Charges via the affine Grassmannian

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Today's plan

Part 1. Kostka–Foulkes polynomials and charge statistics

Part 2. New geometric approach to the charge statistic

- Affine Grassmannian and Hyperbolic Localization
- Wall Crossing on Crystal Graphs

Part 1:
Kostka–Foulkes polynomials and charge statistics

Kostant partition function

Let $\mathfrak{g}$ be a complex semisimple Lie algebra (e.g. $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$).

X weight lattice, $\Phi \subseteq X$ root system.

Definition

The **Kostant partition function** kpf counts the number of way $\mu \in \mathbb{Z}\Phi$ can be written as a sum of positive roots.

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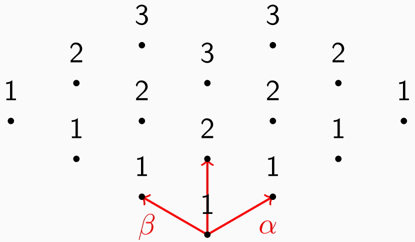
Example

$\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$,
$\Phi^+ = \{\alpha, \beta, \alpha + \beta\}$.
Then $2\alpha + \beta = (\alpha + \beta) + \alpha = \alpha + \alpha + \beta$.
Hence $\text{kpf}(2\alpha + \beta) = 2$.



Kostant's multiplicity formula

Let $\lambda \in X_+$ be a dominant weight.

Let $\Delta(\lambda)$ be the **Verma module** for $\mathfrak{g}$ of highest weight λ. Then

$$\dim \Delta(\lambda)_\mu = \mathrm{kpf}(\lambda - \mu).$$

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Let $L(\lambda)$ be the **irreducible representation** of highest weight λ.

Kostant's multiplicity formula

$$\begin{aligned}\dim L(\lambda)_\mu &= \sum_{w \in W} (-1)^{\ell(w)} \dim \Delta(w(\lambda+\rho)-\rho)_\mu \\ &= \sum_{w \in W} (-1)^{\ell(w)} \mathsf{kpf}(w(\lambda+\rho)-\mu-\rho).\end{aligned}$$

where W is the Weyl group and $\rho = \frac{1}{2}\sum_{\alpha \in \Phi^+} \alpha$.

q-analogue of weight multiplicities

kpf has a q-**analogue** $\mathrm{kpf}_q : \mathbb{Z}\Phi \to \mathbb{Z}[q]$.

The coefficient of q^k in $\mathrm{kpf}_q(\mu)$ counts the number of way $\mu \in \mathbb{Z}\Phi$ can be written as a sum of k positive roots.

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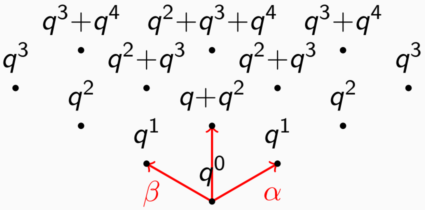
$\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$ as before. $2\alpha + \beta = (\alpha + \beta) + \alpha = \alpha + \alpha + \beta$. Hence $\mathrm{kpf}_q(2\alpha + \beta) = q^2 + q^3$.



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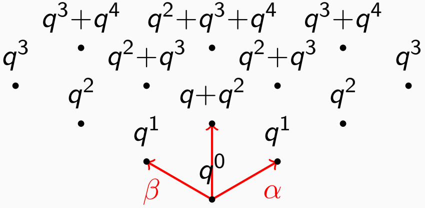

Definition (Lusztig, 1983)

The q-analogue of the weight multiplicities, aka **Kostka-Foulkes polynomials**, are defined by

$$K_{\lambda,\mu}(q) = \sum_{w \in W} (-1)^{\ell(w)} \mathrm{kpf}_q(w(\lambda + \rho) - \mu - \rho).$$

Kostka-Foulkes Polynomials

Clearly, we have $K_{\lambda,\mu}(1) = \dim L(\lambda)_\mu$.

What meaning carry the coefficients of $K_{\lambda,\mu}(q)$ in rep. theory?

