Introduction to differential topology
Homework 3

Problem 1. Let M and N be two disjoint oriented closed submanifolds of $\mathbb{R}^k$ s.t. $\dim M + \dim N = k-1$. Define the map $f\colon M \times N \to S^{k-1} \subset \mathbb{R}^k$ as $f(x,y) := \frac{x-y}{\|x-y\|}$. The number $\deg f$ is called the *linking number* of M and N.

Let $M = N = S^1$, embed the first circle in $\mathbb{R}^3$ as $(\cos\varphi, \sin\varphi, 0)$ and the second one as $(\cos\psi + 1, 0, \sin\psi)$; such a link is called a *Hopf link*. Compute the linking number of the Hopf link.

Problem 2. Let M be a closed oriented submanifold of $\mathbb{R}^{m+1}$ of codimension one. Define the *Gauß map* $G\colon M \to S^m$ by sending a point x to the unit normal vector to M at x pointing outwards.

Embed the torus $S^1 \times S^1$ into $\mathbb{R}^3$ by $((2+\sin\psi)\cos\varphi, (2+\sin\psi)\sin\varphi, \cos\psi)$. Compute the degree of its Gauß map.

Problem 3. Recall from the lecture that a complex polynomial $p(z)$ gives rise to a smooth map $\mathbb{CP}^1 \to \mathbb{CP}^1$. Likewise, if $q(z)$ is another polynomial, then the rational function $\frac{p(z)}{q(z)}$ also gives rise to a map $f\colon \mathbb{CP}^1 \to \mathbb{CP}^1$ (if $q(z_0) = 0$, set $f(z_0) := \infty$). Compute $\deg f$ in terms of p and q.