Light Leaves Basis of Indecomposable Soergel Bimodules

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Soergel Bimodules

A Coxeter group W is a group that can be presented as

$$\langle s \in S \mid (st)^{m_{s,t}} = 1\ \forall s,t \in S\rangle$$

where S is a finite set, $m_{s,s} = 1$ and $m_{s,t}$ is an integer ≥ 2 (or ∞, meaning no relation) if $s \neq t$.
Let $\mathfrak{h} = \bigoplus_{s\in S} \mathbb{R}\alpha_s^\vee$. We can define a representation of W on $\mathfrak{h}$, called the *geometric representation*, through

$$s(v) = v - \alpha_s(v)\alpha_s^\vee \quad \text{where} \quad \alpha_s \in \mathfrak{h}^* \quad \text{and} \quad \alpha_s(\alpha_t^\vee) = -2\cos(\pi/m_{s,t}) \quad \forall s,t \in S.$$

Example. S_n is a Coxeter group: $S = \{s_1 = (1 \leftrightarrow 2), ..., s_{n-1} = (n-1 \leftrightarrow n)\}$ is the set of simple transpositions, $m_{s_i,s_j} = 3$ if $|i-j| = 1$ and $m_{s_i,s_j} = 2$ otherwise. The geometric representation of S_n is isomorphic to $\mathbb{R}^n/\langle(1,...,1)\rangle$, where S_n acts on $\mathbb{R}^n$ permuting the coordinates.

Let $R = \mathrm{Sym}(\mathfrak{h}^*)$, the graded algebra in which $\mathfrak{h}^*$ has degree 2. For an element $s \in S$ we define the bimodule $B_s = R \otimes_{R^s} R(1)$, where R^s is the subalgebra of s-invariants and (1) denotes the grading shift which places the element $1 \otimes 1$ in degree -1. For an expression $\underline{w} = s_1 s_2 \dots s_n$ (not necessarily reduced) we define the **Bott-Samelson bimodule**:

$$BS(\underline{w}) := B_{s_1} \otimes_R B_{s_2} \otimes_R \dots \otimes_R B_{s_n} = R \otimes_{R^{s_1}} R \otimes_{R^{s_2}} \dots \otimes_{R^{s_n}} R(n).$$

We denote by $\mathbb{BS}$Mod the full subcategory of R-Mod-R whose objects are precisely the Bott-Samelson bimodules.
Soergel [Soe07] proved that for any $x \in W$ there is a unique indecomposable bimodule $B_x \overset{\oplus}{\subseteq} BS(\underline{x})$ which does not occur as a summand of $BS(\underline{y})$ for any expression $\underline{y}$ shorter than $\underline{x}$. Moreover, the bimodule B_x does not depend, up to isomorphism, on the choice of the reduced expression $\underline{x}$.

Definition. The category of **Soergel bimodules** $\mathbb{S}$Mod is the additive subcategory of R-Mod-R generated by direct summands of shifts of Bott-Samelson bimodules.

Therefore the bimodules B_x, $x \in W$ and their shifts give a set of representatives of the indecomposable Soergel bimodules.

Remark. *The indecomposable bimodule B_x has an important geometric meaning when W is a Weyl group. It is the equivariant hypercohomology of the IC-sheaf of the Schubert variety corresponding to x inside the flag variety G/B, where G is a reductive group having W as Weyl group and B a Borel subgroup.*

Soergel bimodules give a categorification of the Hecke algebra $\mathcal{H}(W,S)$ of the Coxeter group.

Theorem (Soergel's categorification theorem + Soergel's Conjecture [Soe07, EW14]). *There exists an isomorphism $\mathrm{ch} : \mathfrak{K}^0(\mathbb{S}Mod) \to \mathcal{H}(W,S)$, where $\mathfrak{K}^0$ denotes the split Grothendieck group. Moreover $\mathrm{ch}(B_x)_{x\in W}$ is the Kazhdan-Lusztig basis of $\mathcal{H}(\mathcal{W},\mathcal{S})$ as defined in [KL79].*

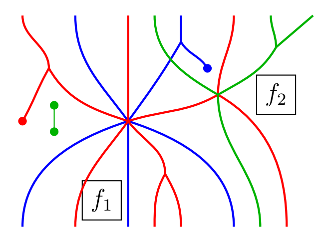
Diagrammatics for Soergel Bimodules

In [EW13] Elias and Williamson gave an alternative description, by generators and relations, of $\mathbb{BS}$Mod using planar diagrams. This also provides, after taking the idempotent completion, a description of the category of Soergel bimodules $\mathbb{S}$Mod.
They first assign a different color to each element of S. Then objects in the category $\mathbb{BS}$Mod correspond to sequences of colored dots:

$$BS(\underline{w}) \text{ where } \underline{w} = s_1 s_2 \dots s_n \longleftrightarrow \bullet\bullet\dots\bullet$$

The morphisms in $\mathbb{BS}$Mod are a linear combination of isotopy classes of some decorated planar diagrams embedded in the strip $\mathbb{R} \times [0,1]$. The edges of this diagram are colored as the elements of S and they may end in a dot with the same color on the boundary of the strip.