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On the weight spaces $L(\lambda)_\mu$ there is a filtration induced by the action of a principal nilpotent element $e \in \mathfrak{g}$, the **Brylinski-Kostant filtration**

$$F_i(L(\lambda)_\mu) = \ker(e^{i+1})$$

Theorem (Brylinski '88)

The coefficient of q^k in $K_{\lambda,\mu}(q)$ is

$$\dim(F_k(L(\lambda)_\mu)/F_{k-1}(L(\lambda)_\mu).$$

Affine Kazhdan-Lusztig polynomials

Kostka-Foulkes polynomials can also be obtained as Kazhdan-Lusztig polynomials $h_{\mu,\lambda}$ for the **affine Weyl group**

$$\widetilde{W} = W \ltimes \mathbb{Z}\Phi.$$

Theorem (Kato '83)

$$K_{\lambda,\mu}(q) = h_{w_\mu, w_\lambda}(q^{\frac{1}{2}}) \quad \textit{where } w_\mu, w_\lambda \in \widetilde{W}$$

In particular, $K_{\lambda,\mu}(q)$ is given by the graded dimension of the stalk in μ of the intersection cohomology of the **Schubert variety** X_λ.

This gives a **geometric meaning** to the KF polynomials. (We will come back to this later on).

Corollary

The polynomials $K_{\lambda,\mu}(q)$ have positive coefficients.

Combinatorial meaning of KF polynomials

The numbers $K_{\lambda,\mu}(1)$ have a combinatorial interpretation.

There are several **combinatorial objects** that enumerate $K_{\lambda,\mu}(1)$:

- Mirkovic–Vilonen polytopes
- Littelmann's paths
- Lakshmibai–Seshadri galleries
- type specific models...

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- type specific models...

In type A, $K_{\lambda,\mu}(1)$ is the number of **semistandard Young tableaux** of shape λ and weight μ.

Question

Can we give a combinatorial interpretation of the coefficients of $K_{\lambda,\mu}(q)$?

This is still an open question in general!

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Definition

A **statistic** for KF pols. is a function $\mathrm{ch} : \mathrm{Tab}(\lambda, \mu) \to \mathbb{Z}_{\geq 0}$ such that

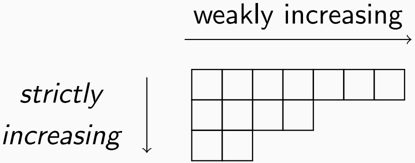
$$K_{\lambda,\mu}(q) = \sum_{T \in \mathrm{Tab}(\lambda,\mu)} q^{\mathrm{ch}(T)}.$$

Lascoux–Schützenberger defined a statistic using **cyclage**.

Semistandard Young tableaux

Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \lambda_k = 0)$ be a partition.

A SSYT of shape λ is a tableau such that



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2	3	4	4		
4	4				

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Example

$\lambda = (3, 1)$, $\mu = (1, 1, 1, 1)$.

$$\text{Tab}(\lambda, \mu) = \left\{ \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1} \end{array}, \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1} \end{array}, \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1} \end{array} \right\}$$

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Remove the box in SW corner and insert it on 1st row.

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This is called the cyclage of T.

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array}$$

Lascoux–Schützenberger's charge statistic

The **cocharge** $co(T)$ is the number of times we need to perform cyclage until we get to a row tableau.

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Definition

The **charge** of a tableau is

$$\mathrm{ch}(T) = \|\mu\| - co(T),$$

where $\|\mu\| = \sum (i-1)\mu_i$

Theorem (Lascoux–Schützenberger '78)

The charge ch : $\mathrm{Tab}(\lambda, \mu) \to \mathbb{Z}$ *is a statistic for the Kostka-Foulkes polynomial* $K_{\lambda,\mu}(q)$.

Part 2:
A geometric approach to the charge statistic

Geometric meaning of the charge?

Kostka-Foulkes polynomials have geometric interpretation:
They compute the stalks of Intersection Cohomology Sheaves of Schubert Varieties X_λ in Affine Grassmannian.

$$K_{\lambda,\mu}(v) = \sum_i \dim IC_\mu^{-i-2(\lambda,\rho)}(X_\lambda, \mathbb{Q}) v^{2i}.$$

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Natural questions

What is the **meaning** of the charge statistic in this geometric setting?

Can this geometric interpretation give another way of thinking about the charge (e.g. avoiding tableaux combinatorics)?

Affine Grassmannian

Let Gr be the **affine Grassmannian**.

$$\mathrm{Gr} = GL_n(\mathbb{C}((t)))/GL_n(\mathbb{C}[[t]])$$

It is a ∞ dim. variety parameterizing $\mathbb{C}[[t]]$-lattices in $\mathbb{C}((t))^n$.

