Introduction to differential topology

Homework 1

Problem 1. Define the *special orthogonal group* SO_n as a group of orientation-preserving linear transformations of $\mathbb{R}^n$ that also preserve the standard inner product. Recall that $SO_n = \{M \in GL_n \mid MM^T = E, \det M = 1\}$.

(1) Prove that both GL_n and O_n are smooth manifolds.

(2) Describe explicitly the tangent spaces $\mathfrak{gl}_n = T_e GL_n$ and $\mathfrak{so}_n = T_e SO_n$.

(3) Define the *matrix exponential* $\exp\colon \mathfrak{gl}_n \to GL_n$ as

$$\exp(M) := \sum_{k=0}^{\infty} \frac{1}{k!} M^k.$$

Prove that it is a local diffeomorphism and conclude the same for $\exp\colon \mathfrak{so}_n \to SO_n$.

(4) Define $\pi\colon SO_n \to S^{n-1}$ as $\pi(M) = Me_1$, where e_1 is the first basis vector of $\mathbb{R}^n$. Compute $d\pi_e\colon \mathfrak{so}_n \to T_{e_1} S^{n-1}$ explicitly in local coordinates[1]. What is the preimage of a point under π?

Problem 2. Define the *complex projective space* $\mathbb{CP}^n$ as the set of lines in the complex vector space $\mathbb{C}^{n+1}$ with coordinates $(z_0, \dots, z_n)$. A line passing through the point $(z_0, \dots, z_n)$ is denoted as $[z_0 : \dots : z_n]$.

(1) For a line l, define $\varphi_i(l)$ as $l \cap \{z_i = 1\} \in \{z_i = 1\} = \mathbb{C}^n$ (if l doesn't intersect $\{z_i = 1\}$, the map φ_i is undefined). Prove that φ_i are charts that turn $\mathbb{CP}^n$ into a smooth manifold of dimension $2n$. (They are called *affine charts.*)

(2) Prove that $\mathbb{CP}^1$ is diffeomorphic to S^2.

(3) Embed S^3 standardly in $\mathbb{C}^2 = \mathbb{R}^4$. Define the *Hopf fibration* $h\colon S^3 \to \mathbb{CP}^1$ as $h(z_0, z_1) = [z_0 : z_1]$. Prove that it is a smooth map, compute explicitly its differential at $(1, 0) \in S^3$ in some local coordinates and conclude that it is surjective.

Problem 3. Define the *swallowtail* map $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ as

$$\begin{aligned} y_1 &= x_1 \\ y_2 &= x_2 \\ y_3 &= x_3^4 + x_1 x_3^2 + x_2 x_3. \end{aligned}$$

(1) Find functions a, b and c (of variables y_1, y_2 and y_3) so that critical values of F correspond precisely to the polynomials $x^4 + ax^2 + bx + c$ with multiple root[2] [3].

(2) Draw the set of critical values of F.

Hint: as in the lecture, it helps to start with cross-sections for various values of a.

(3) Into how many chambers does the set of critical values divide $\mathbb{R}^3$? Interpret these chambers in terms of the above-mentioned polynomials.

Problem 4. Let $F\colon M \to N$ be a smooth map of closed manifolds that is transversal to a closed submanifold $P \subset N$. Prove that $F^{-1}(P)$ is a smooth submanifold of M of dimension $m + p - n$.

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[1] For the sphere, it is convenient to choose local coordinates coming from projection to the hyperplane $\{x_1 = 0\}$.

[2] Recall that a root of a polynomial P is called *multiple* if it is also a root of P'.

[3] It can be shown that by a change of coordinate x any polynomial $a_n x^n + \dots + a_0$ can be reduced to the one of the form $x^n + a_{n-2} x^{n-2} + \dots + a_0$. So no generality is lost.