Nichtkommutative Algebra und Symmetrie
SS 2019 — Ubungsblatt 12

18. Juli 2019

Informationen zur Vorlesung finden Sie unter:
`http://home.mathematik.uni-freiburg.de/soergel/ss19nkas.html`

Exercise 12.1: Let $\mathfrak{g}$ be a semisimple complex Lie algebra with Cartan subalgebra $\mathfrak{h}$. Show that the set of coroots $\alpha^\vee$, for $\alpha \in R(\mathfrak{g}, \mathfrak{h})$ spans $\mathfrak{h}$.

Exercise 12.2: Let $\mathfrak{g}$ be a semisimple complex Lie algebra with Cartan subalgebra $\mathfrak{h}$. For $\alpha \in R(\mathfrak{g}, \mathfrak{h})$ let $\mathfrak{g}^\alpha$ be the subalgebra generated by the weight spaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ (Recall that $\mathfrak{g}^\alpha \cong \mathfrak{sl}_2$).

- Show that if $\beta \in R(\mathfrak{g}, \mathfrak{h})$ with $\beta \neq \pm\alpha$ then $\bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_{\beta+k\alpha}$ is a representation of $\mathfrak{g}^\alpha$.
- Deduce that the set $\{k \in \mathbb{Z} \mid \beta + k\alpha\}$ is an interval in $\mathbb{Z}$.

Exercise 12.3: Let R be the root system of type C_n (i.e. the root system of $\mathfrak{sp}(2n, \mathbb{C})$, cf. Exercise 11.4).

- Find a system of positive root R^+ in R and a set of simple roots $\Pi(R^+) = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$.
- Compute $\langle \alpha_i, \alpha_j^\vee \rangle$ for any i, j.
- Compute the set of fundamental weights $\{\varpi_1, \varpi_2, \ldots, \varpi_n\}$ (Recall that these are defined by $\langle \varpi_i, \alpha_j^\vee \rangle = \delta_{i,j}$)

Exercise 12.4: Let V a vector space over a characteristic 0 field k. Let $R \subset V$ be a root system and, for $\alpha \in R$, let $\alpha^\vee \in V^*$ be the corresponding coroot (i.e. if $s : V \to V$ is the linear map such that $s(\alpha) = -\alpha$ and $s(\beta) - \beta \in \mathbb{Z}\alpha \ \forall \beta \in R$, then $\alpha^\vee$ is defined by $\langle \beta, \alpha^\vee \rangle \alpha = s(\beta) - \beta$). Show that $R^\vee = \{\alpha^\vee \mid \alpha \in R\}$ is a root system in V^*.
Bonus: Is $R^\vee$ always isomorphic to R?

Bonus Exercise 12.5: Repeat Exercise 12.3 for the root system of type B_n and D_n (i.e. for the Lie algebras $\mathfrak{so}(2n+1, \mathbb{C})$ and $\mathfrak{so}(2n, \mathbb{C})$).