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For $\lambda \in X$ weight, let

$$t^\lambda := \begin{pmatrix} t^{\lambda_1} & & & \\ & t^{\lambda_2} & & \\ & & \ddots & \\ & & & t^{\lambda_n} \end{pmatrix} \in \mathrm{Gr}\,.$$

For $\lambda \in X_+$, the **Schubert variety**

$$X_\lambda = \overline{GL_n(\mathbb{C}[[t]]) \cdot t^\lambda}$$

is an irreducible complex variety of dimension $2(\lambda, \rho)$,
where (,) **Killing form** normalized so that $(\alpha, \alpha) = 2$ for $\alpha \in \Phi$

Hyperbolic localization

Let $T \subseteq SL_n(\mathbb{C})$ be the maximal torus. We have an action of the **augmented torus** $\hat{T} = T \times \mathbb{C}^*$ on Gr

where $z \in \mathbb{C}^*$ acts via **loop rotation**:

$$z \cdot t \mapsto zt, \qquad \text{where } t \in \mathbb{C}((t))$$

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Rank 1 subtori $\mathbb{C}^* \subseteq \widehat{T}$ are parametrized by elements

$$\eta \in X_\bullet(\hat{T}) \cong X \oplus \mathbb{Z}.$$

Let

$$Y_\lambda^+ = \{x \in \mathrm{Gr} \mid \lim_{z \mapsto \infty} \eta(z) \cdot x = t^\lambda\}$$

be the **attractive set** of t^λ.

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Definition

Let $\mathcal{F} \in \mathcal{D}^b(Gr)$. The **hyperbolic localization** wrt to η in λ is

$$HL_\lambda^\eta(\mathcal{F}) = H_c^\bullet(Y_\lambda^+, \mathcal{F}).$$

Hyperbolic localization: how does it depend on η?

The torus $\hat{T}$ acts on Gr with fixed points t^{λ}, for $\lambda \in X$, and **one-dimensional orbits** of the form

$\mathcal{O} =$ 

for $\lambda \geq s_\beta(\lambda)$.

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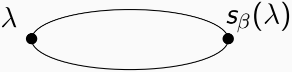
$\mathcal{O} =$ 

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Bijection between real roots and reflections

$$\left\{ \begin{array}{c} \text{reflections in} \\ \text{affine Weyl group } \widetilde{W} \end{array} \right\} \xleftrightarrow{\sim} \left\{ \begin{array}{c} \text{positive real roots in} \\ \text{affine root system } \widehat{\Phi} \end{array} \right\}$$

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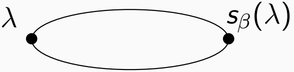
$\mathcal{O} =$ 

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Moreover, $\mu = s_\beta(\lambda)$ iff $\lambda - \mu$ is a multiple of a root in Φ.

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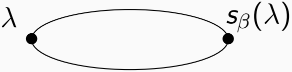
$\mathcal{O} =$ 

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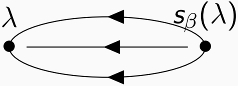
Let $\eta \in X_\bullet(\widehat{T})$ and $\beta \in \widehat{\Phi}$

If $(\eta, \beta) > 0$



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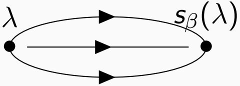
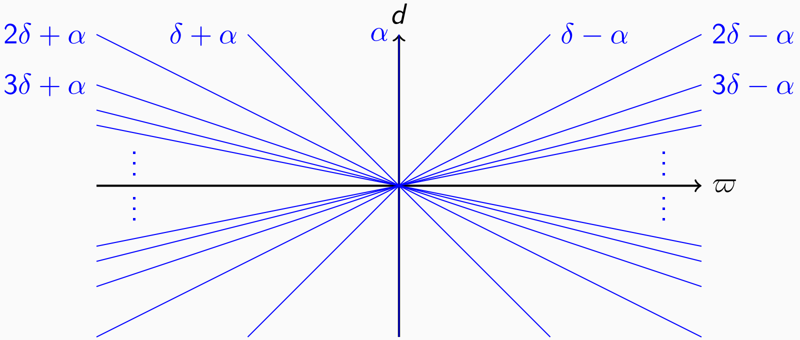


Example: affine root system of type A_1

Assume $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. The affine root system is

$$\widehat{\Phi} = \{n\delta \pm \alpha \mid n \in \mathbb{Z}\}.$$

In this case $X_\bullet(\hat{T}) = \mathbb{Z}\varpi \oplus \mathbb{Z}d$, where $d = \delta^*$.



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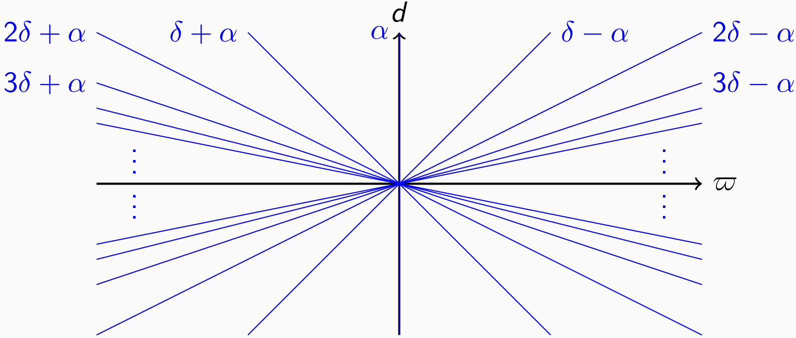

HL^η depends on the alcove of η.

