Lie Groups
SoSe 2020 — Ubungsblatt 5

Ausgabe 15.06.20, **Abgabe** 30.06.19

Solutions are due on Tuesday 30th June at 23:59. Please send it by email at

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Aufgabe 5.1: Write down the Haar measure of the Lie group $\mathbb{C}^*$.
Hint: Combine the case of the groups $\mathbb{R}^*$ and S^1.

(4 Punkte)

Aufgabe 5.2: Let V be an irreducible real representation of a group G. Show that the G-invariant scalar product on V, if exists, is unique up to multiplication by a positive real number.
Hint: Show that $Bil(V) \cong \mathrm{Hom}_k(V, V^*)$, where $Bil(V)$ is the space of bilinear forms on V, and V^* is the dual space on V. Moreover, G-invariant forms correspond to homomorphisms of G-representations. (Recall that V^* can be seen as a G-representation via $g \cdot \phi(v) = \phi(g^{-1}v)$ for any $g \in G$, $\phi \in V^*$ and $v \in V$).

(4 Punkte)

Aufgabe 5.3: Let G be a finite group.

- Show that the Haar measure on G is
$$\int_G f(g)dg = \frac{1}{|G|}\sum_{g\in G} f(g).$$
- Apply Peter-Weyl theorem on G to deduce
$$|G| = \sum_{V \text{ irred}} (\dim V)^2$$
where the sum runs over all the isomorphism classes of irreducible representations of G.

(4 Punkte)

Bonus-Aufgabe 5.4: The goal of this exercise is to deduce from Peter-Weyl theorem that every compact Lie group G has a faithful representation, i.e. an injective representation $\rho : G \to GL(V)$ (and therefore G is isomorphic to a closed subgroup of $GL_n(\mathbb{R})$.

- Using Peter-Weyl theorem, show that for every $g \in G^\circ$ (where G° is the connected component of identity of G) there exists a finite dimensional representation ρ_1 such that $\rho_1(g) \neq Id$.

- Deduce that $\ker \rho_1$ is a subgroup of G which such that $\dim \mathrm{Ker}(\rho_1) < \dim G$.

  Hint: If $\dim \mathrm{Ker}(\rho_1) = \dim G$, $\mathrm{Ker}(\rho_1)$ would contain a neighborhood of $e \in G$.

- Continuing as in the previous point, show that we can find N representations $\rho_1, \rho_2 \ldots, \rho_N$ such that

$$0 = \dim \mathrm{Ker}(\rho_1 \oplus \rho_2 \oplus \ldots \oplus \rho_N) < \dim \mathrm{Ker}(\rho_1 \oplus \rho_2 \oplus \ldots \oplus \rho_{N-1}) < \ldots \dim \mathrm{Ker}(\rho_1)$$

- Deduce that $\mathrm{Ker}(\rho_1 \oplus \rho_2 \oplus \ldots \oplus \rho_N) = \{g_1, \ldots, g_M\}$, so we can find $\rho_1, \rho_2, \ldots, \rho_{N+M}$ such that $\mathrm{Ker}(\rho_1 \oplus \rho_2 \oplus \ldots \oplus \rho_N) = \{e\}$.

(8 Punkte)