Introduction to differential topology

Homework 5

Problems are sorted by logical dependence (not by complexity). You can use statements from the preceding problems, but not from the succeeding ones.

Problem 1. Compute $\chi(\mathbb{RP}^2)$ by constructing a radially-symmetric vector field on S^2.

Problem 2.

(1) (Inclusion-exclusion formula for Euler characteristic.) Suppose that a closed manifold can be represented as $M \cup N$ where M and N are manifolds sharing a common boundary $M \cap N$. Prove that $\chi(M \cup N) = \chi(M) + \chi(N) - \chi(M \cap N)$.

Hint: a vector field on X gives a particular way to construct a vector field on $X \times [0,1]$ pointing outwards at the boundary.

(2) (Parity of Euler characteristic is a cobordism invariant.) Prove that if a manifold with boundary $(M, \partial M)$ is of odd dimension, then $2\chi(M) = \chi(\partial M)$. Conclude that if M_1 and M_2 are cobordant[1], then $\chi(M_1) \equiv \chi(M_2) \mod 2$.

Problem 3. For closed manifolds M and N of the same dimension m their *connected sum* $M\#N$ is defined as follows: choose a point in M and in N, remove m-disks around these points and glue these new manifolds with boundary along some diffeomorphism of their boundary spheres. The homeomorphism (but not diffeomorphism) type of the resulting smooth manifold can be shown to be independent of all the choices made[2][3].

(1) How does orientability of $M\#N$ depend on that of M and N?

(2) Compute $\chi(M\#N)$ in terms of $\chi(M)$ and $\chi(N)$. Compute $\chi(S_g)$, where S_g is a *genus g surface* defined as $\#^g S^1 \times S^1$.

Problem 4. A map $f\colon M \to N$ between manifolds of the same dimension is called a (smooth) *covering* if it has no critical points.

(1) Prove that if M is compact, then the (necessarily finite) number of preimages of $x \in N$ under f, call it d, does not depend on x. One says that f is a *d-sheeted* covering.

Remark: if in addition both M and N are orientable, oriented and f preserves orientation, then $d = \deg f$.

(2) Prove that $\chi(M) = d\chi(N)$ and compute $\chi(\mathbb{RP}^n)$.

(3) Prove that a covering $S_g \to S_h$ between surfaces of genera g and h exists if and only if $g = d(h-1)+1$ for some d.

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[1] Closed manifolds M_1 and M_2 are called *cobordant* if there is a manifold with boundary W s.t. $\partial W = M_2 \sqcup -M_1$.

[2] The construction can be rectified to yield a well-defined diffeomorphism type.

[3] This is enough for the purposes of this exercise.