Introduction to differential topology
Homework 4

Problem 1. Recall that a *projective line* in $\mathbb{CP}^n = \mathbb{P}(\mathbb{C}^{n+1})$ is a projectivization $\mathbb{P}(V) \subset \mathbb{CP}^n$ of a two-dimensional linear subspace $V \subset \mathbb{C}^{n+1}$.

(1) Prove that maps $\iota_V : \mathbb{CP}^1 = \mathbb{P}(V) \hookrightarrow \mathbb{CP}^n$ are homotopic for all the choices of V.

(2) Find the mod 2 intersection number of the standardly embedded $\mathbb{RP}^2$ and a projective line in $\mathbb{CP}^2$ [1].

Problem 2. Denote by π the standard projection $S^n \to \mathbb{RP}^n$.

(1) Prove that any map $f : \mathbb{RP}^n \to \mathbb{RP}^n$ admits a *lift* to S^n, i.e. a map $\widetilde{f} : S^n \to S^n$ s.t. $\pi \circ \widetilde{f} = f \circ \pi$.

Hint: pick a point $x \in S^n$ and make a choice of $\widetilde{f}(x)$. Connect any other $y \in S^n$ with x by a path γ and use γ to define $\widetilde{f}(y)$. Conclude by showing that $\widetilde{f}(y)$ is independent of γ.

(2) Prove that any map $g : \mathbb{RP}^{2n} \to \mathbb{RP}^{2n}$ has a fixed point.

Hint: argue by contradiction.

(3) For any n, construct a map $g : \mathbb{RP}^{2n-1} \to \mathbb{RP}^{2n-1}$ without fixed points.

Problem 3. Let p be a non-degenerate fixed point of $f : M \to M$. Choose a coordinate chart $\varphi : U \xrightarrow{\sim} \mathbb{R}^n$ around p and set $g := \varphi \circ f \circ \varphi^{-1}$. Let $B \ni \varphi(p)$ be a ball small enough that it doesn't contain other fixed points of g. Prove that Lefschetz sign $\operatorname{sgn} x$ equals $\deg \alpha$, where $\alpha : \partial B \to S^{n-1}$ sends x to $\frac{x-g(x)}{|x-g(x)|}$.

Problem 4. Prove that the Lefschetz number of a map $f : S^2 \to S^2$ equals $\deg f + 1$.
Hint: find a nice representative in each homotopy class.

[1] By the previous part, it is well-defined.