^i(Y, A|_Y) \rightarrow \mathcal{H}^i(A)_{u_0} \rightarrow 0$$

and we get

$$\dim H^i(Y, A|_Y) = \dim \mathcal{H}^{i-2}(A)_u + \dim \mathcal{H}^i(A)_{u_0}$$

From which ii) follows. We divide again into two different cases:

- If $l(s\omega) > l(\omega)$ then

$$h^i(\mathcal{J}_s * A)_\omega = \dim H^{i+1}(Y, A|_Y) = h^{i+1}(A)_\omega + h^{i-1}(A)_{s\omega}$$

- If $l(s\omega) < l(\omega)$ then

$$h^i(\mathcal{J}_s * A)_\omega = \dim H^{i+1}(Y, A|_Y) = h^{i+1}(A)_{s\omega} + h^{i-1}(A)_\omega$$

Finally, iii) follows from:

$$\begin{aligned}
C_s h(A) &= C_s \left(\sum_{i,\omega} h^i(A)_\omega q^{\frac{i}{2}} T_\omega \right) = \\
&= \sum_{i,\omega s > \omega} h^i(A)_\omega q^{\frac{i-1}{2}} (T_{s\omega} + T_\omega) + \sum_{i,\omega s < \omega} h^i(A)_\omega q^{\frac{i+1}{2}} (T_{s\omega} + T_\omega) = \\
&= \sum_{i,\omega s > \omega} (h^{i+1}(A)_\omega + h^{i-1}(A)_{s\omega}) q^{\frac{i}{2}} T_\omega + \sum_{i,\omega s < \omega} (h^{i+1}(A)_{s\omega} + h^{i-1}(A)_\omega) q^{\frac{i}{2}} T_\omega = \\
&= \sum_{i,\omega} h^i(\mathcal{J}_s * A)_\omega q^{\frac{i}{2}} T_\omega = h(\mathcal{J}_s * A)
\end{aligned}$$

□

Remark 3.2.3. In the last proof we have shown that $\dim H_c^\bullet(U, A|_U)$ is equal to $\dim H_c^\bullet(\bigoplus_{j \in \mathbb{Z}} (U, \mathcal{H}^j(A|_U)[-j]))$. Actually, we can make a stronger statement: there exists in $\mathcal{D}_c(X)$ an isomorphism

$$A|_U \cong \bigoplus_{j \in \mathbb{Z}} \mathcal{H}^j(A|_U)[-j]$$

We can prove this claim by induction on the cohomological length of $A|_U$. Let's say that $A|_U$ has zero cohomology sheaves in odd degrees and let $2k$ the biggest integer such that $\mathcal{H}^k(A|_U) \neq 0$. Then we have the following distinguished triangle

$$\tau^{\leq 2k-1} A|_U \rightarrow A|_U \rightarrow \tau^{\geq 2k} A|_U \cong \mathcal{H}^{2k}(A|_U)[-2k] \xrightarrow{+1} \quad (3.1)$$

By induction we can assume that $\tau^{\leq 2k-1} A|_U \cong \bigoplus_{j \leq k-1} \mathcal{H}^{2j}(A|_U)[-2j]$. Each sheaf $\mathcal{H}^{2j}(A|_U)$ is locally constant on $U \cong \mathbb{A}_{\mathbb{C}}^1$, that is it is isomorphic to $\mathbb{C}_U^{n_j}$ for some $n_j \in \mathbb{N}$. The boundary map of the triangle above is an element of

$$\begin{aligned}
&\mathrm{Hom}_{\mathcal{D}(U)}(\mathbb{C}_U^{n_k}[-2k], \bigoplus_{j \leq k-1} \mathbb{C}_U^{n_j}[-2j+1]) \cong \\
&\cong \bigoplus_{j \leq k-1} (\mathrm{Hom}_{\mathcal{D}(U)}(\mathbb{C}_U, \mathbb{C}_U[2(k-j)+1])^{n_k n_j}) \cong \bigoplus_{j \leq k-1} (H^{2(k-j)+1}(U))^{n_k n_j}
\end{aligned}$$

So it is 0, since $H^i(U) \cong H^i(\mathbb{A}_{\mathbb{C}}^1) = 0 \forall i \neq 0$. This implies that the distinguished triangle (3.1) is isomorphic to the triangle

$$\tau^{\leq 2k-1} A|_U \rightarrow \tau^{\leq 2k-1} A|_U \oplus \mathcal{H}^{2k}(A|_U)[-2k] \rightarrow \mathcal{H}^{2k}(A|_U)[-2k] \xrightarrow{+1}$$

Hence, in particular $A|_U \cong \tau^{\leq 2k-1} A|_U \oplus \mathcal{H}^{2k}(A|_U)[-2k] \cong \bigoplus_{j \in \mathbb{Z}} \mathcal{H}^j(A|_U)[-j]$ in $\mathcal{D}_c(X)$.

3.3 The Bott-Samelson Decomposition

We want to generalize the previous result to an arbitrary $\omega \in W$. Here, the Bott-Samelson resolution turns out to be a very useful tool. From now on, with a slight abuse of notation, we will use $\widetilde{\mathcal{O}}_\omega$ in place of $\widetilde{\mathcal{O}}(s_1, \dots, s_l)$

Lemma 3.3.1. *Let $\omega = s_1 \cdot \dots \cdot s_l$ be a reduced expression and $\pi : \widetilde{\mathcal{O}}_\omega = \widetilde{\mathcal{O}}(s_1, \dots, s_l) \rightarrow \overline{\mathcal{O}}_\omega$ be the Bott-Samelson resolution defined in section 1.4. Then*

$$R\pi_* \mathbb{C}_{\widetilde{\mathcal{O}}_\omega}[l(\omega)] = \mathcal{J}_{s_1} * \dots * \mathcal{J}_{s_l}$$

Proof. We set $l = l(\omega)$. Working like in the proof of Proposition 3.1.1, if we define the maps

$$\begin{aligned} v : X^{l+1} &\rightarrow X^{2l} & (x_0, x_1, \dots, x_l) &\mapsto (x_0, x_1, x_1, x_2, x_2, \dots, x_{l-1}, x_{l-1}, x_l) \\ \pi : X^{l+1} &\rightarrow X^{2l} & (x_0, x_1, \dots, x_l) &\mapsto (x_0, x_1, \dots, x_l) \end{aligned}$$

we have that

$$\begin{aligned} \mathcal{J}_{s_1} * \dots * \mathcal{J}_{s_l} &\cong R\pi_* v^* (\mathcal{J}_{s_1} \boxtimes \dots \boxtimes \mathcal{J}_{s_l}) = R\pi_* v^* \left(\mathbb{C}_{\overline{\mathcal{O}}_{s_1}}[1] \boxtimes \dots \boxtimes \mathbb{C}_{\overline{\mathcal{O}}_{s_l}}[1] \right) \cong \\ &\cong R\pi_* v^* \mathbb{C}_{\overline{\mathcal{O}}_{s_1} \times \dots \times \overline{\mathcal{O}}_{s_l}}[l] \end{aligned}$$

Since

$$v^{-1}(\overline{\mathcal{O}}_{s_1} \times \dots \times \overline{\mathcal{O}}_{s_l}) = \{(x_0, \dots, x_l) \in X^{l+1} \mid (x_{i-1}, x_i) \in \overline{\mathcal{O}}_{s_i}\} = \widetilde{\mathcal{O}}_\omega$$

we can conclude, as

$$v^* \mathbb{C}_{\overline{\mathcal{O}}_{s_1} \times \dots \times \overline{\mathcal{O}}_{s_l}}[l] \cong \mathbb{C}_{\widetilde{\mathcal{O}}_\omega}[l]$$

□

Before stating the next proposition we need to make some comments on the Decomposition Theorem C.3.6. Given a map $f : X \rightarrow Y$ of algebraic varieties, a *stratification* of f for a morphism is a stratification $Y = \bigsqcup_\beta Y_\beta$ where Y_β are locally closed subsets with the property that the Intersection cohomology complexes of the \overline{Y}_β 's are constructible with respect to it, and with the additional property that, for any β , the restriction $f : f^{-1}(Y_\beta) \rightarrow Y_\beta$ is a topologically locally trivial fibration.

It is not hard to see that the locally closed subvarieties supporting the local systems appearing in the statement of the Decomposition Theorem are a subset of the Y_β 's. In our situation \mathcal{O}_ν , with $\nu \leq \omega$, form a stratification for π , as we have pointed out in the discussion after Theorem 1.4.4.

Proposition 3.3.2. $h(\mathcal{J}_\omega) = C_\omega$

Proof. We recall that

$$h(R\pi_* \mathbb{C}_{\widetilde{\mathcal{O}}_\omega}[l(\omega)]) = h(\mathcal{J}_{s_1} * \dots * \mathcal{J}_{s_l}) = C_{s_1} \cdot \dots \cdot C_{s_l}$$

and, since $\overline{C_s} = C_s$, we get $\overline{h(\pi_* \mathbb{C}_{\tilde{\mathcal{O}}_\omega}[l(\omega)])} = h(\pi_* \mathbb{C}_{\tilde{\mathcal{O}}_\omega}[l(\omega)])$. We can apply the Decomposition Theorem C.3.6 to the proper birational map $\pi : \tilde{\mathcal{O}}_\omega \rightarrow \overline{\mathcal{O}}_\omega$:

$$R\pi_* \mathbb{C}_{\overline{\mathcal{O}}_\omega}[N + l(\omega)] = \bigoplus_{i \in \mathbb{Z}} {}^p \mathcal{H}^i(R\pi_* \mathbb{C}_{\overline{\mathcal{O}}_\omega}[N + l(\omega)])[-i],$$

and each single perverse cohomology sheaf decomposes into simple objects

$${}^p \mathcal{H}^i(R\pi_* \mathbb{C}_{\overline{\mathcal{O}}_\omega}[N + l(\omega)]) = \bigoplus_{\nu \leq \omega} IC_{\overline{\mathcal{O}}_\nu}(L_{\nu,i})$$

The $L_{\nu,i}$ should be local system on a smooth open subset of $\overline{\mathcal{O}}_\nu$, but, as $\mathcal{O}_\nu \cong \mathbb{C}^{l(\nu)}$ is smooth and contractible, every local system on it is trivial. This means that $IC_{\overline{\mathcal{O}}_\nu}(L_{\nu,i}) \cong IC_{\overline{\mathcal{O}}_\nu}(\mathbb{C}_{\overline{\mathcal{O}}_\nu}^{n_i}) \cong IC(\overline{\mathcal{O}}_\nu) \otimes V_\nu^i$ where V_ν^i is a \mathbb{C} -vector space of dimension n_i .

Since π is a birational and in particular is an isomorphism when restricted to $\pi^{-1}(\mathcal{O}_\omega)$ then $L_{\omega,i} = 0$ for every $i \neq 0$, and $L_{\omega,0} \cong \mathbb{C}_{\mathcal{O}_\omega}$ (as in Corollary C.3.7). Shifting, we obtain

$$R\pi_* \mathbb{C}_{\tilde{\mathcal{O}}_\omega}[l(\omega)] = \mathcal{J}_\omega \oplus \bigoplus_{\substack{\nu < \omega \\ i \in \mathbb{Z}}} \mathcal{J}_\nu \otimes V_\nu^i[-i] \quad (3.2)$$

So we have just shown that

$$C_{s_1} \cdot \dots \cdot C_{s_l} = h(\mathcal{J}_\omega) + \sum_{\nu < \omega} P_\nu(q) h(\mathcal{J}_\nu)$$

where $P_\nu(q) = \sum_i \dim V_\nu^{-i} q^{\frac{i}{2}}$. By induction on the length, we assume that $h(\mathcal{J}_\nu) = C_\nu$ for each $\nu < \omega$. From the previous equality, applying the involution $(\overline{\cdot})$ of the Hecke Algebra, we get

$$\overline{h(\mathcal{J}_\omega)} = \overline{C_{s_1} \cdot \dots \cdot C_{s_l}} - \sum_{\nu < \omega} \overline{P_\nu(q) C_\nu} = C_{s_1} \cdot \dots \cdot C_{s_l} - \sum_{\nu < \omega} P_\nu(q^{-1}) C_\nu.$$

$\mathbb{C}_{\tilde{\mathcal{O}}_\omega}[N + l(\omega)] = IC(\tilde{\mathcal{O}}_\omega)$ is fixed by the Verdier duality $\mathbb{D}_{X^{l+1}}$. The map π is proper, so $R\pi_* = R\pi_! = \mathbb{D}_{X^2} R\pi_* \mathbb{D}_{X^{l+1}}$ and also $R\pi_* \mathbb{C}_{\tilde{\mathcal{O}}_\omega}[N + l(\omega)]$ is fixed by the Verdier Duality \mathbb{D}_{X^2} . This means that there is a canonical isomorphism

$${}^p \mathcal{H}^i(R\pi_* \mathbb{C}_{\overline{\mathcal{O}}_\omega}[l(\omega) + N]) \cong {}^p \mathcal{H}^{-i}(R\pi_* \mathbb{C}_{\overline{\mathcal{O}}_\omega}[l(\omega) + N])^\vee$$

which in turn implies that $V_\nu^i \cong (V_\nu^{-i})^\vee$ or $P_\nu(q) = P_\nu(q^{-1})$. Then it follows from the equation above that $\overline{h(\mathcal{J}_\omega)} = h(\mathcal{J}_\omega)$.

On the other hand $h(\mathcal{J}_\omega) = q^{-\frac{l(\omega)}{2}} \sum_{\tilde{\nu} < \omega} \tilde{P}_{\tilde{\nu},\omega}(q) T_{\tilde{\nu}}$. By the support condition for Intersection Cohomology, if we define $\tilde{h}_{\tilde{\nu},\omega}(q) = q^{-\frac{1}{2}(l(\omega) - l(\tilde{\nu}) - 1)} \tilde{P}_{\tilde{\nu},\omega}(q)$, we get $\tilde{h}_{\tilde{\nu},\omega}(q) \in \mathbb{Z}[q^{-\frac{1}{2}}]$. For $h(\mathcal{J}_\omega)$ the two defining conditions of Theorem 2.4.1 hold. From the uniqueness this yields $h(\mathcal{J}_\omega)$ to be exactly C_ω . Furthermore, we obtain $\tilde{h}_{\tilde{\nu},\omega} = h_{\tilde{\nu},\omega}$, hence $\tilde{P}_{\tilde{\nu},\omega} = P_{\tilde{\nu},\omega}$ \square

Summarizing, we have the following important result, conjectured by Kazhdan and Lusztig in [KL79] and proven, for Weyl groups, shortly after in [KL80]. This result has been recently generalized to a general Coxeter group [EW14a].

Corollary 3.3.3. *We have $\tilde{P}_{\nu,\omega}(q) = P_{\nu,\omega}(q)$. So the Kazhdan-Lusztig polynomials $\tilde{P}_{\nu,\omega}(q)$ have non-negative coefficients.*

Proof. This is trivial since the coefficient of $\tilde{P}_{\nu,\omega}$ are the dimensions of the stalks of a certain complex of sheaves. \square

Remark 3.3.4. We can use this result to compute Kazhdan-Lusztig polynomials in some cases. Let $\omega_0 \in W$ the longest element of a Weyl group. Then \mathcal{O}_{ω_0} is an open dense orbit in X , so $\mathcal{J}_{\omega_0} = \mathbb{C}_{X \times X}[N]$ and

$$h(\mathcal{J}_{\omega_0}) = q^{-\frac{N}{2}} \left(\sum_{\nu \in W} T_{\nu} \right)$$

Thus $P_{\nu,\omega_0}(q) = 1$ for any $\nu \in W$. More in general when X_{ω} is smooth, then $P_{\nu,\omega}(q) = 1$ for any $\nu < \omega$.

Corollary 3.3.5. $\mathcal{H}^i(\mathcal{J}_{\omega})_{\nu} = \mathcal{H}^i(\mathcal{L}_{\omega})_{\nu} = 0$ for all odd $i \in \mathbb{Z}$.

Proof. From Theorem 2.4.3 $P_{\nu,\omega}(q) \in \mathbb{Z}[q]$ and all coefficient of terms of the kind q^i for an odd i , are zero. \square

This last corollary implies that all \mathcal{J}_{ω} ($\omega \in W$), as well as their shifts, satisfy the hypothesis of 3.2.2. So the Lemma 3.2.2 holds, in particular, for any $A \in \mathcal{K}$.

Proposition 3.3.6. $h(\mathcal{J}_{\omega} * \mathcal{J}_{\omega'}) = C_{\omega} C_{\omega'}$ for any $\omega, \omega' \in W$.

Proof. We recall the notation from 3.2. Then

$$R\pi_* \mathbb{C}_{\tilde{\mathcal{O}}_{\omega}}[l(\omega)] * \mathcal{J}_{\omega'} = \mathcal{J}_{\omega} * \mathcal{J}_{\omega'} + \bigoplus_{\substack{\nu < \omega \\ i \in \mathbb{Z}}} (\mathcal{J}_{\nu} * \mathcal{J}_{\omega'}) \otimes V_{\nu}^i[-i]$$

Now we can use Lemma 3.2.2 to obtain

$$C_{s_1} \cdot \dots \cdot C_{s_l} \cdot C_{\omega'} = h(\mathcal{J}_{\omega} * \mathcal{J}_{\omega'}) + \sum_{\nu < \omega} P_{\nu}(q) h(\mathcal{J}_{\nu} * \mathcal{J}_{\omega'})$$

Then, by induction on $l(\omega)$

$$h(\mathcal{J}_{\omega} * \mathcal{J}_{\omega'}) = C_{s_1} \cdot \dots \cdot C_{s_l} \cdot C_{\omega'} - \sum_{\nu < \omega} P_{\nu}(q) C_{\nu} C_{\omega'} = C_{\omega} C_{\omega'}$$

\square

Remark 3.3.7. This does not hold for general complexes in $\mathcal{D}_c(X \times X)$. For example, let's consider $j_! \mathbb{C}_{\mathcal{O}_s}$, where $j : \mathcal{O}_s \hookrightarrow X$ is the embedding. We have $h(j_! \mathbb{C}_{\mathcal{O}_s}) = T_s$ but $h(j_! \mathbb{C}_{\mathcal{O}_s} * j_! \mathbb{C}_{\mathcal{O}_s}) \neq T_s^2 = (q-1)T_s + q$, otherwise we would have

$$h^0(j_! \mathbb{C}_{\mathcal{O}_s} * j_! \mathbb{C}_{\mathcal{O}_s})_s = \dim \mathcal{H}^0(j_! \mathbb{C}_{\mathcal{O}_s} * j_! \mathbb{C}_{\mathcal{O}_s})_{(B,sB)} = -1$$

Proposition 3.3.8. *The category \mathcal{K} is closed under $*$*

Proof. We need only to show that $\mathcal{J}_\omega * \mathcal{J}_{\omega'} \in \mathcal{K}$, for any $\omega, \omega' \in \mathcal{K}$. We observe that

$$\mathcal{J}_\omega \boxtimes \mathcal{J}_{\omega'} = IC(\overline{\mathcal{O}_\omega} \times \overline{\mathcal{O}_{\omega'}})[-2N]$$

Now we need to study $\Delta^* IC(\overline{\mathcal{O}_\omega} \times \overline{\mathcal{O}_{\omega'}})[-2N]$. In general there is no functoriality for Intersection Cohomology, however our situation is very peculiar.

We recall that $p_1 : \overline{\mathcal{O}_{\omega'}} \rightarrow X$ is a locally trivial fibration with fibers $X_{\omega'}$ while $p_2 : \overline{\mathcal{O}_\omega} \rightarrow X$ is a locally trivial fibration with fibers $X_{\omega-1}$. This means that locally the inclusion $Z = \Delta(X^3) \cap (\overline{\mathcal{O}_\omega} \times \overline{\mathcal{O}_{\omega'}}) \hookrightarrow \overline{\mathcal{O}_\omega} \times \overline{\mathcal{O}_{\omega'}}$ looks like the inclusion $X_{\omega-1} \times U \times X_{\omega'} \xrightarrow{\Delta} X_{\omega-1} \times U \times U \times X_{\omega'}$, where U is an open set in X .

The diagonal $\Delta(X) \subseteq X^2$ is a smooth subvariety, so it has a tubular neighborhood in $X \times X$. As a consequence we can find a tubular neighborhood T of $Z = \Delta(X^3) \cap (\overline{\mathcal{O}_\omega} \times \overline{\mathcal{O}_{\omega'}})$, i.e. T is open in $\overline{\mathcal{O}_\omega} \times \overline{\mathcal{O}_{\omega'}}$ and there exists a retraction $p : T \rightarrow Z$ which is a locally trivial vector bundle with fibers isomorphic to \mathbb{C}^N .

We call j the inclusion $T \xrightarrow{j} \overline{\mathcal{O}_\omega} \times \overline{\mathcal{O}_{\omega'}}$. Then

$$\Delta^*(\mathcal{J}_\omega \boxtimes \mathcal{J}_{\omega'}) = \Delta^* j^* (IC(\overline{\mathcal{O}_\omega} \times \overline{\mathcal{O}_{\omega'}})[-2N]) = \Delta^* (IC(T)[-2N]) = IC(\Delta^{-1}(Z))[-N]$$

Now we claim that $Rr_* IC(\Delta^{-1}(Z)) \in \mathcal{K}$. We apply the decomposition theorem to r and, arguing as in the proof of Proposition 3.3.2, we obtain

$$Rr_* IC(\Delta^{-1}(Z)) = \bigoplus_{\substack{\nu \in W \\ i \in \mathbb{Z}}} IC(\mathcal{O}_\nu) \otimes V_\nu^i[-i]$$

where V_ν^i are finite dimensional vector spaces. Thus it is in \mathcal{K} . □

Chapter 4

Soergel Bimodules and the “Erweiterungssatz”

4.1 The Cohomology of the Flag Variety

Let V a representation of the Weyl group and let's denote by S the symmetric algebra $\text{Sym}(V)$ and by $\text{Sym}^+(V)$ the ideal of all elements with vanishing constant term. So the Weyl group action on \mathfrak{t} , the Lie algebra of T , induces an action on $S = \text{Sym}(\mathfrak{t})$ and $S^+ = \text{Sym}^+(\mathfrak{t})$.

Definition 4.1.1. $C(V) = S/(S^+)^W S$ is called the co-invariant ring of the representation V .

Let $X = G/B$ the flag variety. The description of the cohomology ring of X in terms of the co-invariant ring is a classical result, due to Borel:

Theorem 4.1.2. [Bor53] *The cohomology ring $H^\bullet(X, \mathbb{C}_X)$ is isomorphic, as a graded ring, to the coinvariant ring $C = C(\mathfrak{t}^\vee)$. Here \mathfrak{t}^\vee is the dual of $\mathfrak{t} = \mathfrak{g}_0 = \text{Lie}(T)$, the maximal toral subalgebra of \mathfrak{g} , and the symmetric algebra is graded in such a way that $\deg(\mathfrak{t}^\vee) = 2$.*

Although we don't give a complete proof of the theorem, it is useful to have an insight into it and to understand the maps involved in it. We start with the *exponential exact sequence* on X .

$$0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O} \rightarrow \mathcal{O}^* \rightarrow 0$$

Here \mathcal{O} (resp. \mathcal{O}^*) stands for the sheaf of holomorphic functions (resp. nonvanishing holomorphic functions). The deriving boundary map $c_1 : \text{Pic}(X) = H^1(X, \mathcal{O}^*) \rightarrow H^2(X, \mathbb{Z})$ is known as *first Chern class*. It is injective, since $H^1(X, \mathcal{O}) \subseteq H^1(X, \mathbb{C}) = 0$. Let's now prove surjectivity.

The Bruhat decomposition is also a cell decomposition of X . From this we see that $H^2(X)$ is generated by Poincaré dual of fundamental class of cells of codimension 2, i.e. by the dual of $(X_{\omega_0 s})$, $s \in S$. The subvariety $X_{\omega_0 s}$ is a divisor and define

a line bundle $\mathcal{O}[X_{\omega_0s}]$. The surjectivity follows from $c_1(\mathcal{O}[X_{\omega_0s}]) = X_{\omega_0s}$ [GH p. 141].

Hence $c_1 : Pic(X) \rightarrow H^2(X, \mathbb{Z})$ is an isomorphism. Furthermore we know that every line bundle on X can be linearized [Lur] and that every linearized line bundles is of the form $G \times_B V$, where V is a character of B , hence of T . We obtain $X(T) \cong Pic(X) \cong H^2(X, \mathbb{Z})$. Tensorizing by \mathbb{C} we get $\mathfrak{t}^\vee \cong H^2(X, \mathbb{C})$. Since classes of even degrees commute in cohomology, so in particular $H^\bullet(X, \mathbb{C})$ is a commutative ring, we can extend it to a morphism $S(\mathfrak{t}^\vee) \rightarrow H^\bullet(X, \mathbb{C})$. To conclude one needs to show that this map is surjective and that the kernel is generated by $(S^+)^W$.

4.2 The Module Structure on the Hypercohomology

Given two objects $\mathcal{F}, \mathcal{G} \in \mathcal{D}(X)$ we define

$$\mathrm{Hom}_{\mathcal{D}(X)}^\bullet(\mathcal{F}, \mathcal{G}) = \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{D}(X)}(\mathcal{F}, \mathcal{G}[i])$$

Moreover, $\mathrm{End}^\bullet(\mathcal{F}) = \mathrm{Hom}_{\mathcal{D}(X)}^\bullet(\mathcal{F}, \mathcal{F})$ has a structure of graded \mathbb{C} -algebra.

Let p the map from X to a point. We have, by adjunction,

$$H^\bullet(X, \mathcal{F}) = p_* \mathcal{F} = \mathrm{Hom}_{\mathcal{D}(\mathrm{pt})}^\bullet(\mathbb{C}_{\mathrm{pt}}, p_* \mathcal{F}) = \mathrm{Hom}_{\mathcal{D}(X)}^\bullet(\mathbb{C}_X, \mathcal{F})$$

In particular $C = H^\bullet(X, \mathbb{C}_X) \cong \mathrm{End}_{\mathcal{D}(X)}^\bullet(\mathbb{C}_X)$ and we get an action of C on $H^\bullet(X, \mathcal{F})$ given by composition on the left.

On the other hand, there exists a canonical isomorphism $r : \mathcal{F} \otimes \mathbb{C}_X \rightarrow \mathcal{F}$ and we get another action of C on \mathcal{F} , that is, there is a canonical map $C = \mathrm{End}_{\mathcal{D}(X)}^\bullet(\mathbb{C}_X) \rightarrow \mathrm{End}_{\mathcal{D}(X)}^\bullet(\mathcal{F})$, $\forall \mathcal{F} \in \mathcal{D}(X)$, and by functoriality C acts also on the pushforward $p_* \mathcal{F} \simeq H^\bullet(\mathcal{F})$. In other words $f \in C = \mathrm{End}_{\mathcal{D}(X)}^\bullet(\mathbb{C}_X)$ acts on $H^\bullet(\mathcal{F}) = \mathrm{Hom}_{\mathcal{D}(X)}(\mathbb{C}_X, \mathcal{F} \otimes \mathbb{C}_X)$ as the composition on the right for $1 \otimes f$.

Let $f \in \mathrm{Hom}_{\mathcal{D}(X)}(\mathbb{C}_X, \mathbb{C}_X[d])$ and $g \in \mathrm{Hom}_{\mathcal{D}(X)}(\mathbb{C}_X, \mathcal{F})$. The following diagram is commutative

$$\begin{array}{ccc}
\mathbb{C}_X & \xrightarrow{g} & \mathcal{F} \\
\downarrow f & \swarrow r & \uparrow r \\
& \mathbb{C}_X \otimes \mathbb{C}_X & \xrightarrow{g \otimes \mathrm{Id}} & \mathcal{F} \otimes \mathbb{C}_X \\
& \mathrm{Id} \otimes f \downarrow & \searrow g \otimes f & \downarrow \mathrm{Id} \otimes f \\
& \mathbb{C}_X \otimes \mathbb{C}_X[d] & \xrightarrow{g \otimes \mathrm{Id}} & \mathcal{F} \otimes \mathbb{C}_X[d] \\
& \swarrow r[d] & & \downarrow r[d] \\
\mathbb{C}_X[d] & \xrightarrow{g[d]} & \mathcal{F}[d]
\end{array}$$

(In general, the lower square is $(-1)^d$ -commutative. However, in our situation, f is nonzero only if d is even). This means that the C -actions we have described are the same.

