PARITY SHEAVES & p-CANONICAL BASIS

G complex reductive group.
$\cup$
B Borel subgroup
$\cup$
T maximal torus
W Weyl group

$$\left(\begin{array}{c} SL_n(\mathbb{C}) \\ \cup \\ \begin{pmatrix} \ast & \ast \\ & \ast \end{pmatrix} \\ \cup \\ \begin{pmatrix} \ast & \\ & \ast \end{pmatrix} \end{array} \right) \quad S_n$$

$$G/B = \coprod_{w \in W} BwB/B \quad \text{Bruhat decomposition.}$$

$$BwB \cong \mathbb{C}^{\ell(w)}$$

$\underline{k}$ field, $D^b_B(G/B)$ derived category of B-equivariant constructible sheaves on G/B

$\cup$

$\mathrm{Parity}_B(G/B)$ **Def** $\mathcal{F} \in D_B(G/B)$ is even if for every $x \in W$ $i_x^* \mathcal{F}, i_x^! \mathcal{F} \in D^b_B(pt)$ vanish in odd degrees ($\mathcal{F}$ odd if $\mathcal{F}[1]$ even)

$\mathcal{F}$ is parity if $\mathcal{F} \cong \underset{\substack{\uparrow \\ \text{even}}}{\mathcal{F}_{even}} \oplus \underset{\substack{\uparrow \\ \text{odd}}}{\mathcal{F}_{odd}}$

YOU SHOULD THINK AS CONDITIONS THAT ENSURE THAT SOME SPECTRAL SEQUENCE DEGENERATES
FOR EXAMPLE DECOMPOSITION THEOREM HOLDS FOR PARITY SHEAVES

FOR EVEN RESOLUTION $H^{odd}(f^{-1}(x)) = 0$

EXAMPLE If $\mathrm{char}\, k = 0$, then intersection cohomology sheaves are parity.

Thm (JMW'14) Indecomposable parity sheaves are (up to shifts) parametrized by $\mathcal{E}_w$
where $\mathcal{E}_w|_{BwB} \cong \underline{k}[\ell(w)]$.

EXAMPLE : $\mathbb{P}^1 = SL_2(\mathbb{C})/B$. We have 2 B-orbits and the indecomp. parity sheaves are $\underline{k}_{\mathbb{P}^1}[1], \underline{k}_\infty$

∞

Prop (JMW) Convolution product $*$ of equivariant sheaves preserves $\mathrm{Parity}_B(G/B)$

Def $\mathcal{H}$ is the category generated by $\underline{k}_{P_s/B}[1]$ under $(1), *, \oplus, \overset{\oplus}{\subset}$ called the Hecke category.

We can take the split Grothendieck group of $\mathcal{H}$ and we obtain the Hecke algebra.

Recall H is the free algebra over $\mathbb{Z}[v,v^{-1}]$ with generators st

H_s, $s \in S$ and relations

$$\begin{cases} (H_s - v)(H_s + v^{-1}) = 0 \\ \underbrace{H_s H_t \text{—}}_{m_{st}} = \underbrace{H_t H_s \text{—}}_{m_{st}}. \end{cases}, \text{ with } m_{st} = \operatorname{ord}(st)$$

If $w = s_1 \dots s_k$ is a reduced expression

$H_w := H_{s_1} \text{—} H_{s_k}$ is well-defined

$\{H_w\}_{w \in W}$ <u>standard basis</u> of $\mathcal{H}$.

We can construct the isomorphism $[\mathcal{H}] \longrightarrow H$

$$\mathcal{F} \longmapsto \sum \operatorname{grdim}(i_x^* \mathcal{F}) \, v^{-\ell(x)} H_x$$

If char $k = 0$

In this case $\mathcal{E}_x = IC(X_x, k)$ and $[\mathcal{E}_x] = [IC(X_x)] = \underline{H}_x$

$\underline{H}_x$ is the Kazhdan-Lusztig basis. It has an intrinsic definition in the Hecke algebra.

If char $k = p$ $[\mathcal{E}_x] = {}^p\underline{H}_x$ is the p-canonical basis. No intrinsic definition in H.

There is no "algebraic" way to compute it so in this case the categorification of H is essential even for the definition.

