Introduction to differential topology
Homework 6

Problem 1. Construct a vector field on S^{2n-1} without zeroes.
Problem 2. Let (X, A) be a topological pair and $f, g\colon A \to Y$ be two homotopic maps. Prove that $X \sqcup_f Y$ is homotopy equivalent to $X \sqcup_g Y$.
Problem 3. Prove that the following topological spaces are homotopy equivalent to the wedge of spheres of the same dimension [1] (in particular, find this dimension and the number of spheres in the wedge) [2].

(1) Oriented surface of genus g without n points.

Hint: play with small values of g and n first.

(2) $\mathbb{R}^3$ without two intersecting lines.

(3) $\mathbb{RP}^3$ without two points.

(4) $\mathbb{C}^4$ without two complex 2-planes: $\{z_1 = z_2 = 0\}$ and $\{z_3 = z_4 = 0\}$.

Problem 4.

(1) Define the real *Grassmanian* $G_{k,n}$ as a set of k-dimensional subspaces in $\mathbb{R}^n$. Prove that $G_{k,n}$ is a smooth manifold.

Hint: Any $X \in G_{k,n}$ is a graph of $A\colon U \to V$, where $U \subset \mathbb{R}^n \supset V$, $\dim U = k$, $\dim V = n-k$.

In the rest of this exercise we will start, but not finish, constructing *Schubert cell decomposition* of $G_{k,n}$.

(2) Start by showing that a map $\mathrm{col}\colon \{A \in \mathrm{Mat}_{n,k} \mid \operatorname{rank} A = k\}/GL_k \to G_{k,n}$ that sends A to its column space is well-defined and a bijection. Then prove that every coset in the source admits a unique representative B s.t.

- If some row of B contains 1, then other entries in this row are 0's.
- All entries of B below any 1 are 0's.
- Suppose $B_{i,j} = B_{i',j'} = 1$. Then $i > i' \implies j < j'$.

Hint: up to minor differences, this is what known in linear algebra as reduced row echelon form.

We define the integer σ_i (for $1 \leqslant i \leqslant k$) by the condition $B_{\sigma_i, k-i} = 1$. The increasing sequence $\sigma = (\sigma_i)_{i=1}^k$ is called the *Schubert symbol* of $\mathrm{col}\, B \in G_{k,n}$.

(3) (Open Schubert cells.) Prove that there are $\binom{n}{k}$ different Schubert symbols that arise in the above way. Show that $\{X \in G_{k,n} \mid \sigma(X) = \dot{\sigma}\}$ is homeomorphic to a Euclidean space and compute its dimension in terms of the Schubert symbol $\dot{\sigma}$.

Hint: it is instructive to work out the case $G_{2,4}$ first.

(4) (Invariant definition of a Schubert symbol.) Prove that for $X \in G_{k,n}$ the number $\sigma_i(X)$ equals the unique number satisfying the condition $\dim X \cap \mathbb{R}^{\sigma_i - 1} + 1 = \dim X \cap \mathbb{R}^{\sigma_i} = i$[3]

(5) Equip $\mathbb{R}^n$ with the standard inner product. Define $H^l := \{(x_1, \ldots, x_l, 0, \ldots, 0) \in \mathbb{R}^l \mid x_l > 0\}$. Prove that each $X \in G_{k,n}$ admits a unique orthogonal basis $(v_1, \ldots, v_k)$ s.t. $v_i \in H^{\sigma_i(X)}$.

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[1] i.e. several disjoint spheres with their north poles all identified
[2] You don't have to write formulas in coordinates for all the homotopies involved, but their definitions should be clearly stated.
[3] We think of $\mathbb{R}^l$ as the span of the first l basis vectors in $\mathbb{R}^n$.