Thus (hyper)cohomology defines a functor $H^\bullet(X, \cdot) : \mathcal{D}(X) \rightarrow C\text{-mod}$. We will adopt the notation \mathbb{H} instead of $H^\bullet(X, \cdot)$ for the cohomology when we will want to emphasize the C -module structure.

Let α a simple root and P_α be the corresponding minimal parabolic group containing B . Let $Y_\alpha = G/P_\alpha$ and $\pi_\alpha : X \rightarrow Y_\alpha$ the projection. We now want to study the C -module structure of the cohomology of sheaves \mathcal{F} which are pullbacks $\pi_\alpha^* \mathcal{G}$ of sheaves \mathcal{G} on Y_α . Let $s = s_\alpha$. This is a fundamental result in this direction.

Theorem 4.2.1. *[BGG73, 5.5] The pullback $\pi_\alpha^* : H^\bullet(Y_\alpha, \mathbb{C}_{Y_\alpha}) \rightarrow H^\bullet(X, \mathbb{C}_X)$ is injective and the image corresponds to $C^s \subseteq C$, where C^s is the subalgebra of s -invariants in C .*

The map $\pi_\alpha : X \rightarrow Y_\alpha$ is a fiber bundle with fibers isomorphic to $\mathbb{P}_\mathbb{C}^1$. By the Leray-Hirsch Theorem [BT82, 5.11] $H^\bullet(X, \mathbb{C}_X)$ is a free module over $H^\bullet(Y_\alpha, \mathbb{C}_{Y_\alpha})$ of rank 2. Thus, in particular, we have the injectivity of π_α^* . To deduce the second statement one has to check that the of the image Poincaré Duals of the fundamental classes $[P_\omega]$ of the generalized Schubert cells of Y_α , which are a basis of $H^\bullet(Y_\alpha, \mathbb{C}_{Y_\alpha})$, are exactly the classes in $H^\bullet(X, \mathbb{C}_X)$ fixed by s . However, we won't prove this.

We can define the functor $\mathbb{H}_\alpha = \text{Hom}_{\mathcal{D}(Y_\alpha)}^\bullet(\mathbb{C}_{Y_\alpha}, \cdot) : \mathcal{D}(Y_\alpha) \rightarrow C^s\text{-mod}$.

For $\mathcal{F} \in \mathcal{D}(Y_\alpha)$ we have a canonical map of C -modules

$$\begin{aligned} C \otimes_{C^s} \mathbb{H}_\alpha \mathcal{F} &= \text{Hom}_{\mathcal{D}(X)}^\bullet(\mathbb{C}_X, \pi_\alpha^* \mathbb{C}_{Y_\alpha}) \otimes_{\text{End}_{\mathcal{D}(Y_\alpha)}} \text{Hom}_{\mathcal{D}(Y_\alpha)}^\bullet(\mathbb{C}_{Y_\alpha}, \mathcal{F}) \\ &\longrightarrow \text{Hom}_{\mathcal{D}(X)}^\bullet(\mathbb{C}_X, \pi_\alpha^* \mathcal{F}) = \mathbb{H}_\alpha^* \mathcal{F} \end{aligned}$$

Theorem 4.2.2. *The canonical map $C \otimes_{C^s} \mathbb{H}_\alpha \mathcal{F} \rightarrow \mathbb{H}_\alpha^* \mathcal{F}$ is an isomorphism for any $\mathcal{F} \in \mathcal{D}(Y_\alpha)$.*

Proof. The morphism π_α is a proper topological submersion, therefore it is a locally trivial fibration with fibers isomorphic to \mathbb{P}^1 .

We first determine $R\pi_{\alpha*} \mathbb{C}_X$ locally on Y_α . We take $U \subseteq Y_\alpha$ an open set which trivializes the fibration. Let $p : \mathbb{P}^1 \rightarrow \{\text{pt}\}$.

$$\begin{aligned} R\pi_{\alpha*} \mathbb{C}_X|_U &= Rp_{1*} \mathbb{C}_{U \times \mathbb{P}^1} = R(\text{Id} \times p)_*(\mathbb{C}_U \boxtimes \mathbb{C}_{\mathbb{P}^1}) = \mathbb{C}_U \boxtimes Rp_* \mathbb{C}_{\mathbb{P}^1} = \\ &= \mathbb{C}_U \boxtimes (\mathbb{C}_{\text{pt}} \oplus \mathbb{C}_{\text{pt}}[-2]) = \mathbb{C}_U \oplus \mathbb{C}_U[-2] \end{aligned}$$

This implies that $\mathcal{H}^0(R\pi_{\alpha*} \mathbb{C}_X)$ and $\mathcal{H}^2(R\pi_{\alpha*} \mathbb{C}_X)$ are the only nonzero cohomology sheaves of $R\pi_{\alpha*} \mathbb{C}_X$. We further notice that these sheaves are local system of rank 1 which are trivialized: $\mathcal{H}^0(\pi_{\alpha*} \mathbb{C}_X)$ by the constant 1, and $\mathcal{H}^2(\pi_{\alpha*} \mathbb{C}_X)$ by the orientation of the fibres. Thus we have a distinguished truncation triangle

$$\mathbb{C}_{Y_\alpha} \cong \mathcal{H}^0(R\pi_{\alpha*} \mathbb{C}_X) \cong \tau^{\leq 0} R\pi_{\alpha*} \mathbb{C}_X \rightarrow R\pi_{\alpha*} \mathbb{C}_X \rightarrow \tau^{\geq 1} R\pi_{\alpha*} \mathbb{C}_X \cong \mathbb{C}_{Y_\alpha}[-2] \xrightarrow{+1}.$$

The last arrows lives in $\text{Hom}(\mathbb{C}_{Y_\alpha}[-2], \mathbb{C}_{Y_\alpha}[1]) \cong H^3(Y_\alpha, \mathbb{C}) = 0$ hence the triangle splits and $R\pi_{\alpha*} \mathbb{C}_X \cong \mathbb{C}_{Y_\alpha} \oplus \mathbb{C}_{Y_\alpha}[-2]$

Furthermore, there is a canonical isomorphism

$$\text{Hom}_{\mathcal{D}(X)}^\bullet(\mathbb{C}_X, \pi_\alpha^* \mathcal{F}) \cong \text{Hom}_{\mathcal{D}(X)}^\bullet(\mathbb{C}_X, \pi_\alpha^* \mathcal{F}[-2]) = \text{Hom}_{\mathcal{D}(X)}^\bullet(\pi_{\alpha*} \mathbb{C}_X[2], \mathcal{F})$$

Now we can conclude:

$$\begin{aligned}
C \otimes_{C^s} \mathbb{H}_\alpha \mathcal{F} &= \mathrm{Hom}_{\mathcal{D}(X)}^\bullet(\mathbb{C}_X, \pi_\alpha^! \mathbb{C}_{Y_\alpha}[-2]) \otimes_{\mathrm{End}_{\mathcal{D}(Y_\alpha)}(\mathbb{C}_{Y_\alpha})} \mathrm{Hom}_{\mathcal{D}(Y_\alpha)}^\bullet(\mathbb{C}_{Y_\alpha}, \mathcal{F}) = \\
&= \mathrm{Hom}_{\mathcal{D}(Y_\alpha)}^\bullet(\pi_{\alpha*} \mathbb{C}_X[2], \mathbb{C}_{Y_\alpha}) \otimes_{\mathrm{End}_{\mathcal{D}(Y_\alpha)}(\mathbb{C}_{Y_\alpha})} \mathrm{Hom}_{\mathcal{D}(Y_\alpha)}^\bullet(\mathbb{C}_{Y_\alpha}, \mathcal{F}) = \\
&= (\mathrm{End}_{\mathcal{D}(Y_\alpha)}(\mathbb{C}_{Y_\alpha}) \oplus \mathrm{End}_{\mathcal{D}(Y_\alpha)}(\mathbb{C}_{Y_\alpha})[-2]) \otimes_{\mathrm{End}_{\mathcal{D}(Y_\alpha)}(\mathbb{C}_{Y_\alpha})} \mathrm{Hom}_{\mathcal{D}(Y_\alpha)}^\bullet(\mathbb{C}_{Y_\alpha}, \mathcal{F}) = \\
&= \mathrm{Hom}_{\mathcal{D}(Y_\alpha)}^\bullet(\mathbb{C}_{Y_\alpha}, \mathcal{F}) \oplus \mathrm{Hom}_{\mathcal{D}(Y_\alpha)}^\bullet(\mathbb{C}_{Y_\alpha}, \mathcal{F})[-2] = \mathbb{H}_{\pi_\alpha^*} \mathcal{F}
\end{aligned}$$

□

4.3 Bimodules from Hypercohomology

4.3.1 Ringoids

Definition 4.3.1. We call *ringoid* a set R equipped with two monoid structure $(R, +, 0)$ and $(R, \cdot, 1)$ such that, $\forall a, b, c \in R$ $a + b = b + a$, we have $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$

Let $\overline{C\text{-Mod-}C}$ be the set isomorphism classes of C -bimodules. It is a ringoid with \oplus and \otimes_C .

For any \mathbb{C} -category \mathcal{A} , the \mathbb{C} -functors $\mathcal{A} \rightarrow \mathcal{A}$ up to natural equivalences form, with sum and composition, a ring. We denote it by $\mathcal{R}\mathcal{A}$.

The same holds, if \mathcal{A} is a [1]-category, for [1]-functors (i.e., functors commuting with [1]) up to natural [1]-equivalence. We denote it by $\mathcal{R}^\bullet \mathcal{A}$

The map

$$\overline{C\text{-Mod-}C} \rightarrow \mathcal{R}^\bullet C\text{-Mod} \quad B \longmapsto B \otimes_C (\cdot)$$

is a homomorphism of ringoids. We recall the following result about functors of modules. This homomorphism is injective. In fact, the map $B \otimes_C C \times C \rightarrow B \otimes_C C$ defined by $(b \otimes x, y) \rightarrow b \otimes xy$ defines a right C -module structure on $B \otimes_C C$, and this makes the canonical map $B \otimes_C C \cong B$ an isomorphism of bimodules. Thus we can recover the bimodule structure of B relying only on the functor $B \otimes_C (\cdot)$.

Let $\overline{\mathcal{K}}$ be the set of isomorphism classes of objects in \mathcal{K} . This is a ringoid with \oplus and $*$. From Prop. 3.3.2 and 3.3.6 the map $h : \overline{\mathcal{K}} \rightarrow \mathfrak{H}$ is an injective homomorphism of ringoids. We observe that no two different objects in \mathcal{K} are isomorphic. Otherwise, if $\bigoplus_i \mathcal{J}_{\omega_i}[s_i] \cong \bigoplus_j \mathcal{J}_{\nu_j}[t_j]$ we would have in \mathfrak{H} , applying h , $\sum_i q^{s_i} C_{\omega_i} = \sum_j q^{t_j} C_{\nu_j}$, but the elements $q^{s_i} C_{\omega_i}$ are linearly independent over \mathbb{Z} . Thus, we can omit the (\cdot) over \mathcal{K} .

We denote by \mathfrak{H}^+ the image of h . It is the subringoid generated by C_ω , $\omega \in W$ and $q^{\frac{n}{2}}$, $n \in \mathbb{Z}$.

We now consider the subringoid \mathcal{K}_S of \mathcal{K} , generated by \mathcal{J}_s , with s simple, and their shifts. The restriction of h to \mathcal{K}_S is still an injective ringoid homomorphism, and we denote by \mathfrak{H}_S^+ its image, that is the subringoid generated by C_s , with s simple, and $q^{\frac{n}{2}}$, $n \in \mathbb{Z}$.

Let's consider the convolution product in the special case $X = Y = G/B$ and $Z = \{\text{pt}\}$. Then convolution defines also a ringoid homomorphism

$$\mathcal{K} \rightarrow \mathcal{R}^\bullet \mathcal{D}(X) \quad \mathcal{J} \mapsto \mathcal{J} * (\cdot)$$

and we have the following Lemma:

Lemma 4.3.2. *Let $s = s_\alpha$ and $\pi_\alpha : X \rightarrow Y_\alpha = G/P_\alpha$. The following functors $\mathcal{D}(X) \rightarrow \mathcal{D}(X)$ are naturally equivalent:*

1. $\mathcal{F} \mapsto \mathcal{J}_s * \mathcal{F}$
2. $\mathcal{F} \mapsto \pi_\alpha^* R\pi_{\alpha*} \mathcal{F}[1]$
3. $\mathcal{F} \mapsto \pi_\alpha^! R\pi_{\alpha!} \mathcal{F}[-1]$

Proof. 2 and 3 are clearly equivalent, since π_α is proper and smooth. The following diagram is Cartesian

$$\begin{array}{ccc} \overline{\mathcal{O}}_s & \xrightarrow{p_1} & X \\ p_2 \downarrow & & \downarrow \pi_\alpha \\ X & \xrightarrow{\pi_\alpha} & Y_\alpha \end{array}$$

hence $\pi_\alpha^* R\pi_{\alpha*} \mathcal{F} \cong Rp_{1*} p_2^* \mathcal{F}$.

Now we consider the commutative diagram

$$\begin{array}{ccccc} \overline{\mathcal{O}}_s & \xrightarrow{\Delta|_{\overline{\mathcal{O}}_s}} & \overline{\mathcal{O}}_s \times X & \xrightarrow{p_3} & X \\ i \downarrow & & \downarrow j & & \\ X^2 & \xrightarrow{\Delta} & X^3 & & \\ \downarrow r = p_1 & & & & \\ X & & & & \end{array}$$

in which i and j are the obvious closed embeddings. The labeled vertical arrows are the inclusions and the square is cartesian. This shows that

$$\begin{aligned} \mathcal{J}_s * \mathcal{F} &= Rr_* \Delta^* (\mathbb{C}_{\overline{\mathcal{O}}_s} \boxtimes \mathcal{F})[1] = Rr_* i_* i^* \Delta^* (\mathbb{C}_{\overline{\mathcal{O}}_s} \boxtimes \mathcal{F})[1] \cong \\ &\cong Rp_{1*} (\Delta|_{\overline{\mathcal{O}}_s})^* (\mathbb{C}_{\overline{\mathcal{O}}_s} \boxtimes \mathcal{F})[1] = Rp_{1*} (\Delta|_{\overline{\mathcal{O}}_s})^* p_3^* \mathcal{F}[1] \cong Rp_{1*} p_2^* \mathcal{F}[1] \end{aligned}$$

and the proof is concluded. \square

We call \mathcal{C} the full subcategory of $\mathcal{D}(X)$ formed by objects that are direct sum of $\mathbb{C}_X[n]$, $n \in \mathbb{Z}$.

Lemma 4.3.3. *The convolution defines a ringoid homomorphism $\Phi : \mathcal{K}_S \rightarrow \mathcal{R}^\bullet \mathcal{C}$*

Proof. It is enough to prove that $\mathcal{J}_s * \mathbb{C}_X \in \mathcal{C}$, $\forall s \in S$ and $\forall \mathcal{F} \in \mathcal{C}$. This is easy:

$$\mathcal{J}_{s_\alpha} * \mathbb{C}_X \cong \pi_\alpha^* \pi_{\alpha*} \mathbb{C}_X[1] \cong \pi_\alpha^* (\mathbb{C}_{Y_\alpha}[1] \oplus \mathbb{C}_{Y_\alpha}[-1]) = \mathbb{C}_X[1] \oplus \mathbb{C}_X[-1] \in \mathcal{C}$$

□

The hypercohomology of an object in \mathcal{C} is a free graded module over C . Actually, it is easy to observe that $\mathbb{H} : \mathcal{C} \rightarrow C\text{-f-Mod}$, the full subcategory of free graded modules, is an [1]-equivalence of [1]-categories.

Lemma 4.3.4. *There exists a ringoid homomorphism $\hat{\mathcal{E}} : \mathfrak{H}_S^+ \rightarrow \mathcal{R}^\bullet C\text{-f-Mod}$ such that $\hat{\mathcal{E}}(q^n) = C[-n] \otimes_C (\cdot)$ and $\hat{\mathcal{E}}(C_s) = C[1] \otimes_{C^s} (\cdot)$ for any simple reflection $s = s_\alpha$.*

Proof. We get the homomorphism by setting $\hat{\mathcal{E}} = (\mathcal{R}^\bullet \mathbb{H}) \circ \Phi \circ (h^{-1})$ to make the following diagram of functors commutative

$$\begin{array}{ccc} \mathfrak{H}_S^+ & \xrightarrow{\hat{\mathcal{E}}} & \mathcal{R}^\bullet C\text{-f-Mod} \\ \uparrow h & & \uparrow \mathcal{R}^\bullet \mathbb{H} \\ \overline{\mathcal{K}}_S & \xrightarrow{\Phi} & \mathcal{R}^\bullet C \end{array}$$

Now it is easy to verify that $(\mathcal{R}^\bullet \mathbb{H})\Phi h^{-1}(q^n) = (\mathcal{R}^\bullet \mathbb{H})\Phi(\mathcal{J}_e[-n])$ is the functor which sends $\mathbb{H}\mathcal{F}$ to $\mathbb{H}(\mathcal{F}[-n])$ and therefore $\mathcal{E}(q^n) = C[-n] \otimes_C (\cdot)$.

It remains to consider $(\mathcal{R}^\bullet \mathbb{H})\Phi(h^{-1})(C_s) = (\mathcal{R}^\bullet \mathbb{H})\Phi(\mathcal{J}_s)$. This is the functor which sends $\mathbb{H}\mathcal{F}$ to

$$\mathbb{H}(\mathcal{J}_s * \mathcal{F}) = \mathbb{H}(\pi_\alpha^* \pi_{\alpha*} \mathcal{F}[1]) = C \otimes_{C^s} \mathbb{H}_\alpha(\pi_{\alpha*} \mathcal{F}[1]) = C[1] \otimes_{C^s} \mathbb{H}\mathcal{F}$$

□

Corollary 4.3.5. *There exists a ringoid homomorphism $\mathcal{E} : \mathfrak{H}_S^+ \rightarrow \overline{C\text{-Mod-}C}$ such that $\mathcal{E}(q^n) = C[-n]$ and $\mathcal{E}(C_s) = C \otimes_{C^s} C$ for any simple reflection s*

Proof. The image of $\hat{\mathcal{E}}$ is a subringoid of $\mathcal{R}^\bullet C\text{-f-Mod}$, whose generators are in the image of the homomorphism $\overline{C\text{-Mod-}C} \rightarrow \mathcal{R}^\bullet C$. In fact, $C[-n] \otimes_C (\cdot)$ and $C[1] \otimes_{C^s} (\cdot)$ are obviously the images of $C[-n]$ and $C \otimes_{C^s} C[1]$. Thus we can lift $\hat{\mathcal{E}}$ to a ringoid homomorphism $\mathcal{E} : \mathfrak{H}_S^+ \rightarrow \overline{C\text{-Mod-}C}$ which satisfies the above conditions. □

We now consider the ringoid homomorphism $\hat{\mathbb{B}} = \mathcal{E} \circ h : \mathcal{K}_S \rightarrow \overline{C\text{-Mod-}C}$.

Lemma 4.3.6. *Let $\mathcal{J} \in \overline{\mathcal{K}}_S$. The following functors, from $\mathcal{D}(X)$ to $C\text{-Mod}$,*

$$\mathcal{F} \longmapsto \mathbb{H}(\mathcal{J} * \mathcal{F}) \quad \text{and} \quad \mathcal{F} \longmapsto \hat{\mathbb{B}}\mathcal{J} \otimes_C \mathbb{H}\mathcal{F}$$

are equivalent.

Proof. Firstly we assume $\mathcal{J} = \mathcal{J}_s$, where $s = s_\alpha$ is a simple reflection. It follows from 4.2.2 that

$$\mathbb{H}(\mathcal{J}_s * \mathcal{F}) = \mathbb{H}(\pi_\alpha^* \pi_{\alpha*} \mathcal{F}[1]) = C[1] \otimes_{C^s} \mathbb{H}\mathcal{F}$$

On the other hand $\hat{\mathbb{B}}\mathcal{J}_s = \mathcal{E}(C_s) = C \otimes_{C^s} C[1]$.

In general, an element of $\overline{\mathcal{K}}_S$ can be written as a direct sum of (shift of) the sheaves $\mathcal{J}_{s_1} * \dots * \mathcal{J}_{s_k}$. By induction on k we have

$$\begin{aligned} \mathbb{H}(\mathcal{J}_{s_1} * \dots * \mathcal{J}_{s_k} * \mathcal{F}) &\cong \hat{\mathbb{B}}\mathcal{J}_{s_1} \otimes_C \hat{\mathbb{B}}(\mathcal{J}_{s_2} * \dots * \mathcal{J}_{s_k}) \otimes_C \mathbb{H}\mathcal{F} = \\ &= \hat{\mathbb{B}}(\mathcal{J}_{s_1} * \dots * \mathcal{J}_{s_k}) \otimes_C \mathbb{H}\mathcal{F} \end{aligned}$$

□

The next step will be to extend this homomorphism to the whole \mathfrak{H} . In order to make this possible we need to change slightly our codomain.

4.3.2 The Split Grothendieck Group

Definition 4.3.7. Let \mathcal{A} an additive category . We denote by $\langle \mathcal{A} \rangle$ its *split Grothendieck group*. It is a free abelian group whose basis is indexed by the objects of \mathcal{A} and subject to the relation

$$A = A' + A'' \quad \text{if} \quad A \cong A' \oplus A''$$

For an object $A \in \mathcal{A}$ we denote by $[A]$ its class in $\langle \mathcal{A} \rangle$.

Lemma 4.3.8. *Let A and B two objects in \mathcal{A} . We denote by $[A]$ its class in $\langle \mathcal{A} \rangle$. Then $[A] = [B]$ if and only if there exists an object C such that $A \oplus C \cong B \oplus C$*

Proof. One direction is immediate. So let's assume that $[A] - [B] = 0$ in $\langle \mathcal{A} \rangle$. This means that, if we denote by \overline{A} the isomorphism class of A , in the free abelian group indexed by object of \mathcal{A} we have

$$\overline{A} - \overline{B} = \sum_{i=1}^n (\overline{X_i \oplus Y_i} - \overline{X_i} - \overline{Y_i}) - \sum_{j=1}^m (\overline{W_j \oplus Z_j} - \overline{W_j} - \overline{Z_j})$$

which we can rewrite as

$$\overline{A} + \sum_{i=1}^n (\overline{X_i} + \overline{Y_i}) + \sum_{j=1}^m (\overline{W_j} \oplus \overline{Z_j}) = \overline{B} + \sum_{i=1}^n (\overline{X_i \oplus Y_i}) + \sum_{j=1}^m (\overline{W_j} + \overline{Z_j})$$

Since the isomorphism class are a basis in the free abelian group, we have that the elements on the left hand side are a permutation of the elements on the right hand side. Setting $C = \bigoplus_i (X_i \oplus Y_i) \oplus \bigoplus_j (W_j \oplus Z_j)$ we have the thesis. □

Example 4.3.9. Let $Vect_{\mathbb{C}}$ the category of \mathbb{C} -vector spaces. Then the split Grothendieck group $\langle Vect_{\mathbb{C}} \rangle = 0$. In fact, for any two vector spaces A and B , we can always find a vector space C , whose basis's cardinality is big enough, such that $A \oplus C = B \oplus C$. On the other hand, if we consider the category $Vect_{\mathbb{C}}^f$ of finite \mathbb{C} -vector spaces, we have $\langle Vect_{\mathbb{C}}^f \rangle \cong \mathbb{Z}$.

We can consider $\langle C\text{-Mod-}C \rangle$. Equipped with the operation \otimes_C , it becomes a ring.

At this point the extension follows from a universal property of \mathfrak{H} .

Definition 4.3.10. Let R^+ be a ringoid. The universal ring $\mathcal{U}(R^+)$ of R^+ is a ring, with a ringoid homomorphism $\phi : R^+ \rightarrow \mathcal{U}(R^+)$ such that, for any ring S and any ringoid homomorphism $\psi : R^+ \rightarrow S$ there exists a unique ring homomorphism $\bar{\psi} : \mathcal{U}(R^+) \rightarrow S$ such that $\psi = \bar{\psi} * \phi$

$$\begin{array}{ccc} R^+ & \xrightarrow{\psi} & S \\ \phi \downarrow & \nearrow \bar{\psi} & \\ \mathcal{U}(R^+) & & \end{array}$$

This universal ring always exists. We start with the free product of R^+ copies of \mathbb{Z} :

$$\bigstar_{x \in R^+} \mathbb{Z}e_x$$

and we quotient it by the relations $e_x e_y = e_{xy}$ and $e_x + e_y = e_{x+y}$, $\forall x, y \in R^+$. Finally we define $\phi(x) = e_x$, $\forall x \in R^+$.

If R^+ is a subringoid of a ring R , and if R^+ generates R as a ring, then $\mathcal{U}(R^+)$. In fact, we obviously have a surjective ring homomorphism $\psi : \mathcal{U}(R^+) \rightarrow R$. $\psi(\sum n_x e_x) = \sum n_x x = 0 \implies \sum_{n_x > 0} n_x x = \sum_{n_x < 0} -n_x x$ and this means that $\sum_{n_x > 0} n_x e_x = \sum_{n_x < 0} -n_x e_x \implies \sum n_x e_x = 0$.

Now we apply this to our situation. \mathfrak{H}^+ is a subringoid of \mathfrak{H} and it generates it as a ring: in fact, by induction, it generates C_ω since it is defined as $C_\omega = C_{\omega s} C_s - \sum_{\nu < \omega} g_\nu(0) C_\nu$.

Example 4.3.11. $\langle \mathcal{K} \rangle$, equipped with the convolution product, is a ring. From Lemma 4.3.8 we obtain that $\overline{\mathcal{K}_S}$ is a subringoid of $\langle \mathcal{K} \rangle$. Moreover, it generates $\langle \mathcal{K} \rangle$ as a ring, therefore $\langle \mathcal{K} \rangle = \mathcal{U}(\overline{\mathcal{K}_S})$. So we can extend h to a ring isomorphism $h : \langle \mathcal{K} \rangle \rightarrow \mathfrak{H}$.

Theorem 4.3.12. *There exists a unique ring homomorphism $\mathcal{E} : \mathfrak{H} \rightarrow \langle C\text{-Mod-}C \rangle$ such that $\mathcal{E}(t) = \langle C[-1] \rangle$, $\mathcal{E}(C_s) = \langle C \otimes_{C^s} C \rangle[1]$ for any simple reflection s*

Now our wish is to prove that the functor $\hat{\mathbb{B}}$ is just the hypercohomology.

4.3.3 The Cohomology of Schubert Varieties

Definition 4.3.13. Let $\mathcal{F} \in \mathcal{D}(X \times X)$. We denote by $\mathbb{B}(\mathcal{F})$ the hypercohomology. \mathbb{B} is a functor into $C\text{-Mod-}C$, the category of C -graded bimodules. Here, the left C -module structure arises from the left copy of X , and the right C -module from the right copy of X .