Hyperbolic localization: how does it depend on η?

There are two regions of cocharacters where hyperbolic localization gives a **relevant answer**.

If we take $\eta_{MV} \in X_\bullet(T) \subset X_\bullet(\hat{T})$ dominant, then

$$HL_\mu^{\eta_{MV}}(IC_\lambda) = H_c^\bullet(Y_\mu^+, IC_\lambda) = H_c^{2(\rho,\mu+\lambda)}(X_\lambda \cap Y_\mu^+)$$

Hence:

- it is concentrated in a single degree.
- a basis is given by classes of irreducible components of $X_\lambda \cap Y_\mu^+$, which are called **MV cycles**.

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If we take $\eta_{KL} \in X_\bullet(\hat{T})$ dominant wrt the affine root system, then Y_μ^+ is an affine space and

$$\operatorname{grdim} HL_\mu^{\eta_{KL}}(IC_\lambda) = K_{\lambda,\mu}(v^2)v^{-2(\rho,\mu)}$$

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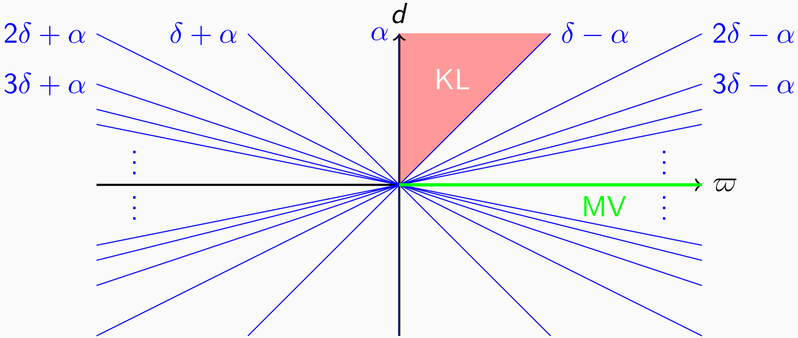
KL region

MV and KL regions in type A_1



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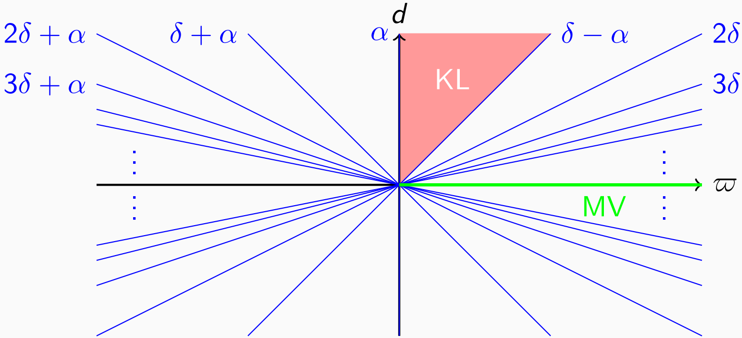

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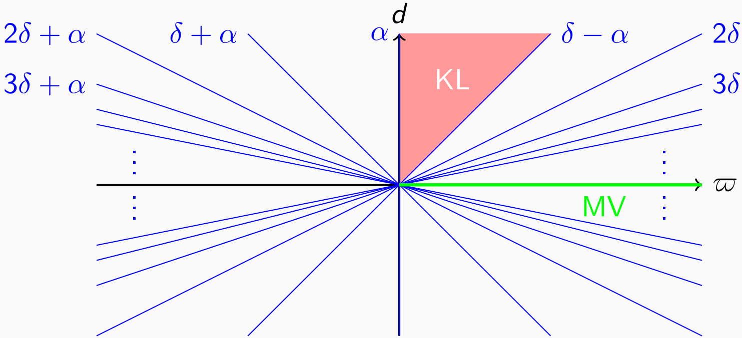

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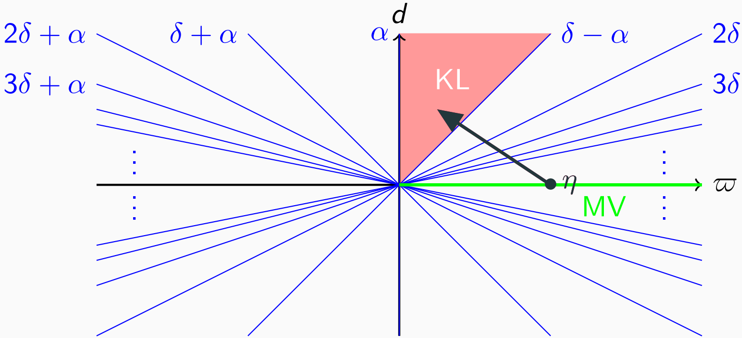

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Idea

We move the cocharacter η from the MV to the KL region!

