Nichtkommutative Algebra und Symmetrie
SS 2019 — Übungsblatt 3

8. Mai 2019

Informationen zur Vorlesung finden Sie unter:
`http://home.mathematik.uni-freiburg.de/soergel/ss19nkas.html`

Exercise 3.1: Let R be a ring and A a right R-module. Let I be a set and for any $i \in I$ let B_i be a left R-module.

1. Show that the tensor product commutes with direct sums:
$$A \otimes_R \left(\bigoplus_{i \in I} B_i \right) \cong \bigoplus_{i \in I} (A \otimes_R B_i) .$$

2. (*) Tensor product does not commute with direct products! Let k be a field, V a vector space over k and I a set. We denote by $\operatorname{Ens}(I, V)$ the vector space of maps $I \to V$.
   - Show that $\prod_{i \in I} V \cong \operatorname{Ens}(I, V)$.
   - Show that the canonical map
   $$V \otimes_k \left(\prod_{i \in I} k \right) \cong V \otimes_k \operatorname{Ens}(I, k) \to \operatorname{Ens}(I, V) \cong \prod_{i \in I} V$$
   defined by $v \otimes \varphi \mapsto (i \mapsto \varphi(i)v)$ is not an isomorphism.
   Hint: All the maps in the image are contained in a finite dimensional subspace of V.

Exercise 3.2: Write the character table over $\mathbb{C}$ of the following groups:

1. $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.
2. The symmetric group S_3.
3. The dihedral group $D_4 = \langle r, s \mid s^2 = r^4 = id, srs = r^{-1} \rangle$.

Bonus: The symmetric group S_4.

Exercise 3.3: Let G be a group and let X be a finite set on which G acts by permutation. Let kX be a vector space with X as a basis (kX can also be thought as the vector spaces of maps $X \to k$) is a representation of G, called the *permutation representation*, defined by

$$G \times kX \to kX$$
$$(g, x) \mapsto g \cdot x$$

Let χ_{kX} denote the character of kX.

1. Show that $\chi_{kX}(g)$ is equal to the number of fixed points of g in X.

2. Show that kX always contain the trivial representation as a subrepresentation.

3. Assume now that G is finite and $|G| \neq 0$ in k. Let m be the number of orbits of G in X. Show that kX contains exactly m copies of the trivial representation, i.e. we have

$$kX \cong \overbrace{triv \oplus triv \oplus \ldots \oplus triv}^{m} \oplus \vartheta$$

as G-representations, where ϑ is a representation of G that does not contain the trivial one.

Bonus Exercise 3.4: We are again in the setting of Exercise 3.3.3,
If the action of G on X is transitive, that is X has a single orbit, we have a decomposition $kX = triv \oplus \vartheta$, where ϑ does not contain the trivial representation. We give a criteria for ϑ to be irreducible.
Assume further that the action of G on X is *doubly transitive*, that is for any x_1, x_2, y_1, y_2 with $x_1 \neq x_2$ and $y_1 \neq y_2$ there exists $g \in G$ such that $g \cdot x_1 = y_1$ and $g \cdot x_2 = y_2$.

1. Show that the action of G on $X \times X$ defined by $g \cdot (x, y) = (g \cdot x, g \cdot y)$ has exactly 2 orbits.

2. Show that $\chi_{k(X\times X)} = \chi^2_{kX}$.

3. Show that the representation ϑ is irreducible.
Hint: we have $\chi_{kX} = 1+\chi_\vartheta$. From $\sum_g \chi_{kX}(g)^2 = 2$ follows $\sum_g \chi_\vartheta(g)^2 = \sum_g \chi_\vartheta(g)\chi_\vartheta(g^{-1}) = 1$.