Lemma 4.3.14. *For any $\mathcal{F} \in \mathcal{D}(X \times X)$, $\mathbb{B}(\mathcal{F}) \cong \mathbb{H}(\mathcal{F} * \mathbb{C}_X)$ as C -bimodules, where the right C -action on $\mathbb{H}(\mathcal{F} * \mathbb{C}_X)$ comes from the action on \mathbb{C}_X .*

Proof. $\mathcal{F} * \mathbb{C}_X = Rr_* \Delta^*(p_{12}^* \mathcal{F}) = Rr_* \mathcal{F} = Rp_{1*} \mathcal{F}$, so clearly $\mathbb{B}(\mathcal{F}) \cong \mathbb{H}(\mathcal{F} * \mathbb{C}_X)$ as complex of vector spaces. The left C -actions clearly coincide. An element $f \in \text{End}_{\mathcal{D}(X)}(\mathbb{C}_X)$, via the right C -action, sends $g \in \text{Hom}_{\mathcal{D}(X)}(\mathbb{C}_X, \mathcal{F} * \mathbb{C}_X)$ into the composition

$$\mathbb{C}_X \xrightarrow{g} \mathcal{F} * \mathbb{C}_X \xrightarrow{\text{Id}_{\mathcal{F}} * f} \mathcal{F} * \mathbb{C}_X$$

where $\text{Id}_{\mathcal{F}} * f = Rr_* \Delta^*(\text{Id}_{\mathcal{F}} \boxtimes f) = Rr_*(\text{Id}_{\mathcal{F}} \otimes (\text{Id}_{\mathbb{C}_X} \boxtimes f))$. This, by adjunction, corresponds to

$$\mathbb{C}_{X \times X} \xrightarrow{g} \mathcal{F} \otimes \mathbb{C}_{X \times X} \xrightarrow{\text{Id}_{\mathcal{F}} \otimes (\text{Id}_{\mathbb{C}_X} \boxtimes f)} \mathcal{F} \otimes \mathbb{C}_{X \times X}$$

As in the discussion in §4.2, we can deduce that the two right C -actions coincide. \square

Proposition 4.3.15. *The functors $\mathbb{B}, \hat{\mathbb{B}} : \mathcal{K}_S \rightarrow C\text{-Mod-}C$ are naturally equivalent.*

Proof. It suffices to prove that, for any $\mathcal{J} \in \mathcal{K}_S$ the functors in $\mathcal{R}^\bullet C\text{-f-Mod}$

$$\Phi(\mathcal{J}) : C \mapsto \mathbb{H}(\mathcal{J} * \mathbb{C}_X) \quad \text{and} \quad \mathbb{B}(\mathcal{J}) \otimes_C (\cdot) : C \mapsto \mathbb{B}(\mathcal{J}) \otimes_C C$$

are naturally equivalent and, for this, we just need to show that $\mathbb{H}(\mathcal{J} * \mathbb{C}_X) \cong \mathbb{B}(\mathcal{J})$, but this is exactly the statement of Lemma 4.3.14. \square

Theorem 4.3.16. *The group homomorphism $\mathbb{B}, \hat{\mathbb{B}} : \langle \mathcal{K} \rangle \rightarrow \langle C\text{-Mod-}C \rangle$ coincide. In particular \mathbb{B} is a ring homomorphism.*

Proof. We already know that they coincide on \mathcal{K}_S . To conclude we just need the second statement, i.e. that $\mathbb{B}(\mathcal{J} * \mathcal{J}') \cong \mathbb{B}(\mathcal{J}) \otimes_C \mathbb{B}(\mathcal{J}')$ for any $\mathcal{J}, \mathcal{J}' \in \mathcal{K}$. Clearly we can assume $\mathcal{J} = \mathcal{J}_\omega, \mathcal{J}' = \mathcal{J}_{\omega'}$.

Firstly we fix $\omega' = s' \in S$ a simple reflection and we show, by induction on $l(\omega)$, that the claim is true for ω . If $\omega = s$ is a simple reflection this descends from the fact that $\mathcal{J}_s * \mathcal{J}_{s'} \in \mathcal{K}_S$ and \mathbb{B} and $\hat{\mathbb{B}}$ coincide on \mathcal{K}_S .

For a general $\omega \in W$, using the Bott-Samelson decomposition

$$\begin{aligned} & \mathbb{B} \left(\mathcal{J}_\omega \oplus \bigoplus_{\substack{\nu < \omega \\ i \in Z}} \mathcal{J}_\nu \otimes V_\nu^i[-i] \right) \otimes_C \mathbb{B}(\mathcal{J}_{s'}) = \mathbb{B}(\mathcal{J}_{s_1} * \dots * \mathcal{J}_{s_l}) \otimes_C \mathbb{B}(\mathcal{J}_{s'}) = \\ & = \mathbb{B}(\mathcal{J}_{s_1} * \dots * \mathcal{J}_{s_l} * \mathcal{J}_{s'}) = \mathbb{B}(\mathcal{J}_\omega * \mathcal{J}_{s'}) \oplus \mathbb{B} \left(\bigoplus_{\substack{\nu < \omega \\ i \in Z}} \mathcal{J}_\nu \otimes V_\nu^i[-i] \right) \otimes_C \mathbb{B}(\mathcal{J}_{s'}) \end{aligned}$$

we obtain $\mathbb{B}(\mathcal{J}_\omega * \mathcal{J}_{s'}) = \mathbb{B}(\mathcal{J}_\omega) \otimes_C \mathbb{B}(\mathcal{J}_{s'})$

For a general ω' we have only to use again the Bott-Samelson decomposition, this time on the second factor, and conclude by induction. \square

Thus we can carry on \mathbb{B} the properties of $\hat{\mathbb{B}}$, obtaining this fundamental result that allows us to effectively compute the hypercohomology of complexes in \mathcal{K} .

Corollary 4.3.17. *Let $\mathcal{J}, \mathcal{J}' \in \mathcal{K}$. Then*

- i) $\mathbb{B}(\mathcal{J} * \mathcal{J}') \cong \mathbb{B}(\mathcal{J}) \otimes_C \mathbb{B}(\mathcal{J}')$ in $C\text{-Mod-}C$.
- ii) The functors $(\mathcal{D}(X) \rightarrow C\text{-Mod}) \mathcal{F} \mapsto \mathbb{H}(\mathcal{J} * \mathcal{F})$ and $\mathcal{F} \mapsto \mathbb{B}(\mathcal{J}) \otimes_C \mathbb{H}(\mathcal{F})$ are naturally equivalent
- iii) $\mathbb{B}(\mathcal{J}_s) \cong C \otimes_{C^s} C[1]$

In the other direction, the theorem implies that for any $\omega \in W$ there exists a bimodule $B_\omega = \mathbb{B}(\mathcal{J}_\omega) \in C\text{-Mod-}C$ such that $\mathcal{E}(C_\omega) = \langle B_\omega \rangle$.

The tensor product \otimes_C defines an action of $\langle C\text{-Mod-}C \rangle$ on $\langle C\text{-Mod} \rangle$. Also \mathfrak{H} , through \mathcal{E} , acts on $\langle C\text{-Mod} \rangle$.

We call $D_\omega = B_{\omega^{-1}} \otimes_C \mathbb{C} \in C\text{-Mod}$. Clearly $\langle D_\omega \rangle = C_{\omega^{-1}} \langle \mathbb{C} \rangle$. Now we put the various pieces together.

Theorem 4.3.18. $\mathbb{H}(\mathcal{L}_\omega)$, as a C -module, is isomorphic to D_ω .

Proof. We have $\mathbb{B}\mathcal{J}_\omega = \hat{\mathbb{B}}\mathcal{J}_\omega = \mathcal{E}h(\mathcal{J}_\omega) = \mathcal{E}(C_\omega) = B_\omega$. Then, from 4.3.17, we get

$$D_\omega = B_{\omega^{-1}} \otimes_C \mathbb{C} = \mathbb{B}\mathcal{J}_{\omega^{-1}} \otimes_C \mathbb{C} = \mathbb{B}\mathcal{J}_{\omega^{-1}} \otimes_C \mathbb{H}\mathcal{L}_e = \mathbb{H}(\mathcal{J}_{\omega^{-1}} * \mathcal{L}_e)$$

$\mathcal{L}_e = \mathbb{C}_{eB}$ is the skyscraper sheaf on $\{eB\}$. It remains to show that $\mathcal{J}_{\omega^{-1}} * \mathcal{L}_e \cong \mathcal{L}_\omega$.

$$\mathcal{J}_{\omega^{-1}} * \mathbb{C}_{eB} = r_* \Delta^*(\mathcal{J}_{\omega^{-1}} \boxtimes \mathbb{C}_{eB}) = p_{1*}(\mathcal{J}_{\omega^{-1}}|_{X \times \{eB\}}) \cong \mathcal{J}_{\omega^{-1}}|_{X \times \{eB\}}$$

Now the map $p_2 : \overline{\mathcal{O}_{\omega^{-1}}} \rightarrow X$ is locally trivial fibration with fiber isomorphic to X_ω . This means that there exists an open neighborhood $U \subseteq X$ of eB , $U \cong \mathbb{C}^N$, such that

$$\overline{\mathcal{O}_{\omega^{-1}}} \cap p_2^{-1}(U) \cong X_\omega \times U$$

Calling i and j the inclusions $X \times \{eB\} \xrightarrow{i} X \times U \xrightarrow{j} X \times X$ we have

$$\begin{aligned} \mathcal{J}_{\omega^{-1}} * \mathcal{L}_e &\cong \mathcal{J}_{\omega^{-1}}|_{X \times \{eB\}} = i^* j^* IC(\overline{\mathcal{O}_{\omega^{-1}}})[-N] = \\ &= i^* IC(X_\omega \times U)[-N] = IC(X_\omega) = \mathcal{L}_\omega \end{aligned}$$

□

4.3.4 The Bott-Samelson bimodule

Now we can apply the result of this section to compute the cohomology of the Bott-Samelson variety with the C -module structure induced by $\tilde{X}_\omega \rightarrow X_\omega \hookrightarrow X$.

Corollary 4.3.19. *The cohomology of the variety $\tilde{\mathcal{O}}(s_1, \dots, s_k) = \tilde{\mathcal{O}}_\omega$, as a C -bimodule, is isomorphic to*

$$\mathbb{B}(\tilde{\mathcal{O}}_\omega) = H^\bullet(\tilde{\mathcal{O}}_\omega, \mathbb{C}_{\tilde{\mathcal{O}}_\omega}) = \mathbb{B}(\pi_* \mathbb{C}_{\tilde{X}_\omega}) \cong C \otimes_{C^{s_1}} C \otimes_{C^{s_2}} \dots \otimes_{C^{s_k}} C$$

Proof. This is an immediate consequence of Lemma 3.3.1 and Corollary 4.3.17 \square

The bimodule $C \otimes_{C^{s_1}} C \otimes_{C^{s_2}} \dots \otimes_{C^{s_k}} C$ is called a *Bott-Samelson bimodule*.

Corollary 4.3.20. *The cohomology of the Bott-Samelson variety $\widetilde{X}(s_1, \dots, s_k) = \widetilde{X}_\omega$, as a C -module, is isomorphic to*

$$\mathbb{H}(\widetilde{X}_\omega) = H^\bullet(\widetilde{X}_\omega, \mathbb{C}_{\widetilde{X}_\omega}) \cong C \otimes_{C^{s_l}} C \otimes_{C^{s_{l-1}}} \dots \otimes_{C^{s_1}} C \otimes_C \mathbb{C}$$

Proof. The variety \widetilde{X}_ω is isomorphic to $\overline{\mathcal{O}_{\omega^{-1}}} \cap (X^l \times \{eB\})$ through the isomorphism $\phi(x_1, \dots, x_l) = (x_l, \dots, x_1, eB)$. Then, the following diagram is Cartesian

$$\begin{array}{ccc} \widetilde{X}_\omega & \xrightarrow{\phi} & \widetilde{\mathcal{O}}_{\omega^{-1}} \\ \pi \downarrow & & \downarrow \pi \\ X \times \{eB\} & \xrightarrow{i} & X \times X \end{array}$$

Hence, as in the proof of Theorem 4.3.18,

$$R\pi_* \mathbb{C}_{\widetilde{X}_\omega} = i^* R\pi_* \mathbb{C}_{\widetilde{\mathcal{O}}_{\omega^{-1}}} = \mathcal{J}_{s_l} * \dots * \mathcal{J}_{s_1} * \mathcal{L}_e$$

and $\mathbb{H}(\pi_* \mathbb{C}_{\widetilde{X}_\omega}) = \mathbb{B}(\mathcal{J}_{s_l}) \otimes_C \dots \otimes_C \mathbb{B}(\mathcal{J}_{s_1}) \otimes_C \mathbb{H}(\mathcal{L}_e)$ \square

Remark 4.3.21. To compute the cohomology of the Bott-Samelson variety, we actually don't need all this machinery. In fact, we have the sequence of locally trivial fibration

$$\widetilde{X}(s_1, \dots, s_k) \rightarrow \widetilde{X}(s_1, \dots, s_{k-1}) \rightarrow \dots \rightarrow \{pt\}$$

all with fibers isomorphic to $\mathbb{P}_{\mathbb{C}}^1 = S^2$. They are all orientable sphere bundles since they are complex smooth varieties, thus we can apply the Leray-Hirsch theorem and it follows that

$$\begin{aligned} H^\bullet(\widetilde{\mathcal{O}}(s_1, \dots, s_k)) &= H^\bullet(P_{\mathbb{C}}^1) \otimes_C \dots \otimes_C H^\bullet(P_{\mathbb{C}}^1) = \\ &= \mathbb{C}[x_1]/(x_1^2) \otimes_C \dots \otimes_C \mathbb{C}[x_k]/(x_k^2) = \mathbb{C}[x_1, \dots, x_k]/(x_1^2, \dots, x_k^2) \end{aligned}$$

However this method does not give information on the C -module structure.

We are now finally able to define Soergel bimodules.

Definition 4.3.22. An indecomposable C -bimodule is a *Soergel bimodule* if it is a direct summand of a (possibly zero) shift of a Bott-Samelson bimodule. A C -bimodule is a Soergel bimodule if there exists a decomposition into indecomposable Soergel bimodules.

We denote by \mathbb{S} the full subcategory of Soergel bimodules.

From the decomposition (3.2), the bimodule $\mathbb{B}(\mathcal{J}_\omega)$, is a Soergel bimodule for any $\omega \in W$, and the same holds for any object in \mathcal{K} .

Example 4.3.23. If $\omega \in S$, then the Bott-Samelson resolution is obviously an isomorphism. This also happens if $l(\omega) = 2$: if $\omega = st$, we consider the morphism $\pi : \tilde{\mathcal{O}}(s, t) \rightarrow \mathcal{O}_\omega$. Then for any $(x, y) \in \mathcal{O}_\omega$ the set $\pi^{-1}(x, y)$ is a single point. In fact if $(x, z, y), (x, z', y) \in \pi^{-1}(x, y)$, then $p_s(z) = p_s(x) = p_s(z') \in G/P_s$ and $p_t(z) = p_t(y) = p_t(z') \in G/P_t$, so $z(z')^{-1} \in P_s \cap P_t/B = \{eB\}$.

The first nontrivial case is for $l(\omega) = 3$. Let $G = SL_3(\mathbb{C})$, so $W = S_3$ and let $S = \{s, t\}$. The longest element is $\omega_0 = sts$ and $X_{\omega_0} = X$. Even though this is a smooth variety, the Bott-Samelson map is not an isomorphism. In fact, from example 2.4.5 we know that $C_{sts} = C_s C_t C_s - C_s$ so we have a decomposition of the Bott-Samelson bimodule

$$C \otimes_{C^s} C \otimes_{C^t} C \otimes_{C^s} C[3] = C[3] \oplus C \otimes_{C^s} C[1]$$

4.4 The “Erweiterungssatz”: Statement of the Theorem and Consequences

In this section we will prove and discuss the Erweiterungssatz due to Soergel [Soe90]. It states that the functor $\mathbb{H} = H^\bullet : \mathcal{D}_c(X) \rightarrow C\text{-Mod}$ is fully faithful on \mathcal{K} , the subcategory of $\mathcal{D}_c(X)$ whose objects are direct sums of shifts of \mathcal{L}_ω , $\omega \in W$. In other words, morphism between intersection cohomology complexes of Schubert varieties on X are just morphism between their cohomology C -modules.

For a graded C -module $M = \bigoplus M^i$ we define its shifted module $M[n]^i = M^{n+i}$. Let M and N two graded C -modules, then we define $\text{Hom}_{C\text{-Mod}}^\bullet(M, N)$ by

$$\text{Hom}^i(M, N) = \text{Hom}_{C\text{-Mod}}(M, N[i])$$

Theorem 4.4.1 (Erweiterungssatz). *The natural map induced by the hypercohomology is an isomorphism of graded vector spaces*

$$\text{Hom}_{\mathcal{D}(X)}^\bullet(\mathcal{L}_\omega, \mathcal{L}_\nu) \cong \text{Hom}_{C\text{-Mod}}^\bullet(\mathbb{H}(\mathcal{L}_\omega), \mathbb{H}(\mathcal{L}_\nu)) \quad \forall \omega, \nu \in W$$

Remark 4.4.2. Since all the objects in \mathcal{K} are direct sum of shifted \mathcal{L}_ω the theorem can be immediately generalized to an arbitrary object in \mathcal{K}

Before discussing the proof of this theorem we point out some of its consequences.

Proposition 4.4.3. *$H^\bullet(\mathcal{L}_\omega) = D_\omega$ is an indecomposable C -module.*

Proof. One of the main results of the theory of perverse sheaves is that minimal extension of simple local system are simple objects in the category of perverse sheaves (Prop B.4.9). Let us assume that D_ω decomposes into $D_1 \oplus D_2$, with D_1 and D_2 non trivial. Then the inclusion $i_j : D_j \rightarrow D_\omega$ and the projection $\pi_j : D_\omega \rightarrow D_j$, $j \in \{1, 2\}$, are homomorphisms of graded modules (of degree 0). Therefore, for example, $i_j \circ \pi_j : D_\omega \rightarrow D_\omega$ is a homomorphism of degree 0, too, and it cannot be invertible. Hence, from the Erweiterungssatz, it would follow that $\text{Hom}^0(\mathcal{L}_\omega, \mathcal{L}_\omega) \cong \text{Hom}^0(D_\omega, D_\omega)$ contains nontrivial non invertible elements. But this is a contradiction: \mathcal{L}_ω is simple and every non-zero endomorphism (of degree 0) should be invertible. \square

Furthermore, again from the Erweiterungssatz, we can recover the direct summand $H^\bullet(\mathcal{L}_\omega)$ of $H^\bullet(\tilde{X}(s_1, \dots, s_k))$ relying only on its algebraic structure. In other words, from the C -module structure on $H^\bullet(\tilde{X}(s_1, \dots, s_k))$, that is $C \otimes_{C^{s_1}} \dots \otimes_{C^{s_k}} C \otimes_C \mathbb{C}$, we can already recover $H^\bullet(L_\omega)$ as a submodule.

From 3.2 we have a decomposition of the cohomology of the Bott-Samelson module

$$C \otimes_{C^{s_1}} C \otimes_{C^{s_2}} \dots \otimes_{C^{s_k}} C \otimes_C \mathbb{C}[l(\omega)] = D_\omega \oplus \bigoplus_{\substack{\nu < \omega \\ i \in \mathbb{Z}}} (D_\nu[-i])^{\dim V_\nu^i} = D_\omega \oplus \bigoplus_{\substack{\nu < \omega \\ i \in \mathbb{Z} \\ 1 \leq j \leq \dim V_\nu^i}} D_{\nu,j}[-i]$$

This is actually unique.

Proposition 4.4.4. *The C -module*

$$H^\bullet(\tilde{X}(s_1, \dots, s_k)) = C \otimes_{C^{s_1}} \dots \otimes_{C^{s_k}} C \otimes_C \mathbb{C}[l(\omega)]$$

has a unique decomposition into indecomposable objects, so in particular all the decompositions are isomorphic to $D_\omega \oplus \bigoplus (D_\nu[-i])^{\dim V_\nu^i}$. Moreover, if

$$C \otimes_{C^{s_1}} \dots \otimes_{C^{s_k}} C \otimes_C \mathbb{C}[l(\omega)] = \bigoplus_{i=1}^m D_i$$

is another decomposition such that D_1 is the submodule containing $1 \simeq 1 \otimes 1 \otimes \dots \otimes 1$, then $D_1 \cong D_\omega$.

We need the following general Lemma.

Lemma 4.4.5. *Let M be a C -module and $M = \bigoplus_{i=1}^n E_i = \bigoplus_{j=1}^n F_j$ two decompositions of M into indecomposable objects. If we assume that for any i , $\text{Hom}_{C\text{-Mod}}(E_i, E_i)$ is a field, then $m = n$ and there exists a permutation σ such that $E_i \cong F_{\sigma(i)}$ for any i .*

Proof. Let $e_i : M \rightarrow E_i$ and $f_j : M \rightarrow F_j$ the projection. Since $\sum_j f_j = \text{Id}_M$ and $f_j^2 = f_j$ we have

$$\sum_j e_1 f_j f_j e_1|_{E_1} = \text{Id}_{E_1}$$

so there exists an index k such that $e_1 f_k f_k e_1$ is an automorphism of E_1 . We call γ its inverse. We have the morphisms $E_1 \xrightarrow{f_k e_1} F_k \xrightarrow{e_1 f_k} E_1$ and $\gamma \circ e_1 f_k$ is a section of $f_k e_1$. So we have $F_k = \text{Im}(f_k e_1) \oplus \text{Ker}(e_1 f_k|_{F_k})$. But F_k is indecomposable, so we have that $e_1 f_k$ is injective, hence is an isomorphism $F_k \cong E_1$.

Furthermore since $\text{Ker}(e_1 f_k|_{F_k}) = 0$ we have that $F_k \cap (E_2 \oplus \dots \oplus E_n) = 0$, so $M = F_k \oplus E_2 \oplus \dots \oplus E_n$. Therefore

$$M/F_k \cong \bigoplus_{i=2}^n E_i \cong \bigoplus_{j=1, j \neq k}^n F_j$$

and we can conclude by induction. \square

Proof. (Proposition). The first statement follows immediately from Lemma 4.4.5. For the second statement firstly we notice that a summand D_1 containing 1 always exists since the degree 0 part of $C \otimes_{C^{s_1}} C \otimes_{C^{s_2}} \dots \otimes_{C^{s_k}} C \otimes \mathbb{C}$ has dimension 1. Furthermore we can see that 1 must belong to D_ω . In fact,

$$IH^0(X_\omega) = H^{-l(\omega)}(\mathcal{L}_\omega) = H^0(\mathcal{H}^{-l(\omega)}(\mathcal{L}_\omega)) = H^0(X) = \mathbb{C}$$

(cfr. Lemma B.5.7) is nonzero. Hence, 1, which spans the $-l(\omega)$ degree part of $\mathbb{H}(\tilde{X}_\omega)$, must belong to D_ω . Then calling $\pi_1 : \mathbb{H}(\tilde{X}_\omega) \rightarrow D_1$ and $i_\omega : D_\omega \rightarrow \mathbb{H}(\tilde{X}_\omega)$ the obvious projection and inclusion. We have that $i_\omega \pi_1 \pi_1 i_\omega$ is nonzero since it sends 1 into 1, hence is an automorphism of D_ω . Now we can conclude, as in the proof of the Lemma, that $D_\omega \cong D_1$. \square

Remark 4.4.6. This result holds more generally for any module of finite length M over a ring R and it is known as Krull-Remak-Schmidt theorem (cfr. [Lan02, 7.5]). Actually the assumption that $\text{Hom}_{C\text{-Mod}}(E_i, E_i)$ is a field is unnecessary since for any indecomposable E of finite length $\text{Hom}_{R\text{-Mod}}(E, E)$ is a local ring.

Corollary 4.4.7. D_ω is the unique summand of $\mathbb{H}(\tilde{X}_\omega)$ which is not a summand of any other module $\mathbb{H}(\tilde{X}_\nu)$, with $\nu < \omega$.

Remark 4.4.8. The proof of the Erweiterungssatz works, up to some minor modifications, also on $X \times X$, i.e. we have

$$\text{Hom}_{\mathcal{D}(X \times X)}^\bullet(\mathcal{J}_\omega, \mathcal{J}_\nu) \cong \text{Hom}_{C\text{-Mod-}C}^\bullet(\mathbb{B}(\mathcal{J}_\omega), \mathbb{B}(\mathcal{J}_\nu)) \quad \forall \omega, \nu \in W$$

The analogue of Prop. 4.4.3 and 4.4.4 hold in this setting, i.e. B_ω is an indecomposable bimodule and the decomposition of the Bott-Samelson bimodule is unique. In particular this implies that the functor $\mathbb{B} : \mathcal{K} \rightarrow \mathbb{S}$ is fully faithful and essentially surjective, hence it is an equivalence of categories, so we have $\mathfrak{H} \cong \langle \mathcal{K} \rangle \cong \langle \mathbb{S} \rangle$. This result is often referred saying that Soergel bimodules are a *categorification* of the Hecke algebra.

Moreover, we notice that for two bimodules $B_1, B_2 \in \mathbb{S}$ we have $[B_1] = [B_2] \in \langle \mathbb{S} \rangle$ if and only if $B_1 \cong B_2$. In fact, in view of Lemma 4.3.8 if $[B_1] = [B_2]$ then there exists a bimodule B such that $B_1 \oplus B \cong B_2 \oplus B$ but since the decomposition of $B_1 \oplus B$ into indecomposable is unique we get $B_1 \cong B_2$

4.5 The ‘‘Erweiterungssatz’’: Proof of the Theorem

We will follow the proof given by Ginsburg [Gin91] which is easier and less technical than Soergel’s original proof. Both these proofs rely substantially on Saito’s weight theory.

We have a filtration of the flag variety by closed subvarieties

$$\{B\} = X_0 \subseteq X_1 \subseteq \dots \subseteq X_N = X$$

where $X_n = \bigsqcup_{l(\omega) \leq n} B\omega B/B$. However we can refine this filtration adding only one Schubert cell at time. In this way $U_n = X_n/X_{n-1}$ is a single stratum $B\omega_n B/B$ and it is isomorphic to the affine space $\mathbb{C}^{l(\omega_n)}$.

Let's denote by v_n and i_n the closed embeddings and by u_n the open embedding

$$X_{n-1} \xrightarrow{v_n} X_n \xleftarrow{u_n} U_n \quad X_n \xrightarrow{i_n} X$$

We fix an element $\omega \in W$ and we define $L_n = i_n^* \mathcal{L}_\omega$. We have the following distinguished triangles in $\mathcal{D}_c(X_n)$

$$u_n! u_n^! L_n \rightarrow L_n \rightarrow v_{n*} v_n^* L_n \xrightarrow{+1} \quad v_n! v_n^! L_n \rightarrow L_n \rightarrow Ru_{n*} u_n^* L_n \xrightarrow{+1}$$

Thus we can obtain the long exact sequences in cohomology. From the first triangle we get:

$$0 \rightarrow H^0(X_n, u_n! u_n^! L_n) \rightarrow H^0(X_n, L_n) \rightarrow H^0(X_n, v_{n*} v_n^* L_n) \rightarrow H^1(X_n, u_n! u_n^! L_n) \rightarrow \dots$$

Since $H^\bullet(X_n, u_n! u_n^! L_n) = H_c^\bullet(U_n, u_n^! L_n)$ and $H^\bullet(X_n, v_{n*} v_n^* L_n) = H^\bullet(X_{n-1}, v_n^* L_n) = H^\bullet(X_{n-1}, L_{n-1})$ we can rewrite it as

$$0 \rightarrow H_c^0(U_n, u_n^! L_n) \rightarrow H^0(X_n, L_n) \rightarrow H^0(X_{n-1}, L_{n-1}) \rightarrow H_c^1(U_n, u_n^! L_n) \rightarrow \dots$$

Similarly, from the second triangle we get the long exact sequence

$$0 \rightarrow H^0(X_{n-1}, v_n^! L_n) \rightarrow H^0(X_n, L_n) \rightarrow H^0(U_n, u_n^* L_n) \rightarrow H^1(X_{n-1}, v_n^! L_n) \rightarrow \dots$$

We claim that in these sequences all the connecting morphisms vanish, so they split into the short sequences

$$0 \rightarrow H_c^\bullet(U_n, u_n^! L_n) \rightarrow H^\bullet(X_n, L_n) \rightarrow H^\bullet(X_{n-1}, L_{n-1}) \rightarrow 0$$

$$0 \rightarrow H^\bullet(X_{n-1}, v_n^! L_n) \rightarrow H^\bullet(X_n, L_n) \rightarrow H^\bullet(U_n, u_n^* L_n) \rightarrow 0$$

We need now some preparatory work before starting the proof of our claim.