Computing ${}^p\underline{H}_x$ is a fundamental problem in rep. theory

$\rightsquigarrow$ this is related to computation of characters of simple modules for algebraic groups in positive characteristic.

H_x very simple. can we find something in the middle? ${}^p\underline{H}_x$ very complicated

HYPERBOLIC LOCALIZATION

$\mathbb{C}^* \circlearrowright X$ smooth variety induces the Białynicki-Birula decomposition

- the connected components of $X^{\mathbb{C}^*}$ are smooth
- $X = \coprod_{\substack{F \text{ connected} \\ \text{component of } X^{\mathbb{C}^*}}} X_F^+$, where $X_F^+ = \{x \in X \mid \lim_{z \to 0} z \cdot x \in F\}$

  X_F^+ is an affine bundle over F.

$\quad \| $

$\coprod X_F^- \qquad X_F^- = \{x \in X \mid \lim_{z \to \infty} z \cdot x \in F\}$

$$\begin{tikzcd}
F \arrow[r, shift left, "v_+"] \arrow[d, shift right, "v_-"'] & X_F^+ \arrow[l, shift left, "p_+"] \arrow[d, hook, "u_+"] \\
X_F^- \arrow[u, shift right, "p_-"'] \arrow[r, "u_-"] & X
\end{tikzcd}$$

If $\mathcal{F} \in D^b_{\mathbb{C}^*}(X)$ then

$$(p_+)_*(u_+)^! \mathcal{F} \cong (v_+)^*(u_+)^! \mathcal{F} \in D^b_{\mathbb{C}^*}(F)$$

<u>Thm</u> (BRADEN'S HYPERBOLIC LOCALIZATION)

$$\mathcal{F}^{*!} := (v_+)^*(u_+)^! \mathcal{F} \cong (v_-)^!(u_-)^* \mathcal{F}$$

HYP. LOC. ON G/B.

We have an action $T \circlearrowright G/B$, so every cochar. $\mathbb{C}^* \to T$ induces a $\mathbb{C}^*$-action.

Fix η a cocharacter. The centralizer of η is L, Levi subgroup of G,

$\Phi_\eta = \langle \alpha \in \Phi \mid \langle \eta, \alpha \rangle = 0 \rangle$ is the root system of L

$W_\eta = \langle s_\alpha \mid \alpha \in \Phi_\eta \rangle$ is a parabolic subgroup of W.

<u>Define</u> ${}^\eta W$ be the set of representatives of $W_\eta \backslash W$ of of minimal length.

Prop The set of η-fixed points

$$(G/B)^{\mathbb{C}^*} = \bigsqcup_{x\in {}^{\eta}W} LxB/B \cong \bigsqcup_{x\in {}^{\eta}W} L/B_L$$

where $B_L = B\cap L$

Thm (SMW'16) In this setting, hyperbolic localisation preserve parity sheaves.

We obtain a functor

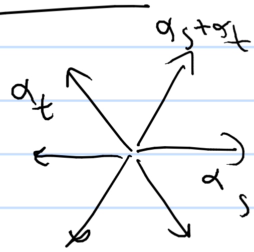
$$\mathcal{H}_{G/B} \xrightarrow{\sim} \bigoplus_{x\in {}^{\eta}W} \mathcal{H}_{L/B_L}$$

NO MONOIDAL STRUCTURE ON THIS SIDE.

We have a map $\bigoplus_{y\in {}^{\eta}W} \mathrm{ch}(\ )H_y : \left[\bigoplus_{y\in {}^{\eta}W} \mathcal{H}_{L/B_L}\right] \longrightarrow H \cong [\mathcal{H}_{G/B}]$

but the functor $(\ )^{!*}$ messes up with the gradings

EXAMPLE A_2 $\quad\alpha_s, \alpha_t$ simple roots, s,t simple reflections

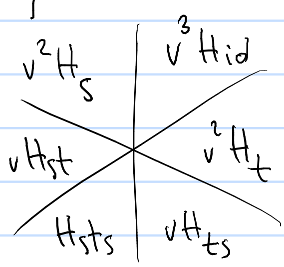


Take η such that $\langle \eta, \alpha_s+\alpha_t\rangle = 0$, $W_\eta = \langle 1, sts\rangle$