Wall crossing in hyperbolic localization

Let $\eta_1, \eta_2 \in X_\bullet(\widehat{T})$ be on opposite sides of a wall

$$H_\beta = \{\eta \in X_\bullet(\widehat{T}) \mid (\eta, \beta) = 0\} \qquad \text{for } \beta \in \widehat{\Phi}.$$

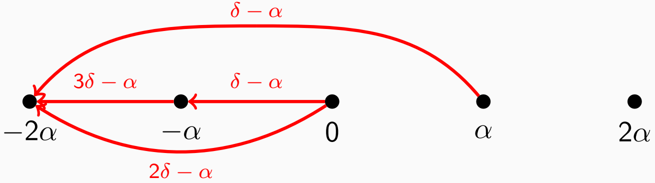
It follows from a computation on $\mathbb{P}^1(\mathbb{C})$ that for any $\mu \leq \lambda$ such that $\mu \geq s_\beta(\mu)$ we have

$$\operatorname{grdim} HL_\mu^{\eta_2}(IC_\lambda) = v^{-2} \cdot \operatorname{grdim} HL_\mu^{\eta_1}(IC_\lambda)$$
$$\operatorname{grdim} HL_{s_\beta(\mu)}^{\eta_2}(IC_\lambda) = \operatorname{grdim} HL_{s_\beta(\mu)}^{\eta_1}(IC_\lambda) + (1 - v^{-2}) \operatorname{grdim} HL_\mu^{\eta_1}(IC_\lambda)$$

Example: Wall crossing in type A_1

Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ and let $\lambda = 2\alpha \in X_+$.

In this case, we only need to cross the walls $H_{\delta-\alpha},\ H_{2\delta-\alpha},\ H_{3\delta-\alpha}$ because these are the only reflections occurring in $X_{2\alpha}$.



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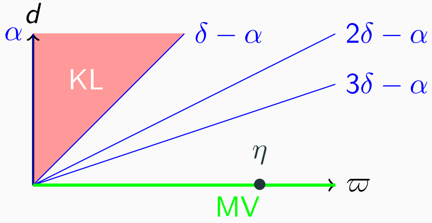
grdim HL^η	v^4	v^2	1	v^{-2}	v^{-4}
	-2α	$-\alpha$	0	α	2α



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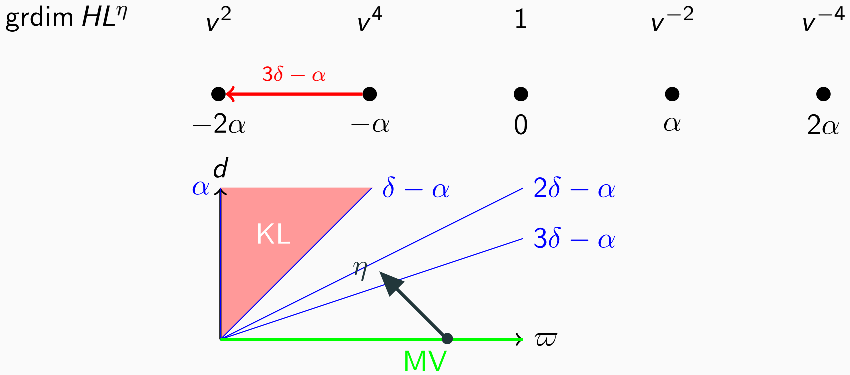

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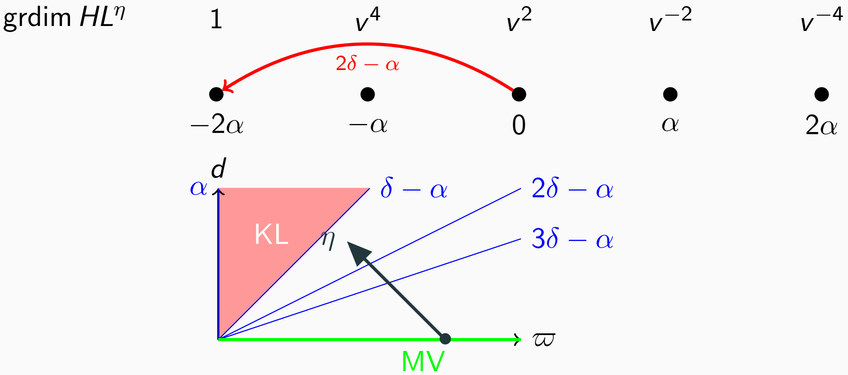