Example. A morphism between $BS(\underline{tstsstus})$ and $BS(\underline{stsutsuu})$, where $m_{s,t} = 5$, $m_{s,u} = 3$ and $m_{t,u} = 2$.

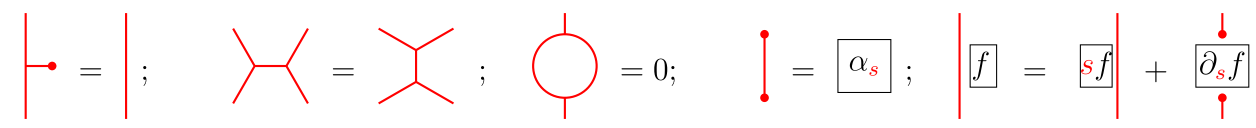


The generating morphisms, i.e. the kinds of vertices allowed in the diagrams, are:

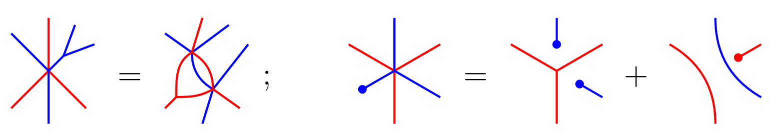
dot | trivalent vertex | $2m_{s,t}$-valent vertex (here $m_{s,t} = 4$)

The connected components of the complement of the diagram can be decorated by elements $f \in R$. Finally one needs to quotient the set of diagrams so obtained by the following relations:

- One color relations:



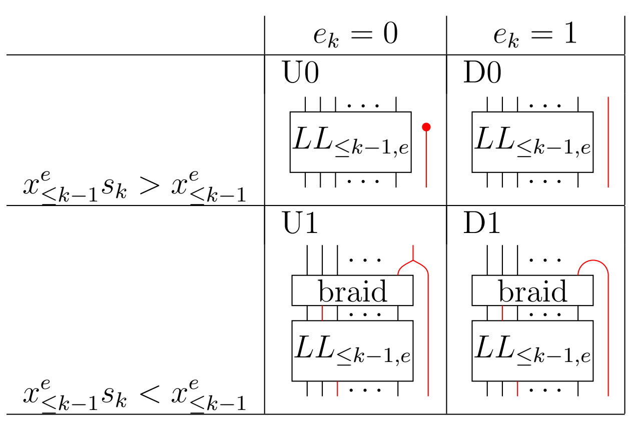
- Two color relations (here we illustrate only the case $m_{s,t} = 3$)



- Three color relations (not described here).

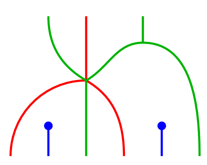
Light Leaves Basis

In [Lib08] Libedinsky defined inductively and explicitly a basis of the free right R-Mod $\mathrm{Hom}(BS(\underline{x}), BS(\underline{y}))$ for any two sequences $\underline{x}$ and $\underline{y}$. Let $\underline{x} = s_1 s_2 \dots s_n$ and $e \in \{0,1\}^n$. For any $k \leq n$ we define $\underline{x}_{\leq k} = s_1 s_2 ... s_k$ and $x^e_{\leq k} = s_1^{e_1} s_2^{e_2} ... s_k^{e_k}$. Assume by induction we have already defined a map $LL_{\leq k-1,e} \in \mathrm{Hom}(BS(\underline{x}_{\leq k-1}), BS(\underline{x^e_{\leq k-1}}))$. Then we obtain $LL_{\leq k,e}$ after dividing in four different cases, as follows:



where "braid" is a subdiagram containing only braid moves, i.e. only $2m$-valent vertices. We call the resulting morphism $LL_{\underline{x},e} = LL_{\leq n,e}$ a **Light Leaf**. The choice of the "braid" part is not canonical, so neither are the light leaves.

Example. $W = \widetilde{A}_2$ the Coxeter group where $S = \{s, t, u\}$ and $m_{s,t} = m_{t,u} = m_{s,u} = 3$. Let $\underline{w} = stustu$ and $e = (1,0,1,1,0,1)$. Then

$LL_{\underline{w},e} =$ 

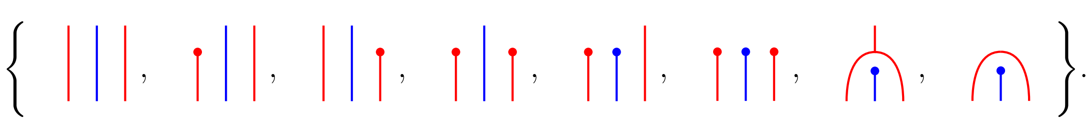
We could flip upside down a light leaf $LL_{\underline{x},e} \in \mathrm{Hom}(BS(\underline{x}), BS(\underline{z}))$ in order to obtain an element of $\mathrm{Hom}(BS(\underline{z}), BS(\underline{x}))$: we will denote it by $\overline{LL}_{x,e}$.