L is the group generated by T, $\begin{pmatrix}1&0&\lambda\\0&1&0\\0&0&1\end{pmatrix}$, $\begin{pmatrix}1&0&0\\0&1&0\\\lambda&0&1\end{pmatrix}$

$${}^{\eta}W = \{1, s, t\}$$

$$[\underline{k}_{G/B}(3)] \cong \underline{H}_{sts} =$$



$$k_{G/B}(3)^{!*} \cong \underbrace{\underline{k}_{L/B_L}(3)}_{1} \oplus \underbrace{\underline{k}_{L/B_L}(1)}_{s} \oplus \underbrace{\underline{k}_{L/B_L}(-1)}_{t}$$

$$\underline{H}_{sts} \neq \underline{H}^{\eta}_{sts} H_{id} + \underline{H}^{\eta}_{sts} H_s + \underline{H}^{\eta}_{sts} H_t$$

where $\underline{H}^{\eta}_{sts} = H_{sts} + v$

Thm (Billey–Braden $p=0$; Williamson $p>0$)

It works for $v=1$. That is the diagram

$$\begin{tikzcd} \mathcal{H}_{G/B} \arrow[r] \arrow[dr, "ch|_{v=1}"'] & \bigoplus_y \mathcal{H}_{L/B_L} \arrow[d, "\oplus ch|_{v=1}"] \\ & \mathbb{Z}[W] \end{tikzcd}$$

commutes

However, if P is a standard parabolic subgroup (i.e. $B \subset P$) and L is its Levi subgroup we do not need to take $v=1$ (equivalently, if the subgroup W_η is generated by a subset I of simple reflections

Thm (Grojnowski–Haiman, $p=0$; Jensen–P., $p>0$)

L as above. Then we have a commuting diagram

$$\begin{tikzcd} \mathcal{H}_{G/B} \arrow[r] \arrow[dr, "ch"'] & \bigoplus_{y \in {}^I W} \mathcal{H}_{L/B_L} \arrow[d, "\oplus ch(-) v^{\ell(y)} H_y"] \\ & H \end{tikzcd}$$

commutes.

Consequence: characters of the indecomposable parity sheaves of L/B_L gives a new basis on H of the form $\{{}^p\underline{H}_x H_y\}_{x \in W_I, y \in {}^I W}$

and $\forall z \in W$ we have

$${}^p\underline{H}_z = \sum n_z^{x,y}(v)\, {}^p\underline{H}_x H_y \quad \text{with } n(v) \in \mathbb{N}[v, v^{-1}]$$

We call $\{{}^p\underline{H}_x H_y\}$ the p-hybrid basis (or p-mixed basis)

This sits between standard and p-can, and it gives a lot of constraint on the form of ${}^p\underline{H}_z$ (you should think as giving a lower bounds)

GENERALIZATIONS

- Everything I said works also for G Kac-Moody group. In particular this covers the case $\widetilde{W} = W \ltimes \mathbb{Z}\Phi$ affine Weyl group; in which case ${}^p\underline{H}_x$ is directly related to characters of algebraic groups.
- One can realize the same functor algebraically in the language of Soergel bimodules (done by Hazi, Williamson)

In this setting one can further generalize to p-good reflection subgroups ("something that becomes like a parabolic subgroup mod p)

EXAMPLE $\widetilde{W}_p = W \ltimes p\mathbb{Z}\Phi \subset W \ltimes \mathbb{Z}\Phi$

OUR MOTIVATIONS p-CELLS

Right p-cell is an equiv. class of W under the relations generated by $x \overset{p}{\leq} y$ if ${}^p\underline{H}_y {}^p\underline{H}_s$ contains ${}^p\underline{H}_x$.

Then one gets a finer statement by writing

$\forall x \in W_I \ \forall y \in {}^I W$ we have

$$\underline{H}_{xy} = \underline{H}_x H_y + \sum_{xy \overset{p}{\sqsubset} zw} r_{xy}^{z,w}(v) \underline{H}_z H_w$$

where $xy \overset{p}{\sqsubset} zw$ if $x \overset{p}{\leq} z$ and $y < w$

right p-cell order on W_I

Bruhat order

$\rightsquigarrow$ we use this to prove

Thm (Geck $p=0$, Jensen-P. $p>0$) Induction of p-cells.

If C is a p-cell in W_I then

$C \cdot {}^I W$ is union of p-cells in W