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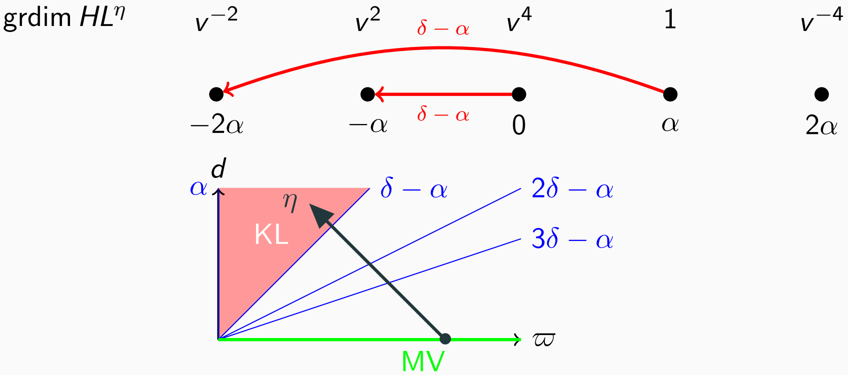
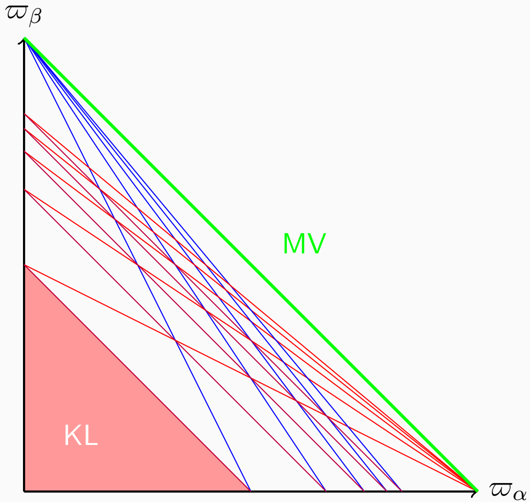


MV and KL regions in type A_2

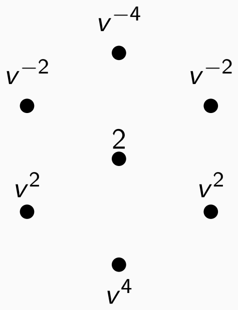
Assume now $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$. In this case $X_\bullet(T)$ has rank 3 but we can take a $2D$ projection that looks like:

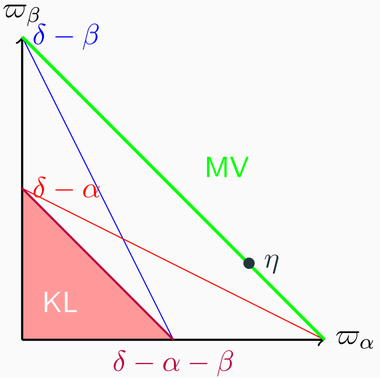


where:
red walls correspond to $n\delta - \alpha$,
blue walls to $n\delta - \beta$,
purple walls to $n\delta - \alpha - \beta$.

Let $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$ and let $\lambda = \alpha + \beta \in X_+$.

In this case, we only need to cross the walls $H_{\delta-\alpha},\ H_{\delta-\beta},\ H_{\delta-\alpha-\beta}$





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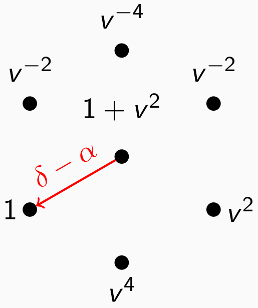

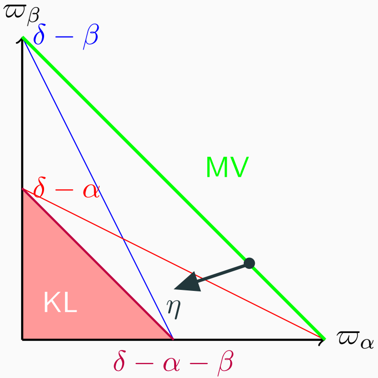

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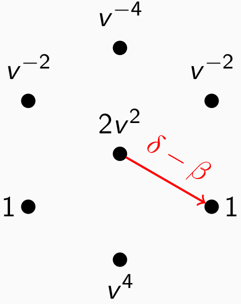

