Nichtkommutative Algebra und Symmetrie
SS 2019 — Übungsblatt 9

28. Juni 2019

Informationen zur Vorlesung finden Sie unter:
`http://home.mathematik.uni-freiburg.de/soergel/ss19nkas.html`

Exercise 9.1: Let $\tilde{e}, \tilde{h}, \tilde{f}$ be a basis of $\mathfrak{sl}(2, \mathbb{C})$ such that $[\tilde{h}, \tilde{e}] = 2\tilde{e}$ and $[\tilde{h}, \tilde{f}] = -2\tilde{f}$. Show that $[\tilde{e}, \tilde{f}] = c\tilde{h}$, for some $c \in \mathbb{C}$.

Exercise 9.2: Let $\times$ be the cross product on $\mathbb{R}^3$ (recall that if $v, w \in \mathbb{R}^3$ and θ is the angle between v and w, then $v \times w$ is the unique vector in $\mathbb{R}^3$ of norm $\|v \times w\| = \|v\|\|w\| \sin(\theta)$, that is orthogonal to both v and w and such that $\det \big(v|w|v \times w\big) > 0$.

- Show that $(\mathbb{R}^3, \times)$ is a Lie algebra isomorphic to $\mathfrak{so}(3, \mathbb{R})$.
- Show that $\mathfrak{so}(3, \mathbb{R})$ is not isomorphic to $\mathfrak{sl}(2, \mathbb{R})$.
- Deduce that $SU(2)$ has no irreducible real representation of dimension 2.

Exercise 9.3: In this exercise we are going to give a different proof of the classification of irreducible representations of $SU(2, \mathbb{C})$ using its character theory.

- Show that $SU(2, \mathbb{C}) \cong S^3$ as differential manifolds, and that the Haar measure on $SU(2, \mathbb{C})$ is the Lebesgue measure on S^3 up to renormalization.
- Compute the character of the representations $L(m)$ of $SU(2, \mathbb{C})$ from the lecture notes, and deduce that these representations are irreducible.
  Hint: Every $g \in SU(2, \mathbb{C})$ is conjugated to a matrix of the form $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$.
- Show that the set $\{L(m) \mid m \in \mathbb{N}\}$ covers all the isomorphism classes of finite dimensional irreducible representations of $SU(2, \mathbb{C})$.
  Hint: The characters of $SU(2, \mathbb{C})$ are determined by their restrictions on $S^1 \subset SU(2, \mathbb{C})$. Then use the classification of representations of S^1 from Exercise 6.5.

Exercise 9.4: Let k be a field and let $\rho : \mathfrak{sl}(2, k) \to \mathrm{End}_k(V)$ be a representation. Show that

$$C := 4\rho(f)\rho(e) + \rho(h)(\rho(h) + 2) \in \mathrm{End}_k(V)$$

is an endomorphism of V commuting with the action of $\mathfrak{sl}(2,k)$. The element C is called the *Casimir operator* of $\mathfrak{sl}(2,k)$.
Compute the action of C on the irreducible representations $L(m)$ of $\mathfrak{sl}(2,\mathbb{C})$.

Exercise 9.5:

- Let G be a compact subgroup of $GL(n,\mathbb{C})$ and let $\mathfrak{g} = Lie(G)$. Show that there exists a scalar product $\langle -,- \rangle$ on $\mathfrak{g}$ such that

$$\langle [X,Y],Z\rangle = \langle X,[Y,Z]\rangle$$

for all $X,Y,Z \in \mathfrak{g}$.

- Show that, if $n \geq 2$, $\mathfrak{sl}_n(\mathbb{R})$ cannot be the Lie algebra of a compact subgroup of $GL(n,\mathbb{C})$.