Theorem. *The set $\{\overline{LL}_{\underline{y},f} \circ LL_{\underline{x},e}\}$ where $x^e = y^f$ is a (cellular) basis for $\mathrm{Hom}(BS(\underline{x}), BS(\underline{y}))$, called the **Double Leaves Basis**.*

Light leaves are also useful to construct a basis of our Bott-Samelson bimodules that is compatible with the support filtration. Let us denote by $1^{\otimes}$ the element $1 \otimes 1 \otimes \dots \otimes 1$ in a BS bimodule.

Corollary. *The set $\{\overline{LL}_{x,e}(1^{\otimes})\}$ where $e \in \{0,1\}^n$ is a basis of $BS(\underline{x})$, called the **Light Leaves Basis**.*

Example. The Light Leaves basis for $BS(\underline{sts}) \in \widetilde{A}_2$ is

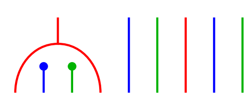


Let $\underline{w} = \underline{s_1 s_2 \dots s_n}$ a reduced expression. Choosing an embedding of the indecomposable module B_w as a summand of $BS(\underline{w})$ is the same thing as choosing a primitive idempotent $\phi_w \in \mathrm{End}(BS(\underline{w}))$ such that $\phi_w(1^{\otimes}) \neq 0$. There is not a unique choice for such an idempotent, but still we can make a natural and convenient choice by induction.
Let $\underline{w'} = s_1 s_2 \dots s_{n-1}$ so that $\underline{w} = \underline{w'} s_n$. We can assume that we have already fixed an idempotent $\phi_{w'} \in \mathrm{End}(BS(\underline{w'}))$. Now B_w still occur as an indecomposable summand in $B_{w'}B_s$ and, moreover, it is also uniquely determined. So there is a unique choice of an indecomposable $\phi_{w',s} \in \mathrm{End}(B_{w'}B_s)$ corresponding to B_w . We choose $\phi_w := \phi_{w',s} \circ (\phi_{w'} \otimes \mathrm{id}_R) \in \mathrm{End}(BS(\underline{w}))$ as the idempotent for B_w.

Light Leaves Basis for Indecomposable Soergel Bimodules

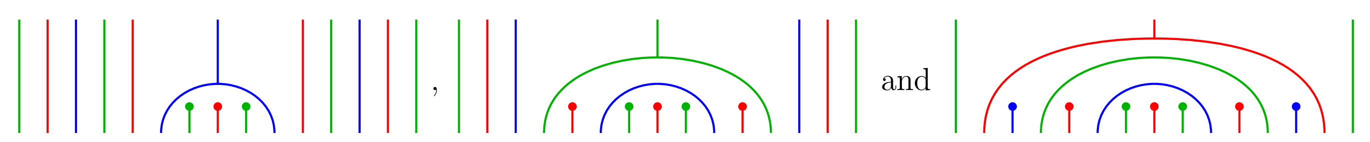
Having fixed an idempotent $\phi_w \in \mathrm{End}(BS(\underline{w}))$ whose image is B_w, it makes sense to ask how, projecting with ϕ_w, one can obtain a basis for B_w starting with the Light Leaves basis for $BS(\underline{w})$.

Example. In the previous example the first 6 elements of the described basis of $BS(\underline{sts})$ form, after projecting, a basis of B_{sts}. These are the Light Leaves where no D's appear: they always give a set of linear independent elements. it is not true in general that they generate, as for the following elements of $\widetilde{A}_2$:
Let $\underline{w} = \underline{stustustu}$. Then a basis for B_w is given by projecting

- All the Light Leaves without D's for $\underline{stustustu}$;
- All the Light Leaves without D's for $\underline{stustu}$ composed with 

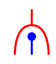
Let $\underline{w} = \underline{ustustusutsutsu}$. Then a basis for B_w is given by projecting

- All the Light Leaves without D's for $\underline{ustustusutsutsu}$;
- All the Light Leaves without D's for $\underline{ustustsutsu}$, $\underline{ustutsu}$ and $\underline{usu}$ composed respectively with



This generalizes to a description (partly conjectured) of a basis of B_w for any $w \in \widetilde{A}_2$.

Some arising questions

1. There are some relations one needs to quotient out to compute a basis for the indecomposable bimodules: for example if in a LL a subdiagram of the form  appears at the bottom, then its projection is 0. Is it possible to understand all such relations?
2. Is there a way to make the construction of this basis "canonical" (e.g. not depending on the reduced expression)?
3. In the ordinary Schubert calculus the Pieri's formula explains how to multiply a cohomology class by a Chern class: can we find a basis of the indecomposable Soergel bimodules where one has an analogous to the Pieri formula?

References

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