4.5.1 \mathbb{C}^* -actions on the Flag Variety

Let $T \subseteq B \subseteq G$ a maximal torus of the reductive group G . T acts naturally on the flag variety $X = G/B$ and, since $W = N_G(T)/T$, the points $\omega B \in X$, $\omega \in W$ are fixed by T . On the other hand, all the fixed points for this action are of this form. In fact if gB is a fixed point we have $tgB = gB$ for any $t \in T$, so $g^{-1}Tg \subseteq B$. But all the maximal torus in B are conjugate, hence there exists $b \in B$ such that $b^{-1}g^{-1}Tgb = T$ and $gb \in N_G(T)$. This means that $gB = \omega B$ for some $\omega \in W$.

Lemma 4.5.1. *For any $\omega \in W$ there exists an open neighborhood V of ωB in X and a one parameter subgroup T_ω in T such that T_ω contracts V to ωB as the parameter goes to 0*

Proof. The statement can be rewritten as follows: there exists a group homomorphism $\chi : \mathbb{C}^* \rightarrow T$, (a cocharacter of T) such that for any $v \in V$

$$\lim_{z \rightarrow 0} \chi(z)(v) = \omega B.$$

We can take $\omega U^- B/B$ as the neighborhood V of ωB . Each point $u \in U$ can be written in an unique way as $u = u_{\alpha_1}(y_1)u_{\alpha_2}(y_2) \cdots u_{\alpha_N}(y_N)$ where $\{\alpha_1, \dots, \alpha_N\} = -R^+$ is the set of negative roots.

$$\begin{aligned} \chi(z)(\omega u B/B) &= \omega(\omega^{-1} \cdot \chi)(z)u_{\alpha_1}(y_1)u_{\alpha_2}(y_2) \cdots u_{\alpha_N}(y_N)B/B = \\ &= \omega \cdot u_{\alpha_1}(\alpha_1(\omega \cdot \chi)(z)y_1) \cdots u_{\alpha_N}(\alpha_N(\omega \cdot \chi)(z)y_N)(\omega^{-1} \cdot \chi)(z)B/B = \\ &= \omega \cdot u_{\alpha_1}(z^{\langle \alpha_1, \omega^{-1} \cdot \chi \rangle} y_1) \cdots u_{\alpha_N}(z^{\langle \alpha_N, \omega^{-1} \cdot \chi \rangle} y_N)B/B \end{aligned}$$

where $(\omega^{-1} \cdot \chi)(z) = \omega^{-1} \chi(x) \omega$ and $\langle \cdot, \cdot \rangle$ is the non-degenerate pairing between characters and cocharacter. Thus the limit, when $z \rightarrow 0$, is ωB for any $u \in U^-$ if and only if $\langle \alpha, \omega^{-1} \cdot \chi \rangle = \langle \omega \cdot \alpha, \chi \rangle \geq 0$ for any negative root α . Equivalently, if $\langle \omega \cdot \alpha, \chi \rangle \leq 0$ for $\alpha \in R^+$. But it is enough to check that $\langle \omega \cdot \alpha, \chi \rangle \leq 0$ for any simple root α . By the non-degeneracy of the pairing $\langle \cdot, \cdot \rangle$ we can always find such a cocharacter χ . \square

The property that every point x has a neighborhood V contracted to x by some \mathbb{C}^* -action has the following consequence for sheaves.

Lemma 4.5.2. *Let V and x as above and let \mathcal{F} a \mathbb{C}^* -equivariant complex of sheaves in $\mathcal{D}_c(V)$. Then, $H^\bullet(V, \mathcal{F}) \cong H^\bullet(\mathcal{F}_x)$*

Proof. Firstly we notice that we can assume that $\mathcal{F}_x = 0$. Otherwise, we denote by $i : \{x\} \hookrightarrow V$ the inclusion and $\mathcal{F}' = \text{Ker}(\mathcal{F} \rightarrow i_* i^* \mathcal{F})$. So we have the exact sequence $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow i_* i^* \mathcal{F} \rightarrow 0$ and in cohomology

$$0 \rightarrow H^0(\mathcal{F}') \rightarrow H^0(\mathcal{F}) \rightarrow H^0(\mathcal{F}_x) \rightarrow H^1(\mathcal{F}') \rightarrow H^1(\mathcal{F}) \rightarrow H^1(\mathcal{F}_x) \rightarrow \dots$$

and $H^\bullet(\mathcal{F}') = 0$ clearly implies the thesis.

By Lemma 4.5.1 above, and the algebraic version of Hartogs' theorem, the action map $\mathbb{C}^* \times V \rightarrow V$ extends to a morphism $\mu : \mathbb{A}_{\mathbb{C}}^1 \times V \rightarrow V$

We call $p_1 : \mathbb{A}^1 \times V \rightarrow \mathbb{A}^1$ and $p_2 : \mathbb{A}^1 \times V \rightarrow V$ respectively the first and the second projection. We also define the morphism τ :

$$\tau : \mathbb{A}^1 \times V \rightarrow \mathbb{A}^1 \times V \quad \tau(z, v) = (z, \mu(z, v))$$

Clearly $p_1 \circ \tau = p_1$.

The assumption that \mathcal{F} is a \mathbb{C}^* -equivariant complex means that there exists an isomorphism between $p_2^* \mathcal{F}|_{\mathbb{C}^* \times V}$ and $\tau^* p_2^* \mathcal{F}|_{\mathbb{C}^* \times V}$. Besides, $\tau^* p_2^* \mathcal{F}|_{\{0\} \times V} = 0$ because $\mathcal{F}_x = 0$ and $p_2 \circ \tau(\{0\} \times V) = \{x\}$. Hence

$$\tau^* p_2^* \mathcal{F} \cong j_! j^! \tau^* p_2^* \mathcal{F} \cong j_!(\tau^* p_2^* \mathcal{F}|_{\mathbb{C}^* \times V}) \cong j_! j^! p_2^* \mathcal{F}$$

($j : \mathbb{C}^* \times V \hookrightarrow \mathbb{A}^1 \times V$ is the inclusion) and there is canonical morphism $\tau^* p_2^* \mathcal{F} \rightarrow p_2^* \mathcal{F}$ arising from the adjunction morphism $j_! j^! \rightarrow \text{Id}$. Finally, applying Rp_{1*} we obtain a morphism

$$\alpha : Rp_{1*} \tau^* p_2^* \mathcal{F} \rightarrow Rp_{1*} p_2^* \mathcal{F}$$

On the other hand, from the commutative diagram

$$\begin{array}{ccc} \mathbb{A}^1 \times V & \xrightarrow{\tau} & \mathbb{A}^1 \times V \\ p_1 \downarrow & & \downarrow p_1 \\ \mathbb{A}^1 & \xrightarrow{\text{Id}} & \mathbb{A}^1 \end{array}$$

we obtain canonically a morphism $\beta : Rp_{1*}(p_2^* \mathcal{F}) = \text{Id}^* Rp_{1*}(\pi_2^* \mathcal{F}) \rightarrow Rp_{1*} \tau^*(\pi_2^* \mathcal{F})$ and composing we obtain a morphism

$$\alpha \circ \beta : Rp_{1*} p_2^* \mathcal{F} \rightarrow Rp_{1*} p_2^* \mathcal{F}$$

From the following Cartesian diagram

$$\begin{array}{ccc} \mathbb{A}^1 \times V & \xrightarrow{p_2} & V \\ p_1 \downarrow & & \downarrow q \\ \mathbb{A}^1 & \xrightarrow{p} & \{\text{pt}\} \end{array}$$

by smooth base change $Rp_{1*} p_2^* \mathcal{F} \cong p^* Rq_* \mathcal{F}$ is a locally constant complex of sheaves on \mathbb{A}^1 . Since

$$(Rp_{1*} \tau^* p_2^* \mathcal{F})_0 = H^\bullet(\{0\} \times V, \tau^* p_2^* \mathcal{F}) = H^\bullet(\{0\} \times V, 0) = 0$$

we get that β is the 0 morphism on the stalk of $0 \in \mathbb{A}^1$, and so is $\alpha \circ \beta$. Thus, by the connectedness of \mathbb{A}^1 it should be 0 everywhere. On the other hand α is an isomorphism on the complement of $0 \in \mathbb{A}^1$. Furthermore in $1 \in \mathbb{A}^1$ also β is an isomorphism since

$$(Rp_{1*} \tau^* p_2^* \mathcal{F})_1 = H^\bullet(\{1\} \times V, \tau^* p_2^* \mathcal{F})$$

and τ is the identity on $\{1\} \times V$. This forces $Rp_{1*} p_2^* \mathcal{F}$ to be 0.

Finally, $0 = Rp_{1*} p_2^* \mathcal{F} = p^* q_* \mathcal{F} = p^* H^\bullet(V, \mathcal{F})$ and so $H^\bullet(V, \mathcal{F}) = 0$. \square

4.5.2 Arguments from Weight Theory

In this section we will use results from Appendix C. We are allowed to do so: indeed, all the complexes we will consider have an additional natural structure as mixed Hodge modules and the morphisms we deal with respect this additional structure.

For a fixed $\omega \in W$ we can take a neighborhood V and a one parameter subgroup T_ω as in Lemma 4.5.1. Now, for any $\nu \in W$, $\mathcal{L}_\nu|_V$ satisfies the hypothesis of the lemma 4.5.2 because it is locally constant on the Schubert cells, which are clearly T_ω -stable. From Prop. C.3.4 we know that \mathcal{L}_ν is a pure complex of weight $l(\nu)$. Let $j_\omega : \{\omega B\} \hookrightarrow X$ be the inclusion.

Remark 4.5.3. We have that $\mathbb{D}_X \mathcal{L}_\nu \cong \mathcal{L}_\nu$ as perverse sheaves. This is not true when we look to \mathcal{L}_ν as a mixed Hodge module, however we have $\mathbb{D}_X \mathcal{L}_\nu \cong \mathcal{L}_\nu(-d_{l(\nu)})$ where $(-d_{l(\nu)})$ is the *Tate twist* (cfr. [Sai90], [PS08]), and it is pure of weight $-l(\nu)$.

While the general theory would only ensure that $j_\omega^* \mathcal{L}_\nu$ is mixed with weights $\leq l(\nu)$, the \mathbb{C}^* -action on a neighborhood of ωB gives us a stronger result.

Proposition 4.5.4. *The complex $j_\omega^* \mathcal{L}_\nu$ is pure of weight $l(\nu)$.*

Proof. On one hand we have $H^\bullet(V, \mathcal{L}_\nu) = Rp_* i_V^* \mathcal{L}_\nu = Rp_* i_V^! \mathcal{L}_\nu$ and both the functors Rp_* and $i_V^!$ increase the weights. On the other hand the functor j_ω^* decreases the weights. This means that $H^\bullet(V, \mathcal{L}_\nu) = j_\omega^* \mathcal{L}_\nu$ should have weights $\geq l(\nu)$ and $\leq l(\nu)$, so it must be pure of weight $l(\nu)$. \square

Corollary 4.5.5. *The complex $j_\omega^! \mathcal{L}_\nu$ is pure of weight $l(\nu)$.*

Proof. This is just the dual statement of Prop. 4.5.4. In fact:

$$j_\omega^! \mathcal{L}_\nu \cong \mathbb{D}_{\text{pt}} j_\omega^* \mathbb{D}_X \mathcal{L}_\nu \cong \mathbb{D}_{\text{pt}} Rp_* i_V^* \mathbb{D}_X \mathcal{L}_\nu \cong Rp_* i_V^* \mathcal{L}_\nu \cong H_c^\bullet(V, \mathcal{L}_\nu)$$

\square

Theorem 4.5.6. *The following short sequence*

$$0 \rightarrow H_c^\bullet(u_n^! L_n) \rightarrow H^\bullet(L_n) \rightarrow H^\bullet(L_{n-1}) \rightarrow 0 \quad (4.1)$$

is exact.

Proof. We need to show that in the distinguished triangle

$$u_n^! u_n^! L_n \rightarrow L_n \rightarrow v_{n*} L_{n-1} \xrightarrow{+1}$$

the boundary maps in cohomology vanish. Let p_n be the map from X_n to a point. We first of all claim that the term of the long exact sequence is pure:

$$H^\bullet(u_n^! u_n^! L_n) = p_{n*} u_n^! u_n^! L_n = (p_n \circ u_n)^! (u_n \circ i_n)^* \mathcal{L}_\nu$$

There exists a one parameter subgroup of T which contracts the Schubert cell U_n to its fixed point ω_n and $(u_n \circ i_n)^* \mathcal{L}_\nu$ satisfies the hypothesis of Lemma 4.5.2 for $V = U_n$, so $H^\bullet(U_n, L_n) = j_{\omega_n}^* \mathcal{L}_\nu$ is pure of weight $l(\nu)$.

The complex $u_n^* L_n = (u_n \circ i_n)^* \mathcal{L}_\nu$ on $U_n \cong \mathbb{A}_{\mathbb{C}}^{l(\omega_n)}$ has constant cohomology sheaves and these are zero in even (or odd) degree. Hence,

$$u_n^* L_n \cong \bigoplus_{j \in \mathbb{Z}} \mathcal{H}^j(u_n^* L_n)[-j]$$

Each $\mathcal{H}^j(u_n^* L_n)[-j]$ is a shifted constant sheaf on U_n and it is pure of weight $l(\omega)$ since this holds punctually, therefore $u_n^* L_n$ is also pure of weight $l(\nu)$. We have

$$H_c^\bullet(u_n^* L_n) \cong \mathbb{D}_{\text{pt}} H^\bullet(\mathbb{D}_{U_n} u_n^* L_n)$$

and it follows that $H_c^\bullet(u_n^*L_n)$ is also pure of weight $l(\nu)$.

Now, by induction, we can assume that also $Rp_{n-1*}L_{n-1} = H^\bullet(L_{n-1})$ is pure of weight $l(\nu)$, the case $n = 0$ being once again essentially the Lemma 4.5.4. Hence, by Lemma C.2.5, it follows that $Rp_{n*}L_n$ is pure and that all connecting morphism in the long exact sequence vanish. \square

Theorem 4.5.7. *The following short sequence*

$$0 \rightarrow H^\bullet(v_n^!L_n) \rightarrow H^\bullet(L_n) \rightarrow H^\bullet(u_n^*L_n) \rightarrow 0$$

is exact.

Proof. It suffices to show that the natural restriction morphism $H^\bullet(L_n) \rightarrow H^\bullet(u_n^*L_n)$ is surjective. Let ω_n the element of the Weyl group in U_n and be the V the open neighborhood of ω_n in X as in 4.5.1. If we denote by $\epsilon : X \setminus V \hookrightarrow X$, $j : V \hookrightarrow X$ the inclusions, from the distinguished triangle

$$\epsilon_! \epsilon^! \mathcal{L}_\nu \rightarrow \mathcal{L}_\nu \rightarrow Rj_* j^* \mathcal{L}_\nu \xrightarrow{+1}$$

we obtain the long exact sequence

$$\dots \rightarrow H^i(\epsilon^! \mathcal{L}_\nu) \rightarrow H^i(X, \mathcal{L}_\nu) \rightarrow H^i(V, \mathcal{L}_\nu) \rightarrow H^{i+1}(\epsilon^! \mathcal{L}_\nu) \rightarrow \dots$$

Now we have already shown that $H^i(V, \mathcal{L}_\nu)$ is pure of weight $l(\nu) + i$ while $H^{i+1}(\epsilon^! \mathcal{L}_\nu)$ is mixed of weights $\geq l(\nu) + i + 1$, thus it can not exists a nonzero homomorphism $H^i(V, \mathcal{L}_\nu) \rightarrow H^{i+1}(\epsilon^! \mathcal{L}_\nu)$. This implies that $H^\bullet(X, \mathcal{L}_\nu) \rightarrow H^\bullet(V, \mathcal{L}_\nu)$ is surjective. We have two different ways to restrict to the point $\{\omega B\}$

$$\begin{array}{ccccc} & & H^\bullet(V, \mathcal{L}_\nu) & & \\ & \alpha \nearrow & & \searrow \beta & \\ H^\bullet(X, \mathcal{L}_\nu) & & & & j_\omega^* \mathcal{L}_\nu \\ & \searrow & & \nearrow \delta & \\ & & H^\bullet(X_n, L_n) & \xrightarrow{\gamma} & H^\bullet(U_n, L_n) \end{array}$$

We have just proved that α is surjective and by Lemma 4.5.2 β is an isomorphism. We can also apply Lemma 4.5.2 to the open $U_n \subseteq X_n$ in order to obtain that also δ is an isomorphism. This yields the surjectivity of γ , hence the theorem. \square

Now we choose another $\mu \in W$ and, for any n , we set $M_n = i_n^! \mathbb{D}_X \mathcal{L}_\mu$. We notice that $j_\omega^! M_n = \mathbb{D}_{\text{pt}}(j_\omega^* \mathcal{L}_\mu)$, hence it is pure of weight $-l(\mu)$ Furthermore, dualizing the statement of Theorems 4.5.6 and 4.5.7 we could see that also the following sequences are exact

$$0 \rightarrow H_c^\bullet(u_n^* M_n) \rightarrow H^\bullet(M_n) \rightarrow H^\bullet(v_n^* M_n) \rightarrow 0$$

$$0 \rightarrow H^\bullet(M_{n-1}) \rightarrow H^\bullet(M_n) \rightarrow H^\bullet(u_n^* M_n) \rightarrow 0$$

By reverse induction, from this we could show that $H^\bullet(u_n^* M_n)$ is pure of weight $l(\mu)$.

Proposition 4.5.8. *For any $n \geq 0$*

- i) $\mathrm{Hom}^\bullet(L_n, M_n)$ is a pure complex of modules, i.e. it is a pure Hodge structure.
ii) There is a natural short exact sequence of complex of modules

$$0 \rightarrow \mathrm{Hom}^\bullet(L_{n-1}, M_{n-1}) \rightarrow \mathrm{Hom}^\bullet(L_n, M_n) \rightarrow \mathrm{Hom}^\bullet(u_n^* L_n, u_n^* M_n) \rightarrow 0$$

Proof. For $n = 0$ it is immediate since $L_0 = i_0^* \mathcal{L}_\nu = j_e^* \mathcal{L}$ and $M_0 = i_0^! \mathcal{L}_\mu = \mathbb{D}(j_e^* \mathcal{L}_\mu)$ are pure of weight $l(\nu)$ and $l(\mu)$, so $\mathrm{Hom}^\bullet(L_0, M_0)$ is pure of weight $-l(\nu) - l(\mu)$.

So we can assume $n > 0$. Using the distinguished triangle $v_{n!} v_n^! M_n \rightarrow M_n \rightarrow u_{n*} u_n^* M_n \xrightarrow{+1}$, and apply the cohomological functor $\mathrm{Hom}(L_n, \cdot)$ we obtain the long exact sequence

$$\dots \rightarrow \mathrm{Ext}^i(L_n, v_{n!} M_{n-1}) \rightarrow \mathrm{Ext}^i(L_n, M_n) \rightarrow \mathrm{Ext}^i(L_n, u_{n*} u_n^* M_n) \rightarrow \dots \quad (4.2)$$

Now we can rewrite the term $\mathrm{Hom}^\bullet(L_n, u_{n*} u_n^* M_n)$ as $\mathrm{Hom}^\bullet(u_n^* L_n, u_n^* M_n)$, by adjunction. We can apply Lemma 4.5.2 to the open neighborhood $U_n \subseteq X_n$ of $\omega = \omega_n$.

$$\begin{aligned} \mathrm{Hom}^\bullet(u_n^* L_n, u_n^* M_n) &\cong H^\bullet(u_n^* R\mathcal{H}om^\bullet(L_n, M_n)) \cong j_\omega^* R\mathcal{H}om^\bullet(L_n, M_n) \cong \\ &\cong \mathrm{Hom}(j_\omega^* L_n, j_\omega^* M_n) \cong \mathrm{Hom}(H^\bullet(u_n^* L_n), H^\bullet(u_n^* M_n)) \end{aligned}$$

hence, it is pure of weight $-l(\nu) - l(\mu)$.

We can also consider the distinguished triangle $u_{n!} u_n^! L_n \rightarrow L_n \rightarrow v_{n*} L_{n-1} \xrightarrow{+1}$ and applying the (contravariant) cohomological functor $\mathrm{Hom}(\cdot, v_{n!} M_{n-1})$. In the resulting exact sequence appears the term

$$\mathrm{Hom}^\bullet(u_{n!} u_n^! L_n, v_{n!} M_{n-1}) = \mathrm{Hom}^\bullet(u_n^! L_n, u_n^! v_{n!} M_{n-1}) = 0$$

because $u_n^! v_{n!} = u_n^* v_{n!}$ is the 0 functor. Thus it follows that

$$\mathrm{Hom}^\bullet(L_n, v_{n!} M_{n-1}) \cong \mathrm{Hom}^\bullet(v_{n*} L_{n-1}, v_{n!} M_{n-1})$$

and since v_n is a closed embedding

$$\mathrm{Hom}^\bullet(v_{n*} L_{n-1}, v_{n!} M_{n-1}) = \mathrm{Hom}^\bullet(v_n^* v_{n*} L_{n-1}, M_{n-1}) = \mathrm{Hom}^\bullet(L_{n-1}, M_{n-1})$$

By induction, we can assume that this is pure of weight $-l(\nu) - l(\mu)$. Immediately follows, looking at (4.2), that also $\mathrm{Hom}^\bullet(L_n, M_n)$ is pure. In addition, we have also seen that we can rewrite (4.2) as

$$\begin{aligned} \dots \rightarrow \mathrm{Ext}^{i-1}(u_n^* L_n, u_n^* M_n) &\rightarrow \mathrm{Ext}^i(L_{n-1}, M_{n-1}) \rightarrow \mathrm{Ext}^i(L_n, M_n) \rightarrow \\ &\rightarrow \mathrm{Ext}^i(u_n^* L_n, u_n^* M_n) \rightarrow \mathrm{Ext}^{i+1}(L_{n-1}, M_{n-1}) \rightarrow \dots \end{aligned}$$

All terms appearing here are pure, so checking the weights we see that the connecting maps must vanish and we obtain the exact sequence of ii). □

Here we can use the results and notations of section 1.5, so let's $Y_\omega = \omega_0 X_{\omega\omega_0}$. Let $\omega B = \omega_n B$ be the fixed point in $X_n \setminus X_{n-1} = U_n$. The fundamental class $[Y_\omega]$ of the subvariety Y_ω defines an element in the homology $H_\bullet(X, \mathbb{C}) \cong H^\bullet(X, \mathbb{C})^\vee$ and, through Poincaré Duality, this corresponds to an element $c_n \in H_c^\bullet(X, \mathbb{C}) = H^\bullet(X, \mathbb{C}) = C$. So we have that

$$c_n([X_n]) = \langle [Y_\omega], [X_{\omega_0}] + \dots + [X_{\omega_n}] \rangle = \langle [Y_\omega], [X_\omega] \rangle = 1$$

($\langle \cdot, \cdot \rangle$ is the intersection pairing on $H_\bullet(X)$) and moreover c_n annihilates any cycle supported in X_{n-1} . From this we can deduce what is the the action of c_n on $H^\bullet(L_n)$. We consider the following commutative diagram

$$\begin{array}{ccccccc} H^\bullet(L_n) & \longrightarrow & H^\bullet(u_n^* L_n) & \longrightarrow & 0 \\ & & \downarrow c_n & & \downarrow c_n \\ 0 & \longleftarrow & H^\bullet(L_{n-1}) & \longleftarrow & H^\bullet(L_n) & \longleftarrow & H_c^\bullet(u_n^* L_n) & \longleftarrow & 0 \end{array}$$

in which the rows are exact. Now c_n is the fundamental class of U_n and multiplying by c_n gives Poincaré Duality $H^\bullet(U_n) \rightarrow H_c^\bullet(U_n)$. This works in the same way for $u_n^* L_n$, since $u_n^* L_n$ is a complex of constant sheaves, hence

$$u_n^* L_n = \bigoplus_i \mathcal{H}^i(u_n^* L_n)[-i] \cong \bigoplus_i \mathbb{C}_{U_n}^{k_i}[-i]$$

for some $k_i \in \mathbb{N}$. Now it is clear that multiplication by c_n gives an isomorphism on each summand.

In a dual way, we get a commutative diagram for M_n

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^\bullet(M_{n-1}) & \longrightarrow & H^\bullet(M_n) & \longrightarrow & H^\bullet(u_n^* M_n) & \longrightarrow & 0 \\ & & & & \downarrow c_n & & \downarrow c_n \\ & & & & H^\bullet(M_n) & \longleftarrow & H_c^\bullet(u_n^* M_n) & \longleftarrow & 0 \end{array}$$

and, again, the rows are exact and the right-hand vertical arrow is an isomorphism. These diagrams provide the following identifications:

- $\text{Coker}(c_n : H^\bullet(L_n) \rightarrow H^\bullet(L_n)) \cong H^\bullet(L_n)/H_c^\bullet(u_n^* L_n) \cong H^\bullet(L_{n-1})$
- $H^\bullet(L_n)/\text{Ker}(c_n : H^\bullet(L_n) \rightarrow H^\bullet(L_n)) \cong H^\bullet(u_n^* L_n)$
- $\text{Ker}(c_n : H^\bullet(M_n) \rightarrow H^\bullet(M_n)) \cong H^\bullet(M_{n-1})$
- $H^\bullet(M_n)/\text{Ker}(c_n : H^\bullet(M_n) \rightarrow H^\bullet(M_n)) \cong H^\bullet(u_n^* M_n)$

4.5.3 Conclusion

Now we have all the tools to prove at last the main theorem of this chapter

Theorem 4.5.9 (Erweiterungssatz). *The canonical morphism*

$$\mathrm{Hom}_{\mathcal{D}(X_n)}^\bullet(L_n, M_n) \rightarrow \mathrm{Hom}_{H^\bullet(X)}^\bullet(H^\bullet(L_n), H^\bullet(M_n))$$

is an isomorphism for every n .

Proof. This is trivial for $n = 0$. So we can assume $n > 0$.

Let $\phi \in \mathrm{Hom}_{H^\bullet(X)}^\bullet(H^\bullet(L_n), H^\bullet(M_n))$. Since ϕ commutes with $c_n \in H^\bullet(X)$, it must send $\mathrm{Ker}(c_n : H^\bullet(L_n) \rightarrow H^\bullet(L_n))$ into $\mathrm{Ker}(c_n : H^\bullet(M_n) \rightarrow H^\bullet(M_n))$. Thus, it induces a morphism

$$\frac{H^\bullet(L_n)}{\mathrm{Ker}(c_n : H^\bullet(L_n) \rightarrow H^\bullet(L_n))} \rightarrow \frac{H^\bullet(M_n)}{\mathrm{Ker}(c_n : H^\bullet(M_n) \rightarrow H^\bullet(M_n))}$$

In this way we get a map

$$\mathrm{Hom}_{H^\bullet(X)}^\bullet(H^\bullet(L_n), H^\bullet(M_n)) \xrightarrow{\pi} \mathrm{Hom}_{H^\bullet(X)}^\bullet(H^\bullet(u_n^*L_n), H^\bullet(u_n^*M_n))$$

Furthermore the projection $H^\bullet(L_n) \twoheadrightarrow H^\bullet(L_{n-1})$ together with the dual injection $H^\bullet(M_{n-1}) \hookrightarrow H^\bullet(M_n)$ give rise to an injective map

$$\mathrm{Hom}_{H^\bullet(X)}^\bullet(H^\bullet(L_{n-1}), H^\bullet(M_{n-1})) \hookrightarrow \mathrm{Hom}_{H^\bullet(X)}^\bullet(H^\bullet(L_n), H^\bullet(M_n))$$

We can fit these maps in the following commutative diagram

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{D}(X_{n-1})}^\bullet(L_{n-1}, M_{n-1}) & \xrightarrow{\psi_{n-1}} & \mathrm{Hom}_{H^\bullet(X)}^\bullet(H^\bullet(L_{n-1}), H^\bullet(M_{n-1})) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{D}(X_n)}^\bullet(L_n, M_n) & \xrightarrow{\psi_n} & \mathrm{Hom}_{H^\bullet(X)}^\bullet(H^\bullet(L_n), H^\bullet(M_n)) \\ \downarrow & & \downarrow \pi \\ \mathrm{Hom}_{\mathcal{D}(U_n)}^\bullet(u_n^*L_n, u_n^*M_n) & \xrightarrow{\tilde{\psi}_n} & \mathrm{Hom}_{H^\bullet(X)}^\bullet(H^\bullet(u_n^*L_n), H^\bullet(u_n^*M_n)) \\ \downarrow & & \\ 0 & & \end{array}$$

We already know that the left column is exact and by induction we can assume that ψ_{n-1} is an isomorphism.