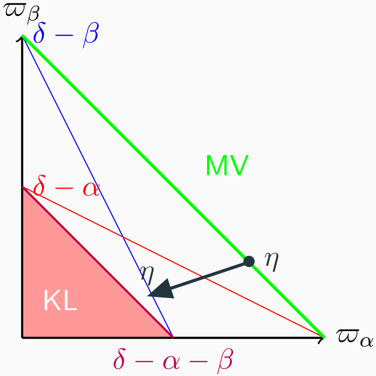

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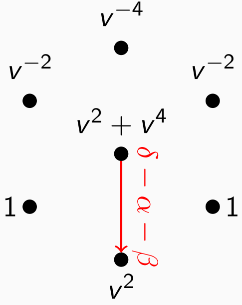

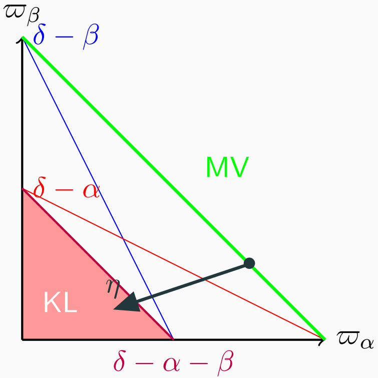

Part 2bis:
Wall Crossing on Crystal Graphs

Recharge statistics

We want to find a way to **combinatorially mimic** wall crossing for HL.

The idea is to define a charge statistic not only in the KL region, but for every cocharacter η.

Definition

We say that $r(\eta, -)$ is a **re**normalized **charge** for η if

$$\operatorname{grdim} HL^{\eta}_{\mu}(IC_{\lambda}) = \sum_{T \in \mathrm{Tab}(\lambda,\mu)} v^{2r(\eta,T)}$$

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It is trivial to define a recharge for η_{MV} in the MV region:

$$r(\eta_{MV}, T) = -(\rho, \mu) \quad \text{for all } T \in \mathrm{Tab}(\lambda, \mu).$$

Crossing wall and recharges

Let $\eta_1, \eta_2 \in X_\bullet(\widehat{T})$ be on opposite sides of a wall H_β, with $\beta \in \widehat{\Phi}$.

Assume we have $r(\eta_1, -)$. How to construct a recharge for η_2?

Definition

Let $\mu > s_\beta(\mu)$. An injective map $\psi : \mathrm{Tab}(\lambda, \mu) \to \mathrm{Tab}(\lambda, s_\beta(\mu))$ is called a **swapping function** for η_1 if

$$r(\eta_1, \psi(T)) = r(\eta_1, T) - 1 \qquad \text{for all } T \in \mathrm{Tab}(\lambda, \mu).$$

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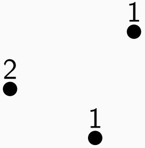
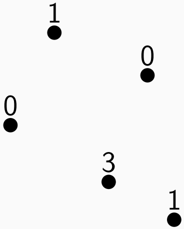
Given a swapping function, we obtain $r(\eta_2, -)$ by swapping the values of $r(\eta_1, -)$ as indicated by ψ.

Example of a swapping function

$\lambda \geq \mu \geq s_\beta(\mu)$

	$\mathrm{Tab}(\lambda, \mu)$	$\mathrm{Tab}(\lambda, s_\beta(\mu))$
$r(\eta_1, -)$		
grdim HL^{η_1}	$v^4 + 2v^2$	$v^6 + 2v^2 + 2$
grdim HL^{η_2}	$v^2 + 2$	$v^6 + v^4 + 3v^2$

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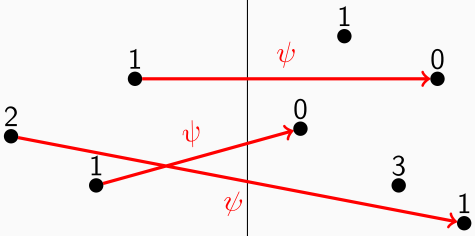
$r(\eta_1, -)$		
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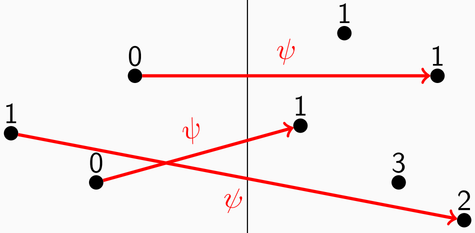

