Lie Groups
SoSe 2020 — Ubungsblatt 3

Ausgabe 25.05.20, **Abgabe** 09.06.19

Solutions are due on Tuesday 9th June at 23:59. Please send it by email at

`leonardo.patimo@math.uni-freiburg.de`

Group work is encouraged!

Lie Algebras

Aufgabe 3.1: Let $\langle -, - \rangle$ be the standard scalar product on $\mathbb{R}^{2n}$ and let

$$J = \begin{pmatrix} 0 & Id_n \\ -Id_n & 0 \end{pmatrix} \in M_{2n}(\mathbb{R}).$$

The standard symplectic form ω on $\mathbb{R}^{2n}$ is defined as $\omega(v, w) = \langle v, Jw \rangle$.
The symplectic group $Sp_{2n}(\mathbb{R})$ is the Lie group of matrices preserving the standard symplectic form ω, i.e.

$$Sp_{2n}(\mathbb{R}) = \{A \in GL_n(\mathbb{R}) \mid A^\top JA = J\}$$

Show that $T_I Sp_{2n}(\mathbb{R}) = \{M \in M_{2n}(\mathbb{R}) \mid M^\top - JMJ = 0\}$.
Hint: Take $M \in T_I Sp_{2n}$. Then for any $t \in \mathbb{R}$ we have $e^{tM^\top} J e^{tM} = J$. Take derivative in t and notice that $J^2 = -Id_{2n}$.

(6 Punkte)

Aufgabe 3.2: Let G be a closed subgroup of $GL_n(\mathbb{R})$ and let N be a closed normal subgroup of G. Show that for any $X \in \mathrm{Lie}(G)$ and $Y \in \mathrm{Lie}(N)$ we have $[X, Y] \in \mathrm{Lie}(N)$.
(A subspace of a Lie algebra with this property is called an *ideal*.)
Hint: For any $s, t \in \mathbb{R}$ we have $e^{tX} e^{sY} e^{-tX} \in N$. Then take derivative in t and s.

(6 Punkte)

Bonus-Aufgabe 3.3: Let G be an abelian Lie subgroup of $GL_n(\mathbb{R})$.

- Show that Lie algebra $\mathrm{Lie}(G)$ is abelian, i.e. for every $X, Y \in \mathrm{Lie}(G)$ we have $[X, Y] = 0$.
- Regard $\mathrm{Lie}(G)$ as a group with $+$. Show that $exp : T_I G \to G$ is a group homomorphism. Moreover, if G is connected show that exp is surjective.

- Show that if G is connected then $G \cong \mathbb{R}^m/\Gamma$, where Γ is a discrete subgroup of $\mathbb{R}^m$.

(6 Punkte)