$\tilde{\psi}_n$ is also an isomorphism: repeatedly using Lemma 4.5.2 we get

$$\begin{aligned} \mathrm{Hom}_{\mathcal{D}(U_n)}^\bullet(u_n^*L_n, u_n^*M_n) &\cong H^0(u_n^*\mathcal{H}\mathrm{om}(L_n, M_n)) \cong j_\omega^*\mathcal{H}\mathrm{om}(L_n, M_n) \cong \\ &\cong \mathrm{Hom}(j_\omega^*L_n, j_\omega^*M_n) \cong \mathrm{Hom}(H^\bullet(u_n^*L_n), H^\bullet(u_n^*M_n)) \end{aligned}$$

Notice that to say that a homomorphism is of $H^\bullet(X)$ -modules for objects on U_n is exactly as to say that it is \mathbb{C} -linear, since the action factorizes through $H^\bullet(X) \rightarrow H^\bullet(U_n) \cong \mathbb{C}$.

The thesis will now follow by applying the Snake Lemma. So it remains just to show that the right-hand column is exact on the middle term. So let us pick $\phi \in \text{Hom}^\bullet(H^\bullet(L_n), H^\bullet(M_n))$ such that $\pi(\phi) = 0$, i.e. such that the composite map

$$H^\bullet(L_n) \xrightarrow{\phi} H^\bullet(M_n) \rightarrow H^\bullet(u_n^*M_n) = H^\bullet(M_n)/H^\bullet(M_{n-1})$$

is 0. Therefore the image of ϕ is contained in $H^\bullet(M_{n-1}) = \text{Ker}(c_n : H^\bullet(M_n) \rightarrow H^\bullet(M_n))$. But ϕ commutes with c_n , so ϕ is 0 on $\text{Im}(c_n : H^\bullet(L_n) \rightarrow H^\bullet(M_n)) = H_c^\bullet(u_n^*L_n)$. This finally means that ϕ comes from a morphism in $\text{Hom}^\bullet(L_{n-1}, M_{n-1})$. \square

Appendix A

Functors on Derived Category of Sheaves

Let X be a complex algebraic variety of dimension d . X is naturally endowed with two different topologies, the Zariski topology and the complex topology. We will usually consider it a topological space using the latter, unless otherwise specified.

Let $\mathrm{Sh}(\mathbb{C}_X)$ the category of sheaves of \mathbb{C}_X -modules on X . We denote it by $\mathcal{D}^{\natural}(\mathbb{C}_X)$ or $\mathcal{D}^{\natural}(X)$ its *derived category* (here \natural stands for $b, +, -$ or \emptyset meaning, respectively, the bounded, bounded-below, bounded-above or unbounded derived category). A fairly complete introduction to derived category of sheaves could be found in the first two chapter of [KS94].

A.1 The Direct and Inverse Image Functors

Let $f : X \rightarrow Y$ a morphism of complex algebraic varieties. This induces a pullback functor $f^* : \mathrm{Sh}(\mathbb{C}_Y) \rightarrow \mathrm{Sh}(\mathbb{C}_X)$. This is an exact functor, hence it induces a functor, also denoted by f^* , $f^* : \mathcal{D}^{\natural}(Y) \rightarrow \mathcal{D}^{\natural}(X)$.

The direct image functor $f_* : \mathrm{Sh}(\mathbb{C}_X) \rightarrow \mathrm{Sh}(\mathbb{C}_Y)$ is left exact. Thus, it admits a right derived functors $Rf_* : \mathcal{D}^+(X) \rightarrow \mathcal{D}^+(Y)$. If there is no risk of confusion, we will usually write f_* in place of Rf_* .

Any bounded-below complex of sheaves F^\bullet admits a injective resolution $0 \rightarrow F^\bullet \rightarrow J^\bullet$: J^\bullet is a complex of injective sheaves and the induced map between the cohomology sheaves $\mathcal{H}^i(F^\bullet) \rightarrow \mathcal{H}^i(J^\bullet)$ is an isomorphism for any i (when this happens for a general map of complexes, the map is said to be a *quasi-isomorphism*). Furthermore, J^\bullet is unique up to homotopy. To compute $Rf_*(F^\bullet)$, where F^\bullet is a bounded-below complex of sheaves on X , one chooses an injective resolution $0 \rightarrow F^\bullet \rightarrow J^\bullet$, and sets $Rf_*(F^\bullet) := f_*(J^\bullet)$. This construction is possible more generally for any left-exact functor.

Furthermore, each sheaf F on X admits an injective resolution $0 \rightarrow F \rightarrow J^\bullet$ such that $J^m = 0$ for any $m > n$. This means that every bounded complex of sheaves has an injective resolution which is still bounded. Thus, we can also consider the functor $Rf_* : \mathcal{D}^b(X) \rightarrow \mathcal{D}^b(Y)$.

For example if F is a single sheaf on X (which we can think of as a complex of sheaves concentrated in degree 0) and $p : X \rightarrow \{\text{pt}\}$ then $Rp_*(F) = R\Gamma(F)$ and we have an isomorphism $R\Gamma(F) \cong \bigoplus_i H^i(F)[-i]$, which we use to identify $R\Gamma(F)$ with $H^\bullet(F)$. One can abbreviate $R^i f_*$ for $H^i \circ Rf_*$.

The functors (f_*, f^*) are a pair of adjoint functors: for any $F \in \text{Sh}(\mathbb{C}_X)$ and $G \in \text{Sh}(\mathbb{C}_Y)$

$$\text{Hom}_{\text{Sh}(\mathbb{C}_Y)}(G, f_* F) \cong \text{Hom}_{\text{Sh}(\mathbb{C}_X)}(f^* G, F)$$

There is also a derived version of this fact:

Proposition A.1.1. *Let $F \in \mathcal{D}(X)$ and $G \in \mathcal{D}^+(Y)$. Then,*

$$R\text{Hom}_{\mathbb{C}_Y}(G, Rf_* F) = R\text{Hom}_{\mathbb{C}_X}(f^* F, G)$$

Here $\text{Hom}^\bullet(\cdot, \cdot)$ is the bifunctor on complex of sheaves defined as

$$\text{Hom}^n(X^\bullet, Y^\bullet) = \prod_k \text{Hom}_{\mathbb{C}_X\text{-Mod}}(X^k, Y^{n+k})$$

$$(d^n f)^k = d_Y^{n+k} \circ f^k + (-1)^{n+1} f^{k+1} \circ d_X^k \in \text{Hom}_{\mathbb{C}_X\text{-Mod}}(X^k, Y^{n+k+1})$$

and $R\text{Hom} : \mathcal{D}^-(X)^{op} \times \mathcal{D}^+(X) \rightarrow \mathcal{D}^+(\mathbb{C}\text{-Mod})$ is the derived functor of Hom . $R\text{Hom}(X^\bullet, Y^\bullet)$ can be computed using an injective resolution $0 \rightarrow Y^\bullet \rightarrow J^\bullet$

$$R\text{Hom}^n(X^\bullet, Y^\bullet) = \text{Hom}^n(X^\bullet, J^\bullet) = \prod_k \text{Hom}_{\mathbb{C}_X\text{-Mod}}(X^k, J^{n+k})$$

Proof. If F is an injective sheaf, $f_* F$ is also injective. Hence $R\text{Hom}_{\mathbb{C}_Y}(G, Rf_*(\cdot))$ is the derived functor of $\text{Hom}_{\mathbb{C}_Y}(G, f_*(\cdot))$. On the other hand $R\text{Hom}_{\mathbb{C}_X}(f^* G, \cdot)$ is the derived functor $\text{Hom}_{\mathbb{C}_X}(f^* G, \cdot)$ and we can conclude from the underived case. \square

There is also a local statement of the adjointness of f_* and f^* for $F \in \text{Sh}(\mathbb{C}_X)$ and $G \in \text{Sh}(\mathbb{C}_Y)$,

$$\mathcal{H}\text{om}_{\text{Sh}(\mathbb{C}_Y)}(G, f_* F) \cong \mathcal{H}\text{om}_{\text{Sh}(\mathbb{C}_X)}(f^* G, F).$$

With a similar argument we can get also a derived version of this:

$$R\mathcal{H}\text{om}_{\mathbb{C}_Y}(G, Rf_* F) = Rf_* R\mathcal{H}\text{om}_{\mathbb{C}_X}(f^* F, G)$$

One can notice that $H^0(\text{Hom}^\bullet(F^\bullet, G^\bullet))$ is exactly the group of morphisms of complex of sheaves $F^\bullet \rightarrow G^\bullet$ up to algebraic homotopy, in other words is the group of morphism $\text{Hom}_{K(X)}(F^\bullet, G^\bullet)$ in the homotopy category.

Proposition A.1.2. *Let $F, G \in \mathcal{D}^+(X)$. Then,*

$$H^0(R\text{Hom}^\bullet(F^\bullet, G^\bullet)) = \text{Hom}_{\mathcal{D}^+(X)}(F^\bullet, G^\bullet)$$

In particular $Rf_ : \mathcal{D}^+(X) \rightarrow \mathcal{D}^+(Y)$ and $f^* : \mathcal{D}^+(Y) \rightarrow \mathcal{D}^+(X)$ are adjoint functors.*

Proof. Let $0 \rightarrow G^\bullet \rightarrow J^\bullet$ an injective resolution. We have $H^0(\mathrm{RHom}^\bullet(F^\bullet, G^\bullet)) = H^0(\mathrm{Hom}^\bullet(F^\bullet, J^\bullet)) = \mathrm{Hom}_{K^+(X)}^\bullet(F^\bullet, J^\bullet)$. Since J is injective, the canonical map $\mathrm{Hom}_{K^+(X)}^\bullet(F^\bullet, J^\bullet) \rightarrow \mathrm{Hom}_{D^+(X)}^\bullet(F^\bullet, J^\bullet) \cong \mathrm{Hom}_{D^+(X)}^\bullet(F^\bullet, G^\bullet)$ is an isomorphism (cfr. [KS06, 13.4.1]). \square

In general we have $H^n(\mathrm{RHom}^\bullet(F^\bullet, G^\bullet)) \cong \mathrm{Hom}_{D^+(X)}(F^\bullet, G^\bullet[n])$. This group is often denoted as $\mathrm{Ext}^n(F^\bullet, G^\bullet)$.

A.2 The Direct Image with Compact Support

Although the results stated in this and the following sections hold under much more general hypotheses, we will state them only for complex algebraic varieties and algebraic maps between them.

Definition A.2.1. Let $f : X \rightarrow Y$ be a morphism of complex algebraic varieties. The *direct image functor with compact support* $f_! : \mathrm{Sh}(\mathbb{C}_X) \rightarrow \mathrm{Sh}(\mathbb{C}_Y)$ is the functor which to a sheaf $F \in \mathrm{Sh}(\mathbb{C}_X)$ associates the sheaf $f_!F \in \mathrm{Sh}(\mathbb{C}_Y)$ defined as

$$f_!F(V) = \{s \in F(f^{-1}(V)) \mid f|_{\mathrm{supp}(s)} : \mathrm{supp}(s) \rightarrow V \text{ is proper}\}$$

for any open $V \subseteq Y$

Example A.2.2. If $f : X \rightarrow Y$ is a proper morphism, then clearly $f_* = f_!$

$f_!$ is a left-exact functor, so we can defined its right-derived functor

$$Rf_! : \mathcal{D}^+(X) \rightarrow \mathcal{D}^+(Y)$$

Example A.2.3. If $p : X \rightarrow \{\mathrm{pt}\}$, the functor $p_!$ is equivalent to the functor of global sections with global support Γ_c . Deriving this functor we recover the cohomology with compact support

$$R^q p_!F = H_c^q(X, F)$$

Theorem A.2.4 (Proper Base Change). *Let $f : X \rightarrow Y$ a morphism of complex algebraic varieties. Then*

i) *For any $y \in Y$ we have $(f_!F)_y \cong \Gamma_c(f^{-1}(y), F|_{f^{-1}(y)})$ and, $\forall q \in \mathbb{N}$, $(R^q f_!F)_y \cong H_c^q(f^{-1}(y), F|_{f^{-1}(y)})$,*

ii) *If $i : Z \hookrightarrow X$ is the inclusion of a locally closed subvariety, then the functor $i_!$ is exact and $(i_!F)_x = 0$ for any $x \notin Z$.*

iii) *If the diagram*

$$\begin{array}{ccc} X' & \xrightarrow{g} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g'} & Y \end{array}$$

is cartesian, then $(g')^ \circ f_! \cong (f')_! \circ g^*$*

Proof. See [KS94, 2.5.3, 2.5.4, 2.5.11] \square

A.3 The Adjunction Triangles

From the adjointness of the pair (Rf_*, f^*) we get a canonical morphism $F \rightarrow Rf_*f^*F$, called the *adjunction morphism*. This is the image of $\text{Id} \in \text{Hom}(f^*F, f^*F)$ via the adjunction isomorphism.

Let now $i : Z \hookrightarrow X$ be the inclusion of a closed subvariety. We set $U = X \setminus Z$ and we denote by $j : U \hookrightarrow X$ the open embedding.

For a closed subvariety $Z \subseteq X$ we can define $\Gamma_Z(X, F) = \text{Ker}(F(X) \rightarrow F(X \setminus Z))$. More generally, for a locally closed subvariety $Z \subseteq X$, we can define $\Gamma_Z(X, F)$ as $\Gamma_Z(U, F)$, where $U \subseteq X$ is any open subset such that Z is closed in U . This does not depend on the choice of U .

In this way we can define the sheaf of sections of F supported on Z .

$$\Gamma_Z(F)(U) = \Gamma_{Z \cap U}(U, F)$$

Proposition A.3.1. *Let $i : Z \hookrightarrow X$ be the inclusion of a closed subvariety. Then the functor $i^* \circ \Gamma_Z(\cdot)$ is right-adjoint to the functor $i_!$*

$$\text{Hom}(F, i^* \circ \Gamma_Z(G)) \cong \text{Hom}(i_!F, G)$$

$i^* \circ \Gamma_Z(\cdot)$ is a left-exact functor. We denote its right-derived functor $R(i^* \circ \Gamma_Z(\cdot))$ by $i^!$.

For any sheaf F on X , the sequence

$$0 \rightarrow \Gamma_Z(F) \rightarrow F \rightarrow j_*j^*F$$

is exact by definition. Moreover $\Gamma_Z(F) \cong i_*i^*\Gamma_Z(F)$ for any sheaf F (this can be easily checked on the stalk). So we can rewrite it by

$$0 \rightarrow i_!i^*\Gamma_Z(F) \rightarrow F \rightarrow j_*j^*F$$

Furthermore, if F is injective we can add a 0 on the right because injective sheaves are flabby, so $F(V) \rightarrow F(V \cap U)$ is surjective for any open V .

”Deriving” this sequence does not change anything for injective complexes J^\bullet : we have the following exact sequence of complex of sheaves:

$$0 \rightarrow i_!i^!J^\bullet \rightarrow J^\bullet \rightarrow j_*j^*J^\bullet \rightarrow 0$$

We can apply the general fact that any exact sequence gives a distinguished triangle. Hence have the following distinguished triangle in $\mathcal{D}^+(X) = \mathcal{D}^+(\mathcal{I})$ (\mathcal{I} is the subcategory of injective sheaves)

$$\rightarrow i_!i^!F^\bullet \rightarrow F^\bullet \rightarrow Rj_*j^*F^\bullet \xrightarrow{+1}$$

For a sheaf F we can also consider the sequence

$$0 \rightarrow j_!j^*F \rightarrow F \rightarrow i_*i^*F \rightarrow 0$$

This is an exact sequence: for any point $x \in X$ we have

$$(j_!j^*F)_x = \begin{cases} 0 & \text{if } x \in Z \\ F_x & \text{if } x \notin Z \end{cases} \quad (i_*i^*F)_x = \begin{cases} F_x & \text{if } x \in Z \\ 0 & \text{if } x \notin Z \end{cases}$$

This sequence gives the distinguished triangle

$$\rightarrow j_!j^*F^\bullet \rightarrow F^\bullet \rightarrow i_*i^*F^\bullet \xrightarrow{+1}$$

A.4 Poincaré-Verdier duality

Let $f : X \rightarrow Y$ a morphism of complex algebraic varieties. One can define a functor $f^! : \mathcal{D}^+(Y) \rightarrow \mathcal{D}^+(X)$ [KS94, §3] that is the right-adjoint of $Rf_!$.

Theorem A.4.1 (Verdier Duality). *There exists an additive functor of triangulated categories $f^! : \mathcal{D}^+(Y) \rightarrow \mathcal{D}^+(X)$, called exceptional inverse image such that*

$$R\mathrm{Hom}^\bullet(Rf_!F^\bullet, G^\bullet) \cong R\mathrm{Hom}^\bullet(F^\bullet, f^!G^\bullet)$$

for any $F \in \mathcal{D}^+(X)$, $G \in \mathcal{D}^+(Y)$.

The local version

$$R\mathcal{H}\mathrm{om}^\bullet(Rf_!F^\bullet, G^\bullet) \cong Rf_*R\mathcal{H}\mathrm{om}^\bullet(F^\bullet, f^!G^\bullet)$$

holds if we assume $F \in \mathcal{D}^-(X)$.

The construction of $f^!$ is quite demanding and technical in general. However, we can give explicit description in some special cases.

Proposition A.4.2. *Let $j : Z \rightarrow X$ a locally closed immersion, $j^!$ coincides with the functor defined in A.3, that is*

$$j^!(F^\bullet) \cong j^*R\Gamma_Z(F^\bullet)$$

In particular for an open embedding j of an open $U \subseteq X$, we get $j^! = j^*$. This follows from $\Gamma_U = j_*j^*$ (so $R\Gamma_U = Rj_*j^*$) and $\mathrm{Id} = j^*j_*$ (so $\mathrm{Id} = j^*Rj_*$).

The exceptional inverse image well-behaves with respect to composition: $(f \circ g)^! = g^! \circ f^!$. Besides, since $f^! \circ Rg_*$ is the right-adjoint of $g^* \circ Rf_!$ and $Rg_* \circ f^!$ is the right-adjoint of $Rf_! \circ g^*$, for a cartesian diagram as in A.2.4 we have

$$f^! \circ Rg'_* \cong Rg_* \circ (f')^!$$

Another useful formula [KS94, 3.1.13] is the following:

$$f^!R\mathcal{H}\mathrm{om}(F^\bullet, G^\bullet) \cong R\mathcal{H}\mathrm{om}(f^*F^\bullet, f^!G^\bullet) \tag{A.1}$$

for any $F^\bullet \in \mathcal{D}^b(X)$ and $G^\bullet \in \mathcal{D}^+(X)$.

Definition A.4.3. Let $p_X : X \rightarrow \{\text{pt}\}$. Then the complex $p_X^!(\mathbb{C}) \in \mathcal{D}^b(X)$ is called the *dualizing complex* and it is denoted by ω_X . For a general morphism $f : X \rightarrow Y$, we define $\omega_{X/Y} = f^!(\mathbb{C}_Y)$ the *relative dualizing complex* of f .

Example A.4.4. If X is a topological manifold of real dimension d , then $\mathcal{H}^m(\omega_X)$ is 0 for $m \neq -d$ while $\mathcal{H}^{-d}(\omega_X)$ is a local system of rank one. If X is a complex d -dimensional manifold, then it is orientable and $\omega_X = \mathbb{C}_X[2d]$.

A topological submersion is a map $f : X \rightarrow Y$ that locally on a open $U \subseteq Y$ is topologically equivalent to the projection $p_1 : U \times \mathbb{R}^d \rightarrow U$

Theorem A.4.5. Let $f : X \rightarrow Y$ be a topological submersion of complex algebraic varieties with fiber of complex dimension d . Then

- i) $\mathcal{H}^m(\omega_{X/Y}) = 0$ for any $m \neq -2d$ and $\mathcal{H}^{-2d}(\omega_{X/Y}) = \mathbb{C}_X$, so $\omega_{X/Y} = \mathbb{C}_X[2d]$.
- ii) For any $F^\bullet \in \mathcal{D}^+(X)$ there exists a canonical isomorphism $f^*(F^\bullet)[2d] \cong f^!(F^\bullet)$.

We can recover the Poincaré Duality for complex manifolds as a special case of the Verdier Duality

Theorem A.4.6 (Poincaré Duality). Let X a complex manifold of dimension d . Then there is a natural isomorphism

$$H^m(X, \mathbb{C}_X) \cong H_c^{2d-m}(X, \mathbb{C}_X)^\vee$$

Proof. For a complex manifold $\omega_X[-2d] \cong \mathbb{C}_X$. It follows that

$$\begin{aligned} H^m(X, \mathbb{C}_X) &\cong H^m(X, \omega_X[-2d]) \cong H^0(X, \omega_X[m-2d]) \cong \\ &\cong H^0(R\Gamma(X, \omega_X[m-2d])) \cong H^0(R\text{Hom}^\bullet(\mathbb{C}_X, \omega_X[m-2d])) \cong \\ &\text{Hom}_{\mathcal{D}^b(X)}(\mathbb{C}_X, \omega_X[m-2d]) \cong \text{Hom}_{\mathcal{D}^b(X)}(\mathbb{C}_X, p_X^! \mathbb{C}_{\text{pt}}[m-2d]) \end{aligned}$$

and, using the adjunction formula for $p_X^!$,

$$\text{Hom}_{\mathcal{D}^b(X)}(\mathbb{C}_X, p_X^! \mathbb{C}_{\text{pt}}[m-2d]) \cong \text{Hom}_{\mathcal{D}^b(\text{pt})}(p_X! \mathbb{C}_X[2d-m], \mathbb{C}_{\text{pt}})$$

$$\text{Hom}_{\mathcal{D}^b(\text{pt})}(R\Gamma_c(X, \mathbb{C}_X)[2d-m], \mathbb{C}_{\text{pt}}) \cong \text{Hom}_{\mathcal{D}^b(\text{pt})}(H_c^{\bullet+2d-m}(X, \mathbb{C}_X), \mathbb{C}_{\text{pt}})$$

As an immediate consequence of the Universal Coefficient Theorem we get

$$\begin{aligned} \text{Hom}_{\mathcal{D}^b(\text{pt})}(H_c^{\bullet+2d-m}(X, \mathbb{C}_X), \mathbb{C}_{\text{pt}}) &\cong H^0(\text{Hom}^\bullet(H_c^{\bullet+2d-m}(X, \mathbb{C}_X), \mathbb{C}_{\text{pt}})) \cong \\ H^{2d-m}(\text{Hom}^\bullet(H_c^\bullet(X, \mathbb{C}_X), \mathbb{C}_{\text{pt}})) &\cong \text{Hom}(H_c^{2d-m}(X, \mathbb{C}_X, \mathbb{C}_{\text{pt}}) \cong H_c^{2d-m}(X, \mathbb{C}_X)^\vee \end{aligned}$$

□

Definition A.4.7. For a complex $F^\bullet \in \mathcal{D}^b(X)$ we define the *Verdier dual* $\mathbb{D}_X F^\bullet \in \mathcal{D}^b(X)$ to be the complex of sheaves $R\text{Hom}^\bullet(F^\bullet, \omega_X)$.

The functor \mathbb{D} is a (contravariant) functor of triangulated categories: if $A \rightarrow B \rightarrow C \xrightarrow{+1}$ is a distinguished triangle in $\mathcal{D}^b(X)$, then also $\mathbb{D}_X C \rightarrow \mathbb{D}_X B \rightarrow \mathbb{D}_X A \xrightarrow{+1}$ is distinguished. Obviously, $\mathbb{D}_X(F^\bullet[n]) = \mathbb{D}_X(F^\bullet)[-n]$

Proposition A.4.8. *Let $f : X \rightarrow Y$ a morphism of complex algebraic varieties. Then*

$$i) f^!(\mathbb{D}_Y F^\bullet) \cong \mathbb{D}_X(f^* F^\bullet) \text{ for any } F^\bullet \in \mathcal{D}^b(Y)$$

$$ii) Rf_*(\mathbb{D}_X F^\bullet) \cong \mathbb{D}_Y(Rf_! F^\bullet) \text{ for any } F^\bullet \in \mathcal{D}^b(X)$$

Proof. i) From the definition we have $f^! \omega_Y = \omega_X$. Using A.1 we get

$$f^!(\mathbb{D}_Y F^\bullet) = f^!(R\mathcal{H}om^\bullet(F^\bullet, \omega_Y)) \cong R\mathcal{H}om^\bullet(f^* F^\bullet, \omega_X) \cong \mathbb{D}_X(f^* F^\bullet)$$

ii) Using the local form of Poincaré Verdier Duality we get

$$Rf_*(\mathbb{D}_X F^\bullet) \cong Rf_*(R\mathcal{H}om^\bullet(F^\bullet, f^! \omega_Y)) \cong R\mathcal{H}om^\bullet(Rf_! F^\bullet, \omega_Y) \cong \mathbb{D}_Y(Rf_! F^\bullet)$$

□

If X is a complex manifold of dimension d , then $\omega_X \cong \mathbb{C}_X[2d]$, so $\mathbb{D}_X F^\bullet$ is just $R\mathcal{H}om(F^\bullet, \mathbb{C}_X)[2d]$. In particular, if F is a local system, then $\mathbb{D}_X F^\bullet = F^\vee[2d]$, a shift of the dual local system F^\vee .

In the case in which $Y = \{\text{pt}\}$ and $G^\bullet = \mathbb{C}_{\text{pt}}$, using the Poincaré-Verdier Duality we get a natural isomorphism

$$R\Gamma(X, \mathbb{D}_X F^\bullet) \cong \mathbb{D}_{\text{pt}} R\Gamma_c(X, F^\bullet) \cong R\Gamma_c(X, F^\bullet)^\vee$$

or, equivalently,

$$H^m(X, \mathbb{D}_X F^\bullet) \cong H_c^{-m}(X, F^\bullet)^\vee$$

Appendix B

Cohomologically Constructible Sheaves

B.1 Whitney Stratification

Let X be a complex algebraic variety of dimension d .

Definition B.1.1. A *stratification* for X is a locally finite partition $X = \bigsqcup_{\alpha \in A} X_\alpha$ which satisfies the following conditions:

- For any $\alpha \in A$, X_α is a locally closed smooth subvariety
- For any $\alpha \in A$, the boundary $\partial S_\alpha = \overline{S_\alpha} \setminus S_\alpha$ is union of some S_β .

Each S_α is called a *stratum* of the stratification.

Whitney suggested also an additional condition for stratifications.