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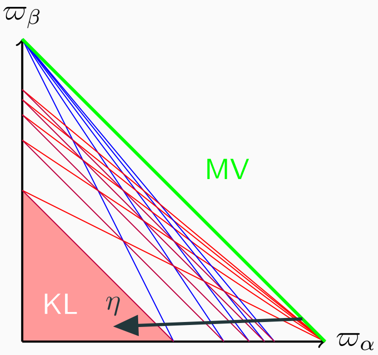

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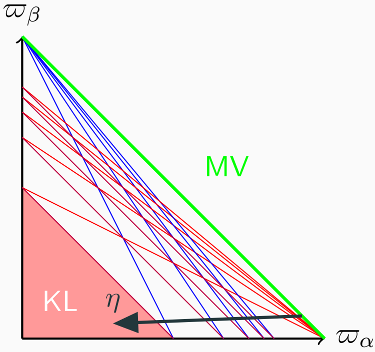

In this case, swapping functions are given by the **modified crystal operators**.

Crystal graphs

We can give to set $\mathrm{Tab}(\lambda, -)$ of SSYT of shape λ the structure of a **crystal graph**.

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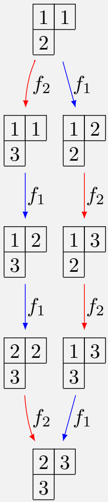

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We have **Crystal operators**

$$f_i, e_i : \mathrm{Tab}(\lambda, -) \to \mathrm{Tab}(\lambda, -) \cup \{0\}$$

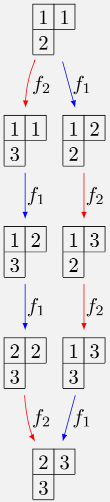
for $1 \leq i \leq n-1$.

$f_i(T)$ can be obtained by changing the label of a box from i to $i+1$ (The box where the functions $\#i - \#(i+1)$ achieves the first maximum in the word reading).

There is an action of W on the crystal: s_i acts by reflecting the f_i string.

Example

$\lambda = (2, 1)$.



Modified crystal operators

We want to attach operators f_α, e_α to each $\alpha \in \Phi_+$.

If $\alpha = \alpha_i + \alpha_{i+1} + \ldots + \alpha_j$ then

$$f_\alpha = wf_i w^{-1} \quad \text{and} \quad e_\alpha = we_i w^{-1}$$

where $w = s_j s_{j-1} \ldots s_{i+1}$.

If $T \in \text{Tab}(\lambda, \mu)$ then $f_\alpha(T) \in \text{Tab}(\lambda, \mu - \alpha)$ and $e_\alpha(T) \in \text{Tab}(\lambda, \mu + \alpha)$.

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Proposition

For the family of cocharacters descrived before, the crystal operators. If $s_\beta(\mu) = \mu - k\alpha$, then

$$\psi = f_\alpha^k : \mathrm{Tab}(\lambda, \mu) \to \mathrm{Tab}(\lambda, s_\beta(\mu))$$

is a swapping function between η_1 and η_2

A new formula for the charge statistic

We are now able to perform "wall crossing" on the crystal and to compute the charge.

For $\alpha \in \Phi_+$ let

$$\epsilon_\alpha(T) = \max\{k \mid e_\alpha^k(T) \neq 0\}$$

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Theorem (P. '21)

The function ch : $\mathrm{Tab}(\lambda, \mu) \to \mathbb{Z}$ *defined as*

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Moreover, it coincides with Lascoux–Schützenberger's charge

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Thank you for your attention!

Bonus slides: Sketch of the Proof

Why do modified crystal operators induce the swapping functions?

Case 0: Assume that $\dim L(\lambda)_\mu \leq 1$ for any μ.
Then there exists a **unique possible** swapping function, so it must coincide with f_α^k.

General case: We try to reduce to case 0.

Definition

An **atomic decomposition** of a crystal $T(\lambda, -)$ is a partition

$$T(\lambda, -) = \bigsqcup \mathcal{A}(\mu)$$

such that in each **atom** $\mathcal{A}(\mu)$ there is exactly an element of weight ν for every $\nu \leq \mu$.

Theorem (Lecouvey-Lenart, '21)

The crystal $T(\lambda, -)$ admits an **atomic decomposition**.

Bonus slides: Sketch of the Proof

Proposition (Lecouvey-Lenart, P.)

The atomic decomposition of $T(\lambda, -)$ is obtained by taking the closure of W-orbits under the operators e_1 and f_1.

If $\alpha = \alpha_1 + \ldots + \alpha_k$, then e_α, f_α preserve the atomic decomposition.

By induction, it suffices to show that the “swapping on each atom” are swapping functions.

Finally, this can be translated in a problem on **twisted Bruhat graphs**