Introduction to differential topology
Homework 2

Problem 1. (Weak Whitney embedding theorem.) Let M be a compact m-manifold.

(1) For a given open ball $U \subset M$ construct a smooth map $F_U \colon M \to S^m$ s.t. dF_x is an isomorphism for all $x \in U$.

(2) Construct an embedding $M \hookrightarrow \mathbb{R}^{(m+1)k}$, where k is the number of balls needed to cover M.

Problem 2. Define the *real projective space* $\mathbb{RP}^n$ as the quotient $\frac{S^n}{x \sim -x}$, where $-x$ denotes the antipodal point.

(1) Prove that $\mathbb{RP}^n$ is an n-manifold.

(2) For which n is $\mathbb{RP}^n$ orientable?

Problem 3. Prove that the tangent space TM is orientable for any M.
Problem 4. Let V and W be complex finite-dimensional vector spaces. For a (complex linear) map $A \colon V \to W$, denote by $A_{\mathbb{R}} \colon V \to W$ the map from V to W viewed as real vector spaces.

(1) Suppose $A \colon V \xrightarrow{\sim} V$ is an invertible operator. Express $\det A_{\mathbb{R}}$ in terms of $\det A$ and conclude that A preserves orientation[1].

(2) Define a natural orientation on V (viewed as a real vector space) s.t. any isomorphism of complex vector spaces $A \colon V \xrightarrow{\sim} W$ would preserve this orientation[2].

[1] Recall that an operator is said to preserve orientation if it maps some (and, therefore, any) basis to an equivalent one. This notion doesn't require vector space to be oriented.

[2] Recall that an isomorphism of oriented vector spaces is said to preserve orientation if it maps some (and, therefore, any) subordinate basis to a subordinate one.