Definition B.1.2. A stratification $X = \sqcup_{\alpha \in A} X_\alpha$ is called a *Whitney Stratification* if the following conditions are satisfied:

- Let $x_i \in S_\alpha$ a sequence of points converging to a point $\bar{x} \in X_\beta$. If the limit of the tangent spaces $T_{x_i} X_\alpha$ exists, then we have $T_{\bar{x}}(X_\beta) \subseteq \lim_i T_{x_i}(X_\alpha)$
- Let $x_i \in X_\alpha$ and $y_i \in X_\beta$ be two sequences of points converging to the same point $\bar{y} \in X_\beta$ and let l_i be the line connecting x_i and y_i . If the limit of the tangent spaces $T_{x_i} X_\alpha$ and the limit of the lines l_i exist, then we have $\lim_i l_i \subseteq \lim_i T_{x_i}(X_\alpha)$

These additional conditions correspond, intuitively, to requiring that the normal structure along the strata is "locally constant". The following example illustrates this property.

Example B.1.3 (Whitney's umbrella). Let X the variety defined by the equation $y^2 = zx^2$ in the affine space $\mathbb{A}_{\mathbb{C}}^3$. The set of singular points of X is the line $x = y = 0$. Thus, $X_1 = \{(x, y, z) \in X \mid x = y = 0\}$, $X_2 = X \setminus X_1$ is a stratification of X .

However, if we consider the sequence of points $x_i = (\frac{1}{i}, 0, 0) \in X_2$, for $i \in \mathbb{N}$ it is clear that $\lim_i T_{x_i} X_2$ does not contain $T_{(0,0,0)} X_1$.

The stratification $Y_1 = \{(0, 0, 0)\}$, $Y_2 = X_1 \setminus \{(0, 0, 0)\}$, $Y_3 = X_2$ is a refinement which is a Whitney stratification.

In general any complex quasi-projective variety of pure dimension admits a Whitney stratification. Moreover, any stratification can be refined to satisfy Whitney conditions. The following is an important consequence of the Whitney condition

Theorem B.1.4. *Let $X = \bigsqcup_{\alpha \in A} X_\alpha$ be a Whitney stratified space of dimension d and let x be a point in the k -dimensional stratum X_β . Then x admits a fundamental system of neighborhoods $\{W_x\}$ homeomorphic, through a stratum-preserving homeomorphism, to the product of an Euclidean space (with a single stratum) and a real cone over a stratified space of smaller dimension L*

$$W_x \cong \mathbb{R}^{2k} \times \mathcal{C}_{\mathbb{R}}(L)$$

Here L is the link of y and it is a stratified space of real dimension $2d - 2k - 1$.

B.2 Constructible Sheaves

Definition B.2.1. Let X a complex algebraic variety. A sheaf F is said to be *constructible* if there exists an algebraic stratification $X = \bigsqcup_{\alpha \in A} X_\alpha$ such that, for every $\alpha \in A$, the restriction $F|_{X_\alpha}$ is a local system on X_α . A complex of sheaves F^\bullet is said *constructible* if all its cohomology sheaves $\mathcal{H}^i(F^\bullet)$ are constructible sheaves.

Remark B.2.2. By *algebraic stratification* we mean that the strata X_α are required to be locally closed subvariety of X , i.e. locally closed in the Zariski topology.

We define $\mathcal{D}_c^b(X)$ the full subcategory of $\mathcal{D}^b(X)$ consisting of bounded constructible complexes of sheaves with respect to an algebraic stratification.

An important feature of this new category it is that it is preserved by the most common functors:

Theorem B.2.3 (Di, 4.1.5). *Let $f : X \rightarrow Y$ a morphism of complex algebraic varieties. Then:*

- i) If $F^\bullet \in \mathcal{D}_c^b(Y)$, then $f^* F^\bullet \in \mathcal{D}_c^b(X)$ and $f^! F^\bullet \in \mathcal{D}_c^b(X)$*
- ii) If $F^\bullet \in \mathcal{D}_c^b(X)$, then $Rf_* F^\bullet \in \mathcal{D}_c^b(X)$ and $Rf_! F^\bullet \in \mathcal{D}_c^b(X)$*
- iii) If $F^\bullet, G^\bullet \in \mathcal{D}_c^b(X)$, then $F^\bullet \overset{L}{\otimes} G^\bullet \in \mathcal{D}_c^b(X)$ and $R\mathcal{H}om(F^\bullet, G^\bullet) \in \mathcal{D}_c^b(X)$*

The main result relating duality and constructibility is the following

Theorem B.2.4 (Di, 4.1.16). *i) Let $F^\bullet \in \mathcal{D}^b(X)$. Then F^\bullet is in $\mathcal{D}_c^b(X)$ if and only if its dual $\mathbb{D}_X F^\bullet$ is in $\mathcal{D}_c^b(X)$.*

ii) Let $F^\bullet \in \mathcal{D}_c^b(X)$. Then there exists a natural isomorphism $F^\bullet \cong \mathbb{D}_X(\mathbb{D}_X F^\bullet)$.
In particular the dualizing complex $\omega_X = \mathbb{D}_X \mathbb{C}_X$ is constructible.

The constructible sheaves, which we may think as complex concentrated in degree zero, obviously form an abelian subcategory of $\mathcal{D}_c^b(X)$. Besides this, $\mathcal{D}_c^b(X)$ admits another abelian subcategory, of the so-called perverse sheaves.

Definition B.2.5. A complex of sheaves $F^\bullet \in \mathcal{D}_c^b(X)$ is called a *perverse sheaf* if

$$\dim(\text{supp}(\mathcal{H}^j(F^\bullet))) \leq -j \quad \text{and} \quad \dim(\text{supp}(\mathcal{H}^j(\mathbb{D}_X F^\bullet))) \leq -j$$

for any $j \in \mathbb{Z}$. We denote by $\text{Perv}(\mathbb{C}_X)$ the subcategory of perverse sheaves.

B.3 Perverse Sheaves

We have just defined the subcategory of perverse sheaves. We introduce two more full subcategories of $\mathcal{D}_c^b(X)$

- ${}^p\mathcal{D}_c^{\leq 0}(X)$ is the subcategory of $\mathcal{D}_c^b(X)$ whose objects are the $F^\bullet \in \mathcal{D}_c^b(X)$ such that $\dim(\text{supp}\mathcal{H}^j(F^\bullet)) \leq -j$ for any $j \in \mathbb{Z}$.
- ${}^p\mathcal{D}_c^{\geq 0}(X)$ is the subcategory of $\mathcal{D}_c^b(X)$ whose objects are the $F^\bullet \in \mathcal{D}_c^b(X)$ such that $\dim(\text{supp}\mathcal{H}^j(\mathbb{D}_X F^\bullet)) \leq -j$ for any $j \in \mathbb{Z}$.

Thus

$$\text{Perv}(\mathbb{C}_X) = {}^p\mathcal{D}_c^{\leq 0}(X) \cap {}^p\mathcal{D}_c^{\geq 0}(X)$$

Since $\mathbb{D}_X \mathbb{D}_X F^\bullet \cong F^\bullet$ for any $F^\bullet \in \mathcal{D}_c^b(X)$, the Verdier duality functor \mathbb{D}_X exchanges ${}^p\mathcal{D}_c^{\leq 0}(X)$ with ${}^p\mathcal{D}_c^{\geq 0}(X)$, so it leaves $\text{Perv}(\mathbb{C}_X)$ fixed.

Lemma B.3.1. Let $F^\bullet \in \mathcal{D}_c^b(X)$. Then,

$$\text{supp}\mathcal{H}^j(\mathbb{D}_X F^\bullet) = \{x \in X \mid \mathcal{H}^{-j}(i_{\{x\}} F^\bullet) \neq 0\}$$

for any $j \in \mathbb{Z}$, where $i_{\{x\}} : \{x\} \hookrightarrow X$ is the inclusion.

Proof. For any $x \in X$, $i_{\{x\}} F^\bullet \cong i_{\{x\}} \mathbb{D}_X \mathbb{D}_X F^\bullet \cong \mathbb{D}_{\{x\}} i_{\{x\}}^* (\mathbb{D}_X F^\bullet)$, so $\mathcal{H}^{-j}(i_{\{x\}} F^\bullet) \cong \mathcal{H}^j(\mathbb{D}_X F^\bullet)_x^\vee$ \square

We can use this Lemma to restate to perversity condition.

Proposition B.3.2. Let $F^\bullet \in \mathcal{D}_c^b(X)$ and let $X = \bigsqcup_{\alpha \in A} X_\alpha$ be a stratification consisting in connected strata such that all the restriction $F^\bullet|_{X_\alpha}$ and $\mathbb{D}_X F^\bullet|_{X_\alpha}$ (or $i_{X_\alpha}^! F^\bullet$) are locally constant for any $\alpha \in A$ (this always exists since both F^\bullet and $\mathbb{D}_X F^\bullet$ are constructible). Then

- i) $F^\bullet \in {}^p\mathcal{D}_c^{\leq 0}(X)$ if and only if $\mathcal{H}^j(F^\bullet|_{X_\alpha}) = \mathcal{H}^j(i_{X_\alpha}^* F^\bullet) = 0$ for any $\alpha \in A$ and $j > -\dim X_\alpha$;
- ii) $F^\bullet \in {}^p\mathcal{D}_c^{\geq 0}(X)$ if and only if $\mathcal{H}^j(i_{X_\alpha}^! F^\bullet) = 0$ for any $\alpha \in A$ and $j < -\dim X_\alpha$.

Proof. i) is trivial since, using the fact that F^\bullet is locally constant on X_α , we have that for $x \in X_\alpha$ $\mathcal{H}^j(F^\bullet)_x \neq 0$ if and only if $\mathcal{H}^j(F^\bullet|_{X_\alpha}) \neq 0$

ii) We have an isomorphism

$$i_{\{x\}}^! F^\bullet \cong i_{\{x\}}^! i_{X_\alpha}^! F^\bullet \cong \mathbb{D}_{\{x\}} i_{\{x\}}^* \mathbb{D}_{X_\alpha} i_{X_\alpha}^! F^\bullet$$

Since $\mathcal{H}^j(i_{X_\alpha}^! F^\bullet)$ is a local system on X_α , $\mathbb{D}_{\{x\}} i_{\{x\}}^* \mathbb{D}_{X_\alpha} i_{X_\alpha}^! F^\bullet \cong i_{\{x\}}^* i_{X_\alpha}^! F^\bullet[-2d_{X_\alpha}]$. Furthermore $\mathcal{H}^j(i_{X_\alpha}^! F^\bullet)$ is locally constant and we have that $\mathcal{H}^{-j}(i_{\{x\}}^! F^\bullet)$ is 0 everywhere on X_α or $\neq 0$ everywhere on X_α , that is the intersection $X_\alpha \cap \text{supp} H^j(\mathbb{D}_X F^\bullet)$ is \emptyset or X_α . If $F^\bullet \in {}^p\mathcal{D}_c^{\geq 0}(X)$, this intersection must be \emptyset if $\dim X_\alpha > -j$ and this happens if and only if $\mathcal{H}^j(i_{X_\alpha}^! F^\bullet) = 0$ for any $j < -\dim X_\alpha$. \square

In particular, if X is a complex manifold and $F^\bullet \in \mathcal{D}_c^b(X)$ is a complex such that all the cohomology sheaves are locally constant on X , then

- $F^\bullet \in {}^p\mathcal{D}_c^{\leq 0}(X)$ if and only if $\mathcal{H}^j(F^\bullet) = 0$ for any $j > -\dim X$
- $F^\bullet \in {}^p\mathcal{D}_c^{\geq 0}(X)$ if and only if $\mathcal{H}^j(F^\bullet) = 0$ for any $j < -\dim X$
- $F^\bullet \in \text{Perv}(\mathbb{C}_X)$ if and only if $F^\bullet \cong \mathcal{H}^{-\dim X}(F^\bullet)[\dim X]$, i.e. if and only if it is the shift of a local system.

B.3.1 t -structures

In order to prove that the Perverse Sheaves on a complex algebraic variety form an abelian variety one can show that the pair $({}^p\mathcal{D}_c^{\leq 0}(X), {}^p\mathcal{D}_c^{\geq 0}(X))$ is a t -structure on $\mathcal{D}_c^b(X)$.

Definition B.3.3. Let \mathcal{D} a triangulated category. Let $\mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq 0}$ full subcategories and we set $\mathcal{D}^{\leq n} = \mathcal{D}^{\leq 0}[-n]$, $\mathcal{D}^{\geq n} = \mathcal{D}^{\geq 0}[-n]$. We say that $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ form a t -structure if the following conditions are satisfied:

- (t1) $\mathcal{D}^{\leq -1} \subseteq \mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq 1} \subseteq \mathcal{D}^{\geq 0}$
- (t2) $\text{Hom}_{\mathcal{D}}(X, Y) = 0$ for any $X \in \mathcal{D}^{\leq 0}$ and $Y \in \mathcal{D}^{\geq 1}$
- (t3) For any $X \in \mathcal{D}$ there exists a distinguished triangle $X_0 \rightarrow X \rightarrow X_1 \xrightarrow{+1}$ such that $X_0 \in \mathcal{D}^{\leq 0}$ and $X_1 \in \mathcal{D}^{\geq 1}$

Example B.3.4. Let \mathcal{C} an abelian category and $\mathcal{D}(\mathcal{C})$ its derived category. Then the pair $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$, defined by

$$\mathcal{D}^{\leq 0} = \{F^\bullet \in \mathcal{D}(\mathcal{C}) \mid H^j(F^\bullet) = 0 \forall j > 0\} \quad \mathcal{D}^{\geq 0} = \{F^\bullet \in \mathcal{D}(\mathcal{C}) \mid H^j(F^\bullet) = 0 \forall j < 0\}$$

form a "standard" t -structure on \mathcal{D} . Similarly we can see that $\mathcal{D}_c^b(X)$ admits a "standard" t -structure.

Definition B.3.5. Let \mathcal{D} a triangulated category and $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ a t -structure. Then we call the full subcategory $\mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ the *core* of the t -structure.

In the example above, the core of the standard t -structure on the derived category of an abelian category \mathcal{C} is equivalent to category \mathcal{C} itself.

We summarize the main general property of t -structure in the following theorem

Theorem B.3.6 ([HTT08], §8.1.). *Let \mathcal{D} a triangulated category and $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ a t -structure of \mathcal{D}*

i) *For any $n \in \mathbb{Z}$ there exists a functor $\tau^{\leq n} : \mathcal{D} \rightarrow \mathcal{D}^{\leq n}$ right-adjoint of the inclusion $\mathcal{D}^{\leq n} \hookrightarrow \mathcal{D}$, i.e.*

$$\text{Hom}_{\mathcal{D}^{\leq n}}(Y, \tau^{\leq n} X) \cong \text{Hom}_{\mathcal{D}}(Y, X) \quad (\text{B.1})$$

for any $Y \in \mathcal{D}^{\leq n}$ and $X \in \mathcal{D}$.

ii) *For any $n \in \mathbb{Z}$ there exists a functor $\tau^{\geq n} : \mathcal{D} \rightarrow \mathcal{D}^{\geq n}$ left-adjoint of the inclusion $\mathcal{D}^{\geq n} \hookrightarrow \mathcal{D}$, i.e.*

$$\text{Hom}_{\mathcal{D}^{\geq n}}(\tau^{\geq n} X, Y) \cong \text{Hom}_{\mathcal{D}}(X, Y) \quad (\text{B.2})$$

for any $Y \in \mathcal{D}^{\geq n}$ and $X \in \mathcal{D}$.

iii) *The triangle*

$$\tau^{\leq n}(X) \rightarrow X \rightarrow \tau^{\geq n+1}(X) \xrightarrow{+1}$$

in which the morphism are the canonical ones coming from the adjunctions B.1 and B.2, is distinguished. In particular if $X_0 \rightarrow X \rightarrow X_1 \xrightarrow{+1}$ is the triangle as in (t3), then $X_0 \cong \tau^{\leq 0} X$ and $X_1 \cong \tau^{\geq 1} X$

iv) *the core $\mathcal{C} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ is an abelian category*

v) *Every exact sequence*

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

in \mathcal{C} gives rise to a distinguished triangle

$$A \rightarrow B \rightarrow C \xrightarrow{+1}$$

in \mathcal{D} .

In the Example B.3.4 the functors $\tau^{\leq n}$ and $\tau^{\geq n}$ are called *truncation functors*. These are defined by

$$\tau^{\leq n} X = \dots \rightarrow X^{n-2} \rightarrow X^{n-1} \rightarrow \text{Ker}(d^n) \rightarrow 0 \rightarrow 0 \rightarrow \dots$$

$$\tau^{\geq n} X = \dots \rightarrow 0 \rightarrow 0 \rightarrow \text{Coker}(d^n) \rightarrow X^{n+1} \rightarrow X^{n+2} \rightarrow \dots$$

Definition B.3.7. We can define a functor ${}^t H^0 : \mathcal{D} \rightarrow \mathcal{C} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ by ${}^t H^0 = \tau^{\geq 0} \tau^{\leq 0}(X) \cong \tau^{\leq 0} \tau^{\geq 0}(X)$. Furthermore we define ${}^t H^n(X) = {}^t H^0(X[n])$.

Proposition B.3.8. *The functor ${}^tH^0$ is a cohomological functor, that is, for any distinguished triangle*

$$X \rightarrow Y \rightarrow Z \xrightarrow{+1}$$

there is a long exact sequence in \mathcal{C}

$$\dots \rightarrow {}^tH^i(X) \rightarrow {}^tH^i(Y) \rightarrow {}^tH^i(Z) \rightarrow {}^tH^{i+1}(X) \rightarrow \dots$$

Definition B.3.9. Let \mathcal{D}_1 and \mathcal{D}_2 be two triangulated categories endowed with t -structures $(\mathcal{D}_i^{\leq 0}, \mathcal{D}_i^{\geq 0})$ ($i = 1, 2$) and let $F : \mathcal{D}_1 \rightarrow \mathcal{D}_2$ be a functor of triangulated categories. We say that F is *left t -exact* if $F(\mathcal{D}_1^{\leq 0}) \subseteq \mathcal{D}_2^{\leq 0}$, that it is *right t -exact* if $F(\mathcal{D}_1^{\geq 0}) \subseteq \mathcal{D}_2^{\geq 0}$ and that it is *t -exact* if it is both left and right t -exact.

Besides, we define ${}^tF = {}^tH \circ F$. This is a functor from the core \mathcal{C}_1 of \mathcal{D}_1 into the core \mathcal{C}_2 of \mathcal{D}_2 .

Proposition B.3.10 ([KS94], 10.1.14.,10.1.18). *Let $\mathcal{D}_1, \mathcal{D}_2$ as above and let $F : \mathcal{D}_1 \rightarrow \mathcal{D}_2$ a functor of triangulated categories. Then*

- i) If F is left (resp. right) t -exact, then ${}^tH^0(F(X)) \cong {}^tF({}^tH^0(X))$ for any $X \in \mathcal{D}_1^{\geq 0}$ (resp. for any $X \in \mathcal{D}_1^{\leq 0}$);*
- ii) If F is left (resp. right) t -exact, then tF is a left (resp. right) exact;*
- iii) If F is t -exact, then F induces a functor $F : \mathcal{C}_2 \rightarrow \mathcal{C}_2$ which is naturally isomorphic to tF . Moreover $F({}^tH^n(X)) \cong {}^tH^n(F(X))$ for any n and X ;*
- iv) If F is left adjoint to $G : \mathcal{D}_2 \rightarrow \mathcal{D}_1$, then F is right t -exact if and only if G is left t -exact.*

This general machinery can be used in our situation in view of the following

Theorem B.3.11. *The pair $({}^p\mathcal{D}_c^{\leq 0}(X), {}^p\mathcal{D}_c^{\geq 0}(X))$ defines a t -structure on $\mathcal{D}_c^b(X)$, called the perverse t -structure,*

In particular we have the perverse truncation functors ${}^p\tau^{\leq 0} : \mathcal{D}_c^b(X) \rightarrow {}^p\mathcal{D}_c^{\leq 0}(X)$, ${}^p\tau^{\geq 0} : \mathcal{D}_c^b(X) \rightarrow {}^p\mathcal{D}_c^{\geq 0}(X)$ and ${}^pH^n : \mathcal{D}_c^b(X) \rightarrow \text{Perv}(\mathbb{C}_X)$, called the n th perverse cohomology. For any functor of triangulated category $F : \mathcal{D}_c^b(X) \rightarrow \mathcal{D}_c^b(Y)$ we can define ${}^pF : \text{Perv}(\mathbb{C}_X) \rightarrow \text{Perv}(\mathbb{C}_Y)$ by ${}^pF = {}^pH^0 \circ F \circ i$, where $i : \text{Perv}(\mathbb{C}_X) \rightarrow \mathcal{D}_c^b(X)$ is the inclusion. For instance, for a morphism of complex algebraic varieties $f : X \rightarrow Y$ we can define the functors ${}^pRf_*, {}^pRf_! : \text{Perv}(\mathbb{C}_X) \rightarrow \text{Perv}(\mathbb{C}_Y)$ and ${}^pf^*, {}^pf^! : \text{Perv}(\mathbb{C}_Y) \rightarrow \text{Perv}(\mathbb{C}_X)$. To shorten the notation we will usually use pf_* and ${}^pf_!$ in place of pRf_* and ${}^pRf_!$.

The following proposition, in view of Prop. B.3.10 is important to investigate the functors considered above for a locally closed embedding.

Proposition B.3.12 ([HTT08], 8.1.41-43). *Let Z be locally closed subvariety of X and let $i : Z \rightarrow X$ the inclusion. Then the functors i^* and $i_!$ are right t -exact while $i^!$ and Ri_* are left t -exact, with respect to the perverse t -structures.*

B.4 Minimal Extension Functor

Let X be an irreducible projective variety of dimension d_X . The Intersection Cohomology Complex is a special example of a perverse sheaf on X . Roughly, Intersection Cohomology may be thought as an homology theory which “works well” for singular spaces, that is a setting in which we can generalize property typical of smooth spaces, such as the Poincaré Duality. The minimal extension functor is a tool necessary to define Intersection Cohomology from the sheaf-theoretic viewpoint.

Let U be a Zariski open dense set of X and let $F^\bullet \in \mathcal{D}_c^b(U)$. We say that a stratification $X = \bigsqcup_{\alpha \in A} X_\alpha$ is *compatible* with F^\bullet if there exists a subset $B \subseteq A$ such that $U = \bigsqcup_{\alpha \in B} X_\alpha$ and both $F^\bullet|_{X_\alpha}$ and $\mathbb{D}_U F^\bullet|_{X_\alpha}$ have locally constant cohomology sheaves for any $\alpha \in B$. Such a stratification always exists.

Let $j : U \hookrightarrow X$ the inclusion. Let $X = \bigsqcup_{\alpha \in A} X_\alpha$ a stratification compatible with $F^\bullet \in \mathcal{D}_c^b(U)$. Up to a refinement, we can assume that it is a Whitney stratification. With this assumption we have that both $Rj_* F^\bullet|_{X_\alpha}$ and $j_! F^\bullet|_{X_\alpha}$ ($j_!$ is exact for a locally closed embedding, so we can forget the R) have locally constant cohomology sheaves.

We have a canonical morphism $j_! \rightarrow j_*$ that gives rise to a morphism between derived functors $j_! = Rj_! \rightarrow Rj_*$. Now, by composing with the functor ${}^p H^0$, we get a canonical morphism ${}^p j_! \rightarrow {}^p Rj_*$ in $\text{Perv}(\mathbb{C}_X)$.

Definition B.4.1. For a perverse sheaf F^\bullet on U , we say that the image of the canonical morphism ${}^p j_! F^\bullet \rightarrow {}^p j_* F^\bullet$ is the *minimal extension* of F^\bullet and we denote it by ${}^p j_{!*} F^\bullet$.

Remark B.4.2. Sometimes ${}^p j_{!*} F^\bullet$ is called *intermediate extension* since it is an extension “between” ${}^p j_! F^\bullet$ and ${}^p Rj_* F^\bullet$. We prefer the term minimal since it is a quotient of ${}^p Rj_! F^\bullet$ and a subobject of ${}^p Rj_* F^\bullet$, so it is “smaller” than both of them. Also, ${}^p j_{!*}$ is minimal amongst extension of F^\bullet in a sense that will be clearer later.

Lemma B.4.3. For any $F^\bullet \in \text{Perv}(\mathbb{C}_U)$, we have $\mathbb{D}_X({}^p j_{!*} F^\bullet) \cong {}^p j_{!*}(\mathbb{D}_U F^\bullet)$

Proof. The functor \mathbb{D}_X sends distinguished triangles into distinguished triangles. Since it is t -exact, it is also an exact functor on $\text{Perv}(\mathbb{C}_X)$. Let’s prove this. If $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is exact on $\text{Perv}(\mathbb{C}_X)$, then $A \rightarrow B \rightarrow C \xrightarrow{+1}$ is distinguished and so is $\mathbb{D}_X C \rightarrow \mathbb{D}_X B \rightarrow \mathbb{D}_X A \xrightarrow{+1}$. In the deriving long exact sequence

$$\dots {}^p H^{-1}(\mathbb{D}_X A) \rightarrow {}^p H^0(\mathbb{D}_X C) \rightarrow {}^p H^0(\mathbb{D}_X B) \rightarrow {}^p H^0(\mathbb{D}_X A) \rightarrow {}^p H^1(\mathbb{D}_X C) \rightarrow \dots$$

is actually a short exact sequence, since all the terms ${}^p H^i(F^\bullet)$ are 0 for any $i \neq 0$ if $F^\bullet \in \text{Perv}(\mathbb{C}_X)$. In this case we have also ${}^p H^0(F^\bullet) \cong F^\bullet$, so we obtain that the sequence $0 \rightarrow \mathbb{D}_X C \rightarrow \mathbb{D}_X B \rightarrow \mathbb{D}_X A \rightarrow 0$ is exact.

Thus, $\mathbb{D}_X({}^p j_{!*} F^\bullet) \rightarrow \mathbb{D}_X({}^p j_! F^\bullet)$ is injective since ${}^p j_! F^\bullet \rightarrow {}^p j_{!*} F^\bullet$ is surjective and $\mathbb{D}_X({}^p Rj_* F^\bullet) \rightarrow \mathbb{D}_X({}^p j_{!*} F^\bullet)$ is injective since ${}^p j_{!*} F^\bullet \rightarrow {}^p Rj_* F^\bullet$ is injective.

Furthermore, we have ${}^p \tau^{\leq 0} \mathbb{D}_X \cong {}^p \tau^{\geq 0} \mathbb{D}_X$ and ${}^p \tau^{\geq 0} \mathbb{D}_X \cong {}^p \tau^{\leq 0} \mathbb{D}_X$. In fact, for any $A \in \mathcal{D}_c^b(X)$, $B \in {}^p \mathcal{D}_c^{\leq 0}$,

$$\text{Hom}_{\mathcal{D}_c^b(X)}(B, A) \cong \text{Hom}_{{}^p \mathcal{D}_c^{\leq 0}(X)}(B, {}^p \tau^{\leq 0} A) \implies$$

$$\implies \mathrm{Hom}_{\mathcal{D}_c^b(X)}(\mathbb{D}_X A, \mathbb{D}_X B) \cong \mathrm{Hom}_{{}^p\mathcal{D}_c^{\geq 0}(X)}(\mathbb{D}_X {}^p\tau^{\leq 0} A, \mathbb{D}_X B)$$

since \mathbb{D}_X is an equivalence of categories. This means that $\mathbb{D}_X {}^p\tau^{\leq 0} \mathbb{D}_X$ is the left-adjoint of the inclusion $\mathcal{D}_c^{\geq 0}(X) \hookrightarrow \mathcal{D}_c^b(X)$, so $\mathbb{D}_X {}^p\tau^{\leq 0} \mathbb{D}_X \cong {}^p\tau^{\geq 0}$. The other isomorphism is analogous. This yields to ${}^pH^0 \mathbb{D}_X = {}^p\tau^{\leq 0} {}^p\tau^{\geq 0} \mathbb{D}_X \cong \mathbb{D}_X {}^p\tau^{\geq 0} {}^p\tau^{\leq 0} \cong \mathbb{D}_X {}^pH^0$

Hence we have

$$\mathbb{D}_X({}^pRj_*F^\bullet) = \mathbb{D}_X({}^pH^0(Rj_*F^\bullet)) \cong {}^pH^0 \mathbb{D}_X(Rj_*F^\bullet) \cong {}^pH^0 j_! \mathbb{D}_U(F^\bullet) \cong {}^p j_! \mathbb{D}_U(F^\bullet)$$

$$\mathbb{D}_X({}^p j_! F^\bullet) = \mathbb{D}_X({}^pH^0(j_! F^\bullet)) \cong {}^pH^0 \mathbb{D}_X(j_! F^\bullet) \cong {}^pH^0 Rj_* \mathbb{D}_U(F^\bullet) \cong {}^p Rj_* \mathbb{D}_U(F^\bullet)$$

Therefore, we have a surjective morphism ${}^p j_! \mathbb{D}_U(F^\bullet) \rightarrow \mathbb{D}_X({}^p j_{!*} F^\bullet)$ and an injective morphism $\mathbb{D}_X({}^p j_{!*} F^\bullet) \rightarrow {}^p j_! \mathbb{D}_U(F^\bullet)$ and this shows that $\mathbb{D}_X({}^p j_{!*} F^\bullet) \cong {}^p j_{!*}(\mathbb{D}_U F^\bullet)$. \square

Lemma B.4.4. *Let U' a Zariski open subset of X containing U and let $j_1 : U \hookrightarrow U'$ and $j_2 : U' \hookrightarrow X$ the inclusions. Then we have ${}^p j_{!*} F^\bullet \cong {}^p j_{2!*} {}^p j_{1!*} F^\bullet$*

Proof. Since Rj_{1*} and Rj_{2*} are left t -exact, we have ${}^p j_* F^\bullet \cong {}^pH^0(Rj_{2*} Rj_{1*} F^\bullet) \cong {}^p j_{2*} {}^p j_{1*} F^\bullet$. Similarly we have ${}^p j_! F^\bullet \cong {}^p j_{2!} {}^p j_{1!} F^\bullet$. The composition morphism

$${}^p j_! F^\bullet = {}^p j_{2!} {}^p j_{1!} F^\bullet \rightarrow {}^p j_{2!} {}^p j_{1!*} F^\bullet \rightarrow {}^p j_{2!*} {}^p j_{1!*} F^\bullet$$

because ${}^p j_!$ is right exact while

$${}^p j_{2!*} {}^p j_{1!*} F^\bullet \rightarrow {}^p j_{2!*} {}^p j_{1*} F^\bullet \rightarrow {}^p j_{2*} {}^p j_{1*} F^\bullet \cong {}^p j_* F^\bullet$$

because ${}^p j_*$ is left exact. Thus we obtain ${}^p j_{2!*} {}^p j_{1!*} F^\bullet \cong \mathrm{Im}({}^p j_! F^\bullet \rightarrow {}^p j_* F^\bullet) = {}^p j_{!*} F^\bullet$. \square

The next theorem will provide a useful characterization of the minimal extension. We denote by $i : Z = X \setminus U \hookrightarrow X$ the inclusion.

Theorem B.4.5. *The minimal extension $G^\bullet = {}^p j_{!*} F^\bullet$ of $F^\bullet \in \mathrm{Perv}(\mathbb{C}_X)$ is the unique perverse sheaf on X satisfying the following conditions:*

- i) $G^\bullet|_U \cong F^\bullet$
- ii) $i^* G^\bullet \in {}^p\mathcal{D}_c^{\leq -1}(Z)$
- iii) $i^! G^\bullet \in {}^p\mathcal{D}_c^{\geq 1}(Z)$

Proof. The first step is to show that the minimal extension G^\bullet satisfies the above conditions. Since $j^! = j^*$ it commutes with ${}^pH^0$ (cfr. [KS94, 5.1.9.]). Then i) follows from

$$\begin{aligned} G^\bullet|_U &= {}^p j_{!*} F^\bullet|_U \cong j^* \mathrm{Im}({}^p j_! F^\bullet \rightarrow {}^p Rj_* F^\bullet) \cong \mathrm{Im}(j^* {}^p j_! F^\bullet \rightarrow j^* {}^p Rj_* F^\bullet) \cong \\ &\cong \mathrm{Im}({}^pH^0(j^* j_! F^\bullet) \rightarrow {}^pH^0(j^* Rj_* F^\bullet)) \cong \mathrm{Im}({}^pH^0 F^\bullet \rightarrow {}^pH^0 F^\bullet) \cong F^\bullet \end{aligned}$$

We recall the adjunction triangle

$$j_!j^*G^\bullet \rightarrow G^\bullet \rightarrow i_*i^*G^\bullet \xrightarrow{+1}$$

which gives rise to the exact sequence

$${}^pH^0(j_!j^*G^\bullet) \rightarrow {}^pH^0(G^\bullet) \rightarrow {}^pH^0(i_*i^*G^\bullet) \rightarrow {}^pH^1(j_!j^*G^\bullet)$$

Clearly ${}^pH^0(G^\bullet) \cong G^\bullet$. From the first point, ${}^pH^0(j_!j^*G^\bullet) = {}^pj_!F^\bullet$ and ${}^pH^1(j_!j^*G^\bullet) = {}^pH^1(j_!F^\bullet) = 0$ since $j_!F^\bullet \in {}^p\mathcal{D}_c^{\leq 0}$. Thus we obtain the exact sequence

$${}^pj_!F^\bullet \rightarrow {}^pj_{!*}F^\bullet \rightarrow {}^pH^0(i_*i^*G^\bullet) \rightarrow 0$$

and this proves ${}^pH^0(i_*i^*G^\bullet) = 0$ because the canonical morphism ${}^pj_!F^\bullet \rightarrow {}^pj_{!*}F^\bullet$ is surjective. But $i_* = i_!$ is t -exact, so ${}^pH^0(i^*G^\bullet) = 0$, while i^* is right t -exact, so $i^*G^\bullet \in {}^p\mathcal{D}_c^{\leq 0}(Z)$, hence $i^*G^\bullet \in {}^p\mathcal{D}_c^{\leq -1}(Z)$.

Similarly, for the condition iii) we can use the distinguished triangle $i_*i^!G^\bullet \rightarrow G^\bullet \rightarrow Rj_*j^*G^\bullet \xrightarrow{+1}$ in order to obtain the exact sequence

$$0 \rightarrow {}^pH^0(i_*i^!G^\bullet) \rightarrow j_{!*}F^\bullet \rightarrow {}^pRj_*F^\bullet$$

and since the canonical morphism $j_{!*}F^\bullet \rightarrow {}^pRj_*F^\bullet$ we have ${}^pH^0(i_*i^!G^\bullet) = 0$, hence ${}^pH^0(i^!G^\bullet) = 0$. Since $i^!$ is left t -exact we have finally $i^!G^\bullet \in {}^p\mathcal{D}_c^{\geq 1}(Z)$.

Viceversa we have to show that if $G^\bullet \in \text{Perv}(\mathbb{C}_X)$ satisfies the three listed conditions, then $G^\bullet \cong {}^pj_{!*}F^\bullet$ canonically.

j^* is left-adjoint to Rj^* and, since $j^! = j^*$, it is also right-adjoint to $j_!$. Thus we obtain canonical morphisms $j_!F^\bullet \rightarrow G^\bullet$, $G^\bullet \rightarrow Rj_*F^\bullet$ from the isomorphisms $F^\bullet \rightarrow j^*G^\bullet$ and $j^*G^\bullet \rightarrow F^\bullet$. Applying ${}^pH^0$, we get ${}^pj_!F^\bullet \rightarrow G^\bullet$ and $G^\bullet \rightarrow {}^pRj_*F^\bullet$. To conclude it suffices to show that the former morphism is surjective, while the latter is injective. The cokernel of ${}^pj_!F^\bullet \rightarrow G^\bullet$ is supported on Z , so there exists an exact sequence

$${}^pj_!F^\bullet \rightarrow G^\bullet \rightarrow i_*E^\bullet \rightarrow 0$$

for some $E^\bullet \in \text{Perv}(\mathbb{C}_Z)$. i^* is right t -exact and this implies that ${}^pi^*$ is right exact. ${}^pi^*G^\bullet \rightarrow {}^pi^*i_*E^\bullet$ is surjective and ${}^pi^*i_*E^\bullet = {}^pH^0i^*i_*E^\bullet \cong {}^pH^0(E^\bullet) = E^\bullet$. But, the condition ii) assures that ${}^pi^*G^\bullet = 0$, so $E^\bullet = 0$.

Similarly, the kernel of $G^\bullet \rightarrow {}^pRj_*F^\bullet$ is supported on Z so we have an exact sequence

$$0 \rightarrow i_*E^\bullet \rightarrow G^\bullet \rightarrow {}^pRj_*F^\bullet$$

for some $E^\bullet \in \text{Perv}(\mathbb{C}_Z)$. $i^!$ is left t -exact, so we get an injective morphism ${}^pi^!i_*E^\bullet \rightarrow {}^pi^!G^\bullet$. By the condition iii) ${}^pi^!G^\bullet = 0$, hence

$$0 = {}^pi^!i_*E^\bullet \cong {}^pH^0i^*R\Gamma_Z i_*E^\bullet \cong {}^pH^0i^*i_*E^\bullet \cong E^\bullet$$

.

□

Corollary B.4.6. *If X is a smooth variety of dimension d , U a Zariski open subset of X , then for every local system L on X we have $L[d] \cong {}^pj_{!*}(L|_U[d])$*

Proof. We have to show that the three conditions of Theorem B.4.5 are satisfied. i) clearly holds. Let $Z = \bigsqcup_{\alpha \in A} Z_\alpha$ a stratification of Z . Each $i_{Z_\alpha}^* L$ is a local system on Z_α , so $\mathcal{H}^j(i_{Z_\alpha}^* L[d]) = \mathcal{H}^{j+d}(i_{Z_\alpha}^* L) = 0$ for any $j > -d$, and since $d > \dim Z_\alpha$ we can use Proposition B.3.2 to deduce $i^* L[d] \in {}^p\mathcal{D}_c^{\leq -1}(Z)$. Furthermore, we have $i^! L[d] = \mathbb{D}_{Z^*} i^* \mathbb{D}_X(L[d])$. As before, $i^* \mathbb{D}_X(L[d]) \in {}^p\mathcal{D}_c^{\leq -1}(Z)$, hence $i^! L[d] \in {}^p\mathcal{D}_c^{\geq 1}(X)$ \square

Proposition B.4.7. *Let $F^\bullet \in \text{Perv}(\mathbb{C}_U)$. Then ${}^p j_{!*} F^\bullet$ is the unique perverse sheaf such that it has neither a non-trivial subobject nor a non-trivial quotient object supported in Z .*

Proof. We want to show that ${}^p Rj_* F^\bullet$ has no non-trivial subobject supported in Z and that ${}^p j_! F^\bullet$ has no non-trivial quotient supported in Z . Then the thesis will follow as an immediate corollary, using the definition of minimal extension.

Let's assume that there exists a subobject $G^\bullet \subseteq {}^p Rj_* F^\bullet$ such that $\text{supp}(G^\bullet) \subseteq Z$. Then $i^! G^\bullet = i^* R\Gamma_Z G^\bullet \cong i^* G^\bullet$ is perverse on Z , thus ${}^p i^! G^\bullet \cong i^! G^\bullet$. ${}^p i^!$ is left-exact so ${}^p i^! G^\bullet$ is a subobject of ${}^p i^! {}^p Rj_* F^\bullet$. But ${}^p i^! {}^p Rj_* F^\bullet \cong {}^p H^0(i^! Rj_* F^\bullet) \cong 0$. Then G^\bullet is 0 since $G^\bullet \cong i_* i^* G^\bullet \cong i_* {}^p i^! G^\bullet$.

Similarly, if ${}^p j_! F^\bullet \rightarrow G^\bullet \rightarrow 0$ is exact and $\text{supp}(G^\bullet) \subseteq Z$, then using the right-exact functor ${}^p i^*$, we have ${}^p i^* G^\bullet = 0$ and we can conclude that $G^\bullet = 0$ as before.

We prove now the uniqueness statement. Let M a perverse sheaf that satisfies the hypothesis. From the adjunction triangle for M we get the following exact sequences

$$\begin{aligned} 0 \rightarrow i_* {}^p H^0(i^! M) \rightarrow M \rightarrow {}^p j_*(j^* M) \rightarrow i_* {}^p H^1(i^! M) \rightarrow 0 \\ 0 \rightarrow i_* {}^p H^{-1}(i^* M) \rightarrow {}^p j_!(j^* M) \rightarrow M \rightarrow i_* {}^p H^0(i^* M) \rightarrow 0 \end{aligned}$$

Then ${}^p H^0(i^! M)$ and ${}^p H^0(i^* M)$ must be 0. Since we already know that $i^! M \in {}^p\mathcal{D}^{\leq 0}$, we get $i^! M \in {}^p\mathcal{D}^{\leq -1}$. Similarly we also get $i^* M \in {}^p\mathcal{D}^{\geq 1}$. Now the thesis follows from Theorem B.4.5. \square

The minimal extension functor is not exact. However the following holds

Proposition B.4.8. *The minimal extension functor ${}^p j_{!*}$ preserves injective and surjective morphisms.*

Proof. Let $0 \rightarrow F^\bullet \rightarrow G^\bullet$ exact in $\text{Perv}(\mathbb{C}_U)$. Then ${}^p j_{!*} F^\bullet \rightarrow {}^p j_{!*} G^\bullet$ is an isomorphism, so the kernel must be supported on Z . Since ${}^p j_{!*}$ can not have non trivial subobject supported in Z , this kernel must be 0. Similarly, if $F^\bullet \rightarrow G^\bullet \rightarrow 0$ is exact in $\text{Perv}(\mathbb{C}_U)$, then the cokernel of $F^\bullet \rightarrow G^\bullet$ should be supported in Z , hence it is 0. \square

Proposition B.4.9. *The minimal extension functor ${}^p j_{!*}$ sends simple objects into simple objects.*

Proof. Let F^\bullet be a simple object in $\text{Perv}(\mathbb{C}_U)$ and let's assume that there exists an exact sequence $0 \rightarrow G^\bullet \rightarrow {}^p j_{!*} F^\bullet \rightarrow H^\bullet \rightarrow 0$ in $\text{Perv}(\mathbb{C}_X)$ such that G^\bullet and H^\bullet are both non-trivial. Then we can apply the exact functor $j^* \cong {}^p j^*$ and we obtain the exact sequence $0 \rightarrow j^* G^\bullet \rightarrow F^\bullet \rightarrow j^* H^\bullet \rightarrow 0$. From the simplicity of F^\bullet , $j^* G^\bullet$ or $j^* H^\bullet$ is 0, hence G^\bullet or H^\bullet is supported in Z . Now we can conclude by Proposition B.4.7 \square

Now let's assume that U is a smooth open subvariety of X and L a local system on U . In view of B.4.6 we can assume that U is maximal, i.e. U is the regular part $X_{\text{reg}} \subseteq X$. We can also choose a Whitney stratification $X = \bigsqcup_{\alpha \in A} X_\alpha$ such that U is the unique open stratum of the stratification. We set $X_k = \bigsqcup_{\dim X_\alpha \leq k} X_\alpha$ and we obtain the filtration of $X = X_d \supseteq X_{d-1} \supseteq \dots \supseteq X_0 \supseteq \emptyset$. In a dual way, we have the following chain of inclusions of opens subset

$$U = U_d \xrightarrow{j_d} U_{d-1} \xrightarrow{j_{d-1}} \dots \xrightarrow{j_2} U_1 \xrightarrow{j_1} U_0 = X$$

where $U_k = X \setminus X_{k-1}$.

Proposition B.4.10. *In this situation we have*

$${}^p j_{!*}(L[d]) \cong (\tau^{\leq -1} Rj_{1*}) \circ \dots \circ (\tau^{\leq -d} Rj_{d*})(L[d])$$

Proof. Since the minimal extensions of a composition is the composition of minimal extensions, it suffices to prove it for a single inclusion, i.e. it suffices to prove that, for any k ,

$${}^p j_{k!*} F^\bullet \cong \tau^{\leq -k} Rj_{k*}(F^\bullet)$$

where F is a perverse sheaf such that each restriction to the strata X_α has locally constant cohomology sheaves. So we need to show that the three conditions of Theorem B.4.5 are satisfied by $G^\bullet = \tau^{\leq -k} Rj_{k*}(F^\bullet)$.

U_k is union of strata having dimension at least k . This, in view of B.3.2, means that $\mathcal{H}^r(F^\bullet) = 0$ for $r > -k$. Thus $j_k^* \tau^{\leq -k} Rj_{k*}(F^\bullet) \cong \tau^{\leq -k} j_k^* Rj_{k*}(F^\bullet) \cong \tau^{\leq -k} F^\bullet \cong F^\bullet$ and the condition i) holds.

Now we set $Z = U_{k-1} \setminus U_k = \bigsqcup_{\dim X_\alpha = k-1} X_\alpha$ and $i : Z \hookrightarrow U_{k-1}$ the embedding. $i^* G^\bullet$, has locally constant cohomology sheaves on the $(k-1)$ -dimensional strata of Z , and from the definition of G^\bullet we have that $\mathcal{H}^r(i^* G^\bullet) = 0$ for $r > -k$. So we can apply B.3.2 to deduce that $i^* G^\bullet \in {}^p \mathcal{D}_c^{\leq -1}$.

Let's now proof that condition iii) holds. In $\mathcal{D}_c(U_{k-1})$ we have the following distinguished triangle,

$$G^\bullet = \tau^{-k} Rj_{k*} F^\bullet \rightarrow Rj_{k*} F^\bullet \rightarrow \tau^{\geq -k+1} Rj_{k*} F^\bullet \xrightarrow{+1}$$

This triangle comes from a short exact sequence of complexes of sheaves. $i^! = i^* R\Gamma_Z$ is exact on injective sheaves, so it give rises to the triangle

$$i^! G^\bullet \rightarrow i^! Rj_{k*} F^\bullet \rightarrow i^! \tau^{\geq -k+1} Rj_{k*} F^\bullet \xrightarrow{+1}$$

But $i^! Rj_{k*} F^\bullet = 0$, hence $i^! G^\bullet \cong i^! \tau^{\geq -k+1} Rj_{k*} F^\bullet[-1]$. In particular this means that $\mathcal{H}^r(i^! G^\bullet) = 0$ for $r \leq -k+1$ and that $i^! G^\bullet$ has locally constant cohomology sheaves on each X_α . Thus we can apply Proposition B.3.2 to obtain $i^! G^\bullet \in {}^p \mathcal{D}_c^{\geq 1}(Z)$ \square

B.5 Intersection Cohomology

Definition B.5.1. Let X an irreducible complex algebraic variety of dimension d . We define the *Intersection Cohomology Complex* $IC(X) \in \text{Perv}(\mathbb{C}_X)$ as

$$IC(X) = {}^p j_{!*}(\mathbb{C}_{X_{\text{reg}}}[d])$$

where X_{reg} is the regular part of X . We also define

$$IH^i(X) = H^i(IC(X)[-d]) = R^i\Gamma(X, IC(X)[-d])$$

the i th *Intersection Cohomology Group* of X and $IH_c^i(X) = \mathbb{H}_c^i(IC(X)[-d]) = R^i\Gamma_c(X, IC(X)[-d])$ the i th *Intersection Cohomology Group with compact supports* of X .

More generally, for a local system L on X_{reg} we define

$$IC_X(L) = {}^p j_{!*}(L[d])$$

and call it a *Twisted Intersection Cohomology Complex* of X .

Theorem B.5.2 (Poincaré Duality for Intersection Cohomology). *Let X an irreducible complex algebraic variety of dimension d . Then we have*

$$IH^i(X) \cong (IH_c^{2d-i}(X))^\vee$$

for any $0 \leq i \leq 2d$,

Proof. First we notice that $\mathbb{D}_X(IC(X)) \cong IC(X)$. In fact, this is an immediate consequence of Lemma B.4.3, since

$$\mathbb{D}_X({}^p j_{!*}(\mathbb{C}_{X_{\text{reg}}})) \cong {}^p j_{!*}(\mathbb{D}_{X_{\text{reg}}}(\mathbb{C}_{X_{\text{reg}}})) \cong {}^p j_{!*}(\mathbb{C}_{X_{\text{reg}}}^\vee) \cong {}^p j_{!*}(\mathbb{C}_{X_{\text{reg}}})$$

Let $p_X : X \rightarrow \{\text{pt}\}$. By the Poincaré-Verdier Duality we get an isomorphism

$$\begin{aligned} R\mathcal{H}om(Rp_{X!}IC(X), \mathbb{C}) &\cong Rp_{X*}R\mathcal{H}om(IC(X), \omega_X) = \\ &= Rp_{X*}\mathbb{D}_X(IC(X)) \cong Rp_{X*}(IC(X)) \end{aligned}$$

This gives an isomorphism

$$(R\Gamma_c(X, IC(X)))^\vee \cong R\Gamma(X, IC(X))$$

and by taking the $(i - d)$ th cohomology groups of both sides we get the thesis \square

Remark B.5.3. For $IC(X)$ we have stricter support condition than a general perverse sheaf.

Let $U = X_{\text{reg}}$ and $Z = X \setminus U$. As a consequence of Theorem B.4.5, for $j \neq -d$, $\mathcal{H}^{-j}(IC(X))$ is supported on Z . Since $i^*IC(X) \in {}^p\mathcal{D}_c^{\leq -1}(Z)$, we have

$$\dim(\text{supp}\mathcal{H}^{-j}(IC(X))) < j \quad \forall j \neq -d$$

We know, from Prop. B.4.9, that if $L[d]$ is a simple object in $\text{Perv}(\mathbb{C}_{X_{\text{reg}}})$, then $IC_X(L)$ is simple as a perverse sheaf. Conversely, since ${}^p j_{!*}$ preserves monomorphism and epimorphism, we see that $IC_X(L)$ is simple only if $L[d]$ is simple. Actually, any simple perverse sheaf is of this kind.

Proposition B.5.4. *Every perverse sheaf has a finite composition series made of twisted intersection cohomology complexes $IC_Y(L)$, where Y is an irreducible closed subvariety of X and L is an irreducible local system on the smooth part of Y .*

In particular, the simple object in $\text{Perv}(\mathbb{C}_X)$ are exactly the objects $IC_Y(L)$.

Proof. Let $F \in \text{Perv}(\mathbb{C}_X)$. We can assume, by induction on the dimension of the support of F , that $\text{supp}(F) = X$. There exists a Zariski open smooth dense set U such that F has locally constant cohomology sheaves on U , hence $F|_U \cong L[d]$, for a local system L on U . Let $j : U \hookrightarrow X$ and $i : Z = X \setminus U$ the embeddings. From the adjunction triangles we get the following exact sequences:

$$0 \rightarrow i_* {}^p H^0(i^! F) \rightarrow F \rightarrow {}^p j_*(j^* F) \rightarrow i_* {}^p H^1(i^! F) \rightarrow 0 \quad (\text{B.3})$$

$$0 \rightarrow i_* {}^p H^{-1}(i^* F) \rightarrow {}^p j_!(j^* F) \rightarrow F \rightarrow i_* {}^p H^0(i^* F) \rightarrow 0 \quad (\text{B.4})$$

If F is simple and supported on X then ${}^p H^0(i^! F)$ and ${}^p H^0(i^* F)$ have to be 0. This means that the canonical functor ${}^p j_!(j^* F) \rightarrow {}^p j_*(j^* F)$ factorize through ${}^p j_!(j^* F) \rightarrow F \hookrightarrow {}^p j_*(j^* F)$, so $F \cong {}^p j_!(j^* F) = {}^p j_{!*}(L[d])$.

Now we claim that $L[d]$ is simple, as a perverse sheaf on U , if and only if L is an irreducible. This will imply the second statement. One direction is obvious. Let assume that L is irreducible and let $0 \rightarrow G \rightarrow L[d] \rightarrow H \rightarrow 0$ be an exact sequence in $\text{Perv}(\mathbb{C}_U)$. We can find a Zariski open set $V \subseteq U$ such that $G|_V \cong M[d]$ and $H|_V \cong N[d]$, where M and N are local system on V . Let's denote by $\bar{j} : V \hookrightarrow U$ the inclusion. Since $M \subseteq L|_V$, $\bar{j}_* M$ is still a local system of the same rank of M . From the irreducibility of L we get that $j_* M$ is 0 or L . This is equivalent to say that M is 0 or N is 0. But if M is 0 then G is supported on $U \setminus V$, but $L[d] \cong \bar{j}_{!*}(L|_V[d])$ has no subobject supported on Z . Similarly if $N = 0$. This proves our claim.

Let's now conclude the proof of the proposition. From B.3 and the induction hypothesis, F has finite length (i.e. has a finite composition series) if and only if ${}^p j_*(L[d])$ does. $L[d]$ has of finite length, since clearly each local system has finite length. The functor ${}^p j_*$ is left exact, so we can assume that L is simple. Otherwise we have an exact sequence $0 \rightarrow L_1 \rightarrow L \rightarrow L_2 \rightarrow 0$, thence ${}^p j_* L_1 \rightarrow {}^p j_* L \rightarrow {}^p j_* L_2$ and we could conclude by induction on the length. Finally, if L is simple, from the adjunction triangle for $Rj_* L[d]$ we obtain the exact sequence

$$0 \rightarrow j_{!*} L[d] \rightarrow {}^p j_* L[d] \rightarrow i_* {}^p H^0(i^* Rj_* L[d]) \rightarrow 0$$

from which ${}^p j_* L[d]$ has finite length. □

Corollary B.5.5. *The category of perverse sheaves is artinian and noetherian.*

B.5.1 Examples

Clearly, the Intersection Cohomology coincides with the Singular Cohomology for smooth variety. In the simples non-trivial case we have the following

Proposition B.5.6. *Let X be a projective variety with isolated singular points. Then*

$$IH^i(X) = \begin{cases} H^i(X_{reg}) & \text{if } 0 \leq i < d \\ \text{Im}(H^d(X) \rightarrow H^d(X_{reg})) & \text{if } i = d \\ H^i(X_{reg}) & \text{if } d < i \leq 2d \end{cases}$$

However, we need a Lemma to be able to prove this

Lemma B.5.7. *There exist canonical morphisms*

$$\mathbb{C}_X \rightarrow IC(X)[-d] \rightarrow \omega_X[-2d]$$

Proof. We use the description given by Prop. B.4.10. We notice that, since Rj_* is left exact, for a complex $F^\bullet \in \mathcal{D}_c^{\geq 0}$ we have $\tau^{\leq 0} \circ Rj_* F^\bullet \cong j_* \circ \tau^{\leq 0} F^\bullet$, where j_* means that we are just applying the functor j_* to the single sheaf $\tau^{\leq 0} F^\bullet \cong \mathcal{H}^0(F^\bullet)$ and regarding the result as a complex concentrated in degree 0. In this way we obtain

$$\tau^{-\leq dp} j_{1*}(L[d]) \cong (j_{1*} \circ j_{2*} \circ \dots \circ j_{d*})(L)[d] \cong (j_* L)[d] \quad (\text{B.5})$$

This means that $\tau^{-\leq d} IC(X) \cong (j_* \mathbb{C}_{X_{reg}})[d] \cong \mathbb{C}_X[d]$ and clearly we get a canonical morphism $\mathbb{C}_X \rightarrow IC(X)[-d]$. Taking the Verdier dual we obtain the the morphism $IC(X)[-d] \rightarrow \omega_X[-2d]$. \square

Proof of the Proposition. Let $U = X_{reg}$ and $j : U \hookrightarrow X$. We call p_1, \dots, p_k the singular points of X . Then $X = U \sqcup \{p_1\} \sqcup \dots \sqcup \{p_k\}$ is a Whitney stratification of X . From Prop. B.4.10 we obtain $IC(X)[d] \cong \tau^{\leq d-1} Rj_* \mathbb{C}_U$. This gives a distinguished triangle

$$IC(X)[-d] \rightarrow Rj_* \mathbb{C}_U \rightarrow \tau^{\geq d}(Rj_* \mathbb{C}_U) \xrightarrow{+1}$$

whence $IH^i(X) \cong H^i(U)$ for any $0 \leq i < d$ while for $i = d$ we have that the canonical morphism $IH^d(X) \rightarrow H^d(U)$ is injective. Furthermore, we can embed the canonical morphism $\mathbb{C}_X \rightarrow IC(X)[-d]$ into the distinguished triangle

$$\mathbb{C}_X \rightarrow IC(X)[-d] \rightarrow F^\bullet \xrightarrow{+1}$$

where $F^\bullet \cong \tau^{\geq 1} \tau^{\leq d-1}(Rj_* \mathbb{C}_U) \cong \tau^{\leq d-1} \tau^{\geq 1}(Rj_* \mathbb{C}_U)$. The triangle

$$\mathbb{C}_X = \tau^{\leq 0}(Rj_* \mathbb{C}_U) \rightarrow (Rj_* \mathbb{C}_U) \rightarrow \tau^{\geq 1}(Rj_* \mathbb{C}_U) \xrightarrow{+1}$$

is isomorphic to the adjunction triangle

$$\mathbb{C}_X \rightarrow Rj_*(j^* \mathbb{C}_X) \rightarrow i_* i^! \mathbb{C}_X[1] \xrightarrow{+1}$$

hence $F^\bullet \cong \tau^{\leq d-1}(i_* i^! \mathbb{C}_X[1])$ is supported on the singular points of X and $H^i(F^\bullet) = 0$ for any $i \geq d$. This implies that $H^i(X) \cong IH^i(X)$ for any $i < d$ and that the map $H^d(X) \rightarrow IH^d(X)$ is surjective. This completes the proof. \square

Example B.5.8. We will now give a counterexample for the exactness of the minimal extension functor. Let $X = \mathbb{C}$ and $U = \mathbb{C}^*$. We can consider on U the local system E of rank 2 defined by the monodromy matrix around the origin

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

So we have the short exact sequence of local systems on U

$$0 \rightarrow \mathbb{C}_U \rightarrow E \rightarrow \mathbb{C}_U \rightarrow 0$$

Clearly we have $j_{!*}\mathbb{C}_U[1] = IC(\mathbb{C}) = \mathbb{C}_X[1]$. However

$$j_{!*}E[1] = IC_X(E) = \tau^{\leq -1}Rj_*E[1] = (j_*E)[1]$$

and the stalk in 0 is given by monodromy invariant section in a neighborhood, so it has dimension 1. This shows that the sequence $0 \rightarrow IC(X) \rightarrow IC_X(E) \rightarrow IC(X) \rightarrow 0$ is not exact in 0.

Appendix C

A Brief Introduction to Mixed Hodge Module

C.1 Pure Hodge Structures

Let V a finite dimensional real vector space and let $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ be its complexification

Definition C.1.1. A *real Hodge structure* on V is a direct sum decomposition

$$V_{\mathbb{C}} = \bigoplus_{p,q \in \mathbb{Z}} V^{p,q}$$

such that $V^{p,q} = \overline{V^{q,p}}$. This decomposition is called the *Hodge decomposition*. A morphism of Hodge structures is a real linear map $f : V \rightarrow W$ such that its complexification $f_{\mathbb{C}}$ preserves types, i.e. $f_{\mathbb{C}}(V^{p,q}) \subseteq W^{p,q}$.

The numbers $h^{p,q} = \dim V^{p,q}$ are called *Hodge numbers* of the decomposition.

Let V a Hodge structure. We say that the *weight k part* $V^{(k)}$ of V is the real vector space $\bigoplus_{p+q=k} V^{p,q}$. If $V = V^{(k)}$ we say that V has a Hodge structure of weight k .

Example C.1.2. Let X a Kähler compact manifold. The Hodge Theorem holds for X [GH94, §0.7], and we have a the classical Hodge decomposition of the cohomology of X .

$$H^k(X, \mathbb{C}_X) = H_{DR}^k(X) \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p,q} H^{p,q}(X) \quad \text{with} \quad H^{p,q} = \overline{H^{q,p}}$$

Hence $H^{\bullet}(X, \mathbb{C}_X)$ has a Hodge structure and the k th cohomology group has a Hodge structure of weight k . A morphism $f : X \rightarrow Y$ of Kähler compact manifolds induces a morphism of Hodge structures $f : H^{\bullet}(Y) \rightarrow H^{\bullet}(X)$.

We can give an equivalent and more convenient definition of Hodge structure in terms of a filtration of real vector spaces.

Definition C.1.3. A *Hodge structure of weight k* on V is a decreasing filtration of $V_{\mathbb{C}}$

$$V_{\mathbb{C}} = F^0(V) \supseteq F^1(V) \supseteq F^2(V) \supseteq \dots$$

such that $F^p(V) \cap \overline{F^q(V)} = 0$ and $F^p(V) \oplus \overline{F^q(V)} = 0$ when $p + q \geq k + 1$. The filtration is called the *Hodge filtration*.

$f : V \rightarrow W$ is a morphism of Hodge structures if and only if $f(F^p(V)) \subseteq F^p(W)$ for any $p \in \mathbb{N}$.

Hodge filtrations and the Hodge decompositions are linked as follows

$$F^p(V) = \bigoplus_{r \geq p} V^{r,s} \quad V^{p,q} = F^p(V) \cap \overline{F^q(V)}$$

So it's easy to pass from one definition of Hodge structure to the other.

Example C.1.4. Let V and W be real vector space with Hodge structure respectively of weight k and l . We can define a Hodge filtration, hence a Hodge structure, on $V \otimes W$ and on $\text{Hom}(V, W)$ by

$$F^p(V \otimes W) = \sum_s F^m(V) \otimes_{\mathbb{R}} F^{p-m}(W)$$

$$F^p \text{Hom}(V, W) = \{f : V_{\mathbb{C}} \rightarrow W_{\mathbb{C}} \mid f(F^s(V)) \subseteq F^{s+p}(W) \forall s\}$$

This gives a Hodge structure of weight $k + l$ on $V \otimes W$ and of weight $k - l$ on $\text{Hom}(V, W)$.

Definition C.1.5. A Hodge structure V of weight k is said to be *polarizable* if there exists a real bilinear form $Q : V_{\mathbb{C}} \times V_{\mathbb{C}} \rightarrow \mathbb{C}$ such that

- $Q(u, v) = (-1)^k Q(v, u) \quad \forall u, v \in V_{\mathbb{C}}$;
- $Q(V^{p,q}, V^{p',q'}) = 0$ if $(p, q) \neq (q', p')$ (or, equivalently, $Q(F^p(V), F^{k+1-p}(V)) = 0$ for any p);
- The hermitean form $i^{p-q} Q(u, \bar{v})$ is positive definite on $V^{p,q}$.

Such a form Q is called a *polarization*.

For example, if X is compact Kähler manifold the Hodge-Riemann bilinear relations (cfr. [GH94, §0.7] give a polarization of each cohomology group $H^k(X)$.

C.2 Mixed Hodge Structures

Definition C.2.1. Let V a finite dimensional real vector space. A *Mixed Hodge Structure* on V is consists of two filtrations

- An increasing filtration of V , called the *Weight filtration* W_{\bullet} .

- A decreasing filtration of $V_{\mathbb{C}}$, called the *Hodge filtration* F^{\bullet}

such that the F^{\bullet} induces on $\text{Gr}_k^W(V) = W_k(V)/W_{k-1}(V)$ a pure Hodge structure of weight k .

If each $\text{Gr}_k^W(V)$ is a polarizable Hodge structure, we say that V has a polarizable mixed Hodge structure.

A morphism of mixed Hodge structure is a morphism $f : V \rightarrow W$ compatible with the two filtrations. For every $k \in \mathbb{Z}$ it induces a morphism of pure weight structures $\text{Gr}_k^W(f) : \text{Gr}_k^W(V) \rightarrow \text{Gr}_k^W(W)$.

Example C.2.2. If V and W are mixed Hodge structures, then $V \otimes W$ and $\text{Hom}(V, W)$ have natural mixed Hodge structures. Both the filtrations on these spaces are defined as in the pure case.

We have the following fundamental lemmas about mixed structures.

Lemma C.2.3. *Let $f : V \rightarrow W$ a morphism of mixed Hodge modules. If f is an isomorphism as a vector space, then f is also an isomorphism as mixed Hodge structures.*

Lemma C.2.4. *Let $V \hookrightarrow W$ be an injective morphism of mixed Hodge structures. Then there exists a unique mixed Hodge structure on the quotient vector space W/V such that the quotient map $W \rightarrow W/V$ is a morphism of mixed Hodge structures.*

Furthermore, the category of mixed Hodge structures is abelian.

Lemma C.2.5. *Let $V \rightarrow W \rightarrow Z$ an exact sequence of mixed Hodge structures. Then the sequences*

$$\begin{aligned} \text{Gr}_k^W(V) &\rightarrow \text{Gr}_k^W(W) \rightarrow \text{Gr}_k^W(Z) \\ \text{Gr}_k^F(V) &\rightarrow \text{Gr}_k^F(W) \rightarrow \text{Gr}_k^F(Z) \\ \text{Gr}_l^F \text{Gr}_k^W(V) &\rightarrow \text{Gr}_l^F \text{Gr}_k^W(W) \rightarrow \text{Gr}_l^F \text{Gr}_k^W(Z) \end{aligned}$$

are also exact, for any k, l

Example C.2.6 (PS, 5.33). Let X a complex algebraic variety, not necessarily neither smooth nor compact. Then we can construct a mixed Hodge structure on the cohomology of X . This construction is functorial: if $f : X \rightarrow Y$ is a morphism of complex algebraic varieties, then the induced morphism on cohomology is a morphism of mixed Hodge structures.

The numbers

$$h^{p,q} = \dim_{\mathbb{C}} \text{Gr}_F^p \text{Gr}_{p+q}^W(V_{\mathbb{C}})$$

are the *Hodge numbers* of the mixed Hodge structures. We have some restrictions on the hodge numbers of a complex algebraic variety.

Proposition C.2.7 (PS, 5.39). *Let X a complex algebraic variety of dimension n . We assume that the Hodge number $h^{p,q}$ of $H^k(X)$ is nonzero. Then*

- i) $0 \leq p, q \leq k$*

ii) If $k > n$, then $k - n \leq p, q \leq n$

iii) If X is smooth, then $p + q \geq k$

iv) If X is compact, then $p + q \leq k$

Definition C.2.8. Let (V, W, F) be a mixed Hodge structure. We say that the weight m occurs in the structure if $\text{Gr}_m^W \neq 0$. We say that it is *pure of weight m* if m is the only occurring weight.

In the language of weights, the last proposition means that

- all the weights are $\leq 2k$
- If $k > n$, then all the weights are $\geq 2k - 2n$
- If X is smooth, then all the weights are $\leq k$
- If X is compact, then all the weights are $\geq k$

C.3 Mixed Hodge Modules: an Axiomatic Approach

In his paper [Sai90] Saito introduced mixed Hodge modules. The definition of mixed Hodge modules is very hard and difficult to use. Discussing it is beyond the purposes of this thesis. However some results in Chapter 3 and 4 lie on the theory of mixed Hodge modules. So we will follow a pragmatic approach, stating the axioms that mixed Hodge modules respect and from which we can recover the required properties.

Axiom 1. For any complex algebraic variety X there exists an abelian category $\text{MHM}(X)$, called the category of *mixed Hodge modules* on X such that there exists a faithful functor

$$\text{rat} : \mathcal{D}^b\text{MHM}(X) \longrightarrow \mathcal{D}_c^b(X)$$

Under this functor the subcategory $\text{MHM}(X)$ corresponds to $\text{Perv}(X)$. For $M \in \text{MHM}(X)$, $\text{rat}(M)$ is the underlying perverse sheaf of M .

We denote by $\mathcal{H}^j(M^\bullet)$, $j \in \mathbb{Z}$ the cohomology groups of an object $M^\bullet \in \mathcal{D}^b\text{MHM}(X)$. Since $\text{MHM}(X)$ is an abelian category the cohomology groups are still in $\text{MHM}(X)$. We notice that this first axiom implies

$$\text{rat}(\mathcal{H}^q(M)) = {}^p H^q(\text{rat}(M)) \quad \forall q \in \mathbb{Z}$$

Axiom 2. If X is a single point, then the category of mixed Hodge modules is the category of polarizable mixed Hodge structures. In this case, for a mixed Hodge structure M , $\text{rat}(M)$ is the underlying vector space.

Axiom 3. Every object $M \in \text{MHM}(X)$ has a weight filtration W such that

- each morphism of mixed Hodge modules preserve the weight filtration strictly
- $\text{Gr}_k^W(M)$ is semisimple in $\text{MHM}(X)$ for any k
- If X is a single points, then the weight filtration is the weight filtration for mixed Hodge structures

A morphism of filtered mixed Hodge modules f is strict means the induced morphism $\text{Coim}(f) \rightarrow \text{Im}(f)$ is an isomorphism of filtered mixed Hodge modules. From strictness we can deduce that the functors \mathcal{H}^j and Gr_i^W commute (cfr. [PS08, A.34], i.e.

$$\text{Gr}_i^W \mathcal{H}^j(M^\bullet) = \mathcal{H}^j \text{Gr}_i^W(M^\bullet) \quad \forall i, j \in \mathbb{Z} \quad \forall M^\bullet \in \mathcal{D}^b\text{MHM}(X)$$

We say that $M \in \mathcal{D}^b(X)$ has weights $\leq n$ if $\text{Gr}_i^W \mathcal{H}^j(M^\bullet) = 0$ for any $i > j + n$. We say that it has weights $\geq n$ if $\text{Gr}_i^W \mathcal{H}^j(M^\bullet) = 0$ for any $i < j + n$. We say that M^\bullet is *pure of weight n* if it has both weights $\geq n$ and $\leq n$.

Axiom 4. There exists a duality functor \mathbb{D}_X which lifts the Verdier duality from $\mathcal{D}_c^b(X)$ to $\mathcal{D}^b\text{MHM}(X)$, i.e. $\mathbb{D}_X \circ \text{rat} = \text{rat} \circ \mathbb{D}_X$.

Axiom 5. For any morphism of complex algebraic varieties $f : X \rightarrow Y$ there exist functors

$$\begin{aligned} Rf_*, Rf_! : \mathcal{D}^b\text{MHM}(X) &\rightarrow \mathcal{D}^b\text{MHM}(Y) \\ f^*, f^! : \mathcal{D}^b\text{MHM}(Y) &\rightarrow \mathcal{D}^b\text{MHM}(X) \end{aligned}$$

which lift the analogous functors between constructible complexes of sheaves. Furthermore they are interchanged under \mathbb{D}_X , that is

$$Rf_* \circ \mathbb{D}_X = \mathbb{D}_X \circ f_! \quad f_! \circ \mathbb{D}_X = \mathbb{D}_X \circ f^!$$

Axiom 6. The functors $Rf_!$ and f^* decrease the weights, that is if M^\bullet has weights $\leq n$, the same holds for $Rf_!M^\bullet$ and f^*M^\bullet in

The functors Rf_* and $f^!$ decrease the weights, that is if M^\bullet has weights $\leq n$, the same holds for Rf_*M^\bullet and $f^!M^\bullet$.

Axiom 7. If $M^\bullet \in \mathcal{D}^b\text{MHM}(X)$ has weights $\geq n$, then $\mathbb{D}_X M^\bullet$ has weights $\leq -n$.

C.3.1 Homomorphisms between Mixed Hodge Modules

Lemma C.3.1. *Let M^\bullet is a bounded complex of objects in $\text{MHM}(X)$ which has weights ≥ 0 . Then there exists another bounded complex \overline{M}^\bullet and a surjective quasi-isomorphism $M^\bullet \rightarrow \overline{M}^\bullet$ such that $\text{Gr}_p^W \overline{M}^q = 0$ for $q > p$.*

Analogously, if M^\bullet is a bounded complex of objects in $\text{MHM}(X)$ which has weights ≤ 0 there exists a bounded complex \overline{M}^\bullet and an injective quasi-isomorphism $M^\bullet \rightarrow \overline{M}^\bullet$ such that $\text{Gr}_p^W \overline{M}^q = 0$ for $q < p$.

Proof. We will prove only the first part, the second being similar. Firstly we notice that the cohomology of $\mathrm{Gr}_p^W(M^\bullet)$ vanish in degrees $> p$. Further we know that $\mathrm{Gr}_p^W(M^p)$ is semisimple, so there exists a decomposition $\mathrm{Gr}_p^W(M^p) = Z^q \oplus C^p$, where $Z^p = \mathrm{Ker}(d : \mathrm{Gr}_p^W(M^p) \rightarrow \mathrm{Gr}_p^W(M^{p+1}))$. We can define the complex C_p^\bullet as:

$$C_p^q = \begin{cases} 0 & \text{if } q < p \\ C^p & \text{if } q = p \\ \mathrm{Gr}_p^W(M^q) & \text{if } q > p \end{cases}$$

C_p^\bullet is an acyclic complex. Let p_0 the smallest integer such that $W_{p_0}M^\bullet \neq 0$. In this case $C_{p_0}^\bullet$ is a subcomplex of $W_{p_0}M^\bullet/W_{p_0-1}M^\bullet = W_{p_0}M^\bullet$, hence of M^\bullet . We can take the quotient $M^\bullet/C_{p_0}^\bullet$ and from now on we will call it M^\bullet . By construction we have that $W_{p_0}M^q = 0$ for $q > p_0$.

Now we consider the subcomplex $C_{p_0+1}^\bullet$ of $\mathrm{Gr}_{p_0+1}^W M^\bullet$. $C_{p_0+1}^q$ is nonzero only for $q \geq p_0 + 1$ and in this case we have $\mathrm{Gr}_{p_0+1}^W M^q = W_{p_0}M^q$. Therefore we can regard $C_{p_0+1}^\bullet$ as a subcomplex of M^\bullet . We can again take the quotient $M^\bullet/C_{p_0+1}^\bullet$ to obtain a complex, which we rename M^\bullet , such that $\mathrm{Gr}_{p_0+1}^W M^q = 0$ for $q > p_0 + 1$. We can reiterate this procedure until we get a complex \overline{M}^\bullet with the property $\mathrm{Gr}_p^W \overline{M}^q = 0$ for $q > p$. \square

Proposition C.3.2. *Let $M^\bullet, N^\bullet \in \mathcal{D}^b \mathrm{MHM}(X)$ and $n \in \mathbb{Z}$ such that M^\bullet has weights $\leq n$ and N^\bullet has weights $\geq n + p + 1$. Then*

$$\mathrm{Ext}_{\mathcal{D}^b \mathrm{MHM}(X)}^p(M^\bullet, N^\bullet) = \mathrm{Hom}_{\mathcal{D}^b \mathrm{MHM}(X)}(M^\bullet, N^\bullet[p]) = 0$$

Proof. By shifting, we can easily reduce to the case $p = 0$, $n = -1$. Let assume that M^\bullet and N^\bullet are representatives in the derived category such that there exists a nontrivial morphism (of complexes) $f : M^\bullet \rightarrow N^\bullet$. We can now apply the Lemma C.3.1 taking \overline{M}^\bullet and \overline{N}^\bullet (more precisely $\overline{M}^\bullet = \overline{M^\bullet[1]}[-1]$). f induces the morphism \overline{f}

$$\overline{M}^\bullet \xrightarrow{i} M^\bullet \xrightarrow{f} N^\bullet \xrightarrow{\pi} \overline{N}^\bullet \quad \overline{f} = \pi \circ f \circ i$$

If we show that $\overline{f} = 0$, we would get a contradiction.

We have $\mathrm{Gr}_p^W \overline{M}[1]^q = 0$ for any $p > q$, hence

$$W_q \overline{M}[1]^q = W_{q+1} \overline{M}[1]^q = \dots = \overline{M}[1]^q$$

or, equivalently,

$$W_{q-1} \overline{M}^q = W_q \overline{M}^q = \dots = \overline{M}^q$$

On the other hand we have $\mathrm{Gr}_p^W \overline{N}^q = 0$ for any $p < q$, hence

$$W_{q-1} \overline{N}^q = W_{q-2} \overline{N}^q = \dots = 0$$

The morphism \overline{f} must factorize through $\overline{M}^q = W_{q-1} \overline{M}^q \rightarrow W_{q-1} \overline{N}^q \hookrightarrow \overline{N}^\bullet$, so it is 0. \square

Corollary C.3.3. *If M^\bullet is pure of weight n , there is a non-canonical isomorphism*

$$M^\bullet \cong \bigoplus_{i \in \mathbb{Z}} \mathcal{H}^p(M^\bullet)[-p]$$

Proof. If M^\bullet is pure of weight n , then $\tau^{\leq p} M^\bullet$ and $\mathcal{H}^p(M^\bullet)[-p]$ are pure of weight n as well. Thus $\tau^{\leq p-1} M^\bullet[1]$ is of weight $n+1$ and, applying Prop. C.3.2, we get

$$\mathrm{Ext}^1(\mathcal{H}^p(M^\bullet)[-p], \tau^{\leq p-1} M^\bullet) = 0$$

On the other hand we have

$$\tau^{\leq p} M^\bullet / \tau^{\leq p-1} M^\bullet = [\dots 0 \rightarrow M^{p-1} / \mathrm{Ker}(d^{p-1}) \rightarrow \mathrm{Ker}(d^{p-1}) \rightarrow 0 \rightarrow \dots]$$

and this is quasi-isomorphic to $\mathcal{H}^p(M^\bullet)[-p]$. This means that the exact sequence

$$0 \rightarrow \tau^{\leq p-1} M^\bullet \rightarrow \tau^{\leq p} M^\bullet \rightarrow \mathcal{H}^p(M^\bullet)[-p] \rightarrow 0$$

splits as $\tau^{\leq p} M^\bullet \cong \tau^{\leq p-1} M^\bullet \oplus \mathcal{H}^p(M^\bullet)[-p]$. Now we can easily conclude. \square

C.3.2 Purity of Intersection Cohomology and Decomposition Theorem

We need an Hodge theoretic version of Intersection Cohomology. We start by defining $\mathbb{C}^H \in \mathrm{MHM}(\mathrm{pt})$ as the pure Hodge structure of type $(0,0)$ on the point. In general we define

$$\mathbb{C}_X^H = p_X^* \mathbb{C}^H$$

where p_X is the unique map sending X to a point.

Similarly to the complexes of sheaves situation, we can define, for an open embedding $j : U \hookrightarrow X$ of complex algebraic varieties, the minimal extension functor $j_{!*}$.

$$j_{!*} M^\bullet = \mathrm{Im}(\mathcal{H}^0 Rj_! M^\bullet \rightarrow \mathcal{H}^0 Rj_* M^\bullet)$$

Thus we define the Hodge theoretic version of Intersection Cohomology as

$$IC^H(X) = j_{!*}(\mathbb{C}_{X^{\mathrm{reg}}}^H)[\dim X]$$

where X^{reg} is the smooth part of X and j is the embedding. We have $\mathrm{rat}(IC^H(X)) = IC(X)$ and it restricts to $\mathbb{C}_{X^{\mathrm{reg}}}[\dim X]$ on X^{reg} .

Proposition C.3.4 (Sa, Pag. 325). *We have*

$$Gr_d^W \mathcal{H}^d(\mathbb{C}_X^H) = IC^H(X)$$

where d is the dimension of X . In particular $IC^H(X)$ is a pure mixed Hodge module of weight d .

Proof. We can see that there is an isomorphism restricting on $U = X^{\text{reg}}$. To show that it is an isomorphism we will prove that $\text{Gr}_d^W \mathcal{H}^d(\mathbb{C}_X^H)$ is the unique object in $\text{MHM}(X)$ such that its restriction to U is $\mathbb{C}_U^H[d]$ and which has no trivial subobject or subquotient supported on $Z = X \setminus U$. Using Axiom 3 we know that it is semisimple, thus it suffices to show that it has no nontrivial quotient supported on Z .

Let $M \in \text{MHM}(X)$ supported on Z and let $i : Z \hookrightarrow X$ the inclusion. We have

$$\text{Hom}(\mathcal{H}^n(\mathbb{C}_X^H), M) = \text{Hom}(\mathcal{H}^n(\mathbb{C}_X^H), i_* i^* M) = \text{Hom}(\mathcal{H}^n(\mathbb{C}_Z^H), i^* M)$$

We have that ${}^p H^k(\mathbb{C}_X[d]) = 0$ for $k > 0$ and, since rat is faithful, we have also that $\mathcal{H}^k(\mathbb{C}_X) = 0$ for $k > n$. In the same way, since $n > \dim Z$, we get $\mathcal{H}^n(\mathbb{C}_Z^H) = 0$, hence $\text{Hom}(\mathcal{H}^n(\mathbb{C}_X^H), M) = 0$

Furthermore $\mathbb{C}_X^H = p_X^* \mathbb{C}^H$ has weights ≤ 0 and this yields $\text{Gr}_k^W \mathcal{H}^n(\mathbb{C}_X^H)$ to be 0 for $k > n$. Thus $\text{Gr}_n^W \mathcal{H}^n(\mathbb{C}_X^H)$ is a quotient of $\mathcal{H}^n(\mathbb{C}_X^H)$, so also $\text{Hom}(\text{Gr}_n^W \mathcal{H}^n(\mathbb{C}_X^H), M)$ is 0 for any M supported on Z . \square

Corollary C.3.5. *Let X a compact complex algebraic variety. Then the intersection cohomology group $IH^k(X)$ has a pure Hodge structure of weight k*

Proof. In this case the functor $Rp_{X!} = Rp_{X*}$ both increases and decreases the weights. So it sends pure complexes into pure Hodge structures. \square

Theorem C.3.6 (Decomposition Theorem). *Let $f : X \rightarrow Y$ be a proper morphism of complex algebraic varieties. Then*

$$Rf_* IC(X) \cong \bigoplus_{i \in \mathbb{Z}} {}^p H^i(Rf_* IC(X))[-i]$$

Furthermore each summand ${}^p H^i(Rf_* IC(X))[-i]$ is semisimple and there is a finite collection of pairs (S_β, L_β) , where S_β is a locally closed subvariety of Y and L_β is a semisimple local system on S_β , such that

$${}^p H^i(Rf_* IC(X))[-i] \cong \bigoplus_{\beta} IC_{\overline{S_\beta}}(L_\beta)$$

Putting together these two parts we have

$$Rf_* IC(X) \cong \bigoplus_{\beta, i} IC_{\overline{S_{\beta, i}}}(L_{\beta, i})[-i] \tag{C.1}$$

Proof. Since f is proper and $IC^H(X)$ is pure, the first part follows immediately from Corollary C.3.3 after applying rat to both sides.

Furthermore $\mathcal{H}^i(Rf_* IC^H(X))[-i]$ is a pure mixed Hodge module, so it is semisimple from Axiom 3. Applying the functor rat we obtain ${}^p H^i(Rf_* IC(X))[-i]$ which is still semisimple (as a perverse sheaf) and we can conclude using the fact that Intersection Cohomology of simple local system are the unique simple object in the category of perverse sheaves. \square

Corollary C.3.7. *Let $f : \tilde{X} \rightarrow X$ be a proper resolution of singularities of a projective variety X . Then $IH^i(X)$ is a direct summand of $H^i(\tilde{X})$ for any $i \in \mathbb{Z}$.*

Proof. We can restrict the decomposition C.1 to the regular part $U = X_{\text{reg}}$ of X . $Rf_*IC(\tilde{X})|_U = Rf_*\mathbb{C}_{\tilde{X}}[d]|_U \cong \mathbb{C}_U[d]$ is a simple object in $\text{Perv}(\mathbb{C}_U)$. Thus only one term of the right hand side of decomposition can survive and this has to be $\mathbb{C}_U = IC(X)|_U$.

Hence the summand $IC(X)$ appears in the decomposition. We obtain the desired result by taking the (global) cohomology of both sides. \square

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