Introduction to differential topology
Homework 7

Problem 1. Choose an integer $m > 1$ and integers q_1, q_2 s.t. $\gcd(q_i, m) = 1$. The generator of the cyclic group $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$ acts on $S^3 \subset \mathbb{C}^2$ by $(z, w) \mapsto (e^{2\pi i q_1/m} z, e^{2\pi i q_2/m} w)$. This extends to the action of $\mathbb{Z}_m$ on S^3 and the quotient is called the *lens space* $L(q_1, q_2)$. Find the cell decomposition of $L(q_1, q_2)$ with one cell in each dimension and describe the attaching maps.

Problem 2. For a topological space X, define the *n-th symmetric power of* X, $SP^n(X)$, as X^n/Σ_n, where Σ_n is the symmetric group on n letters, which naturally acts on the n-fold Cartesian product X^n.

Find the cell decompositions of $SP^2(S^1)$ and $SP^3(S^1)$ and describe the attaching maps.

Problem 3. Let X and Y be two homotopy equivalent CW complexes s.t. both don't have $(n+1)$-cells. Prove that their n-skeletons are also homotopy equivalent.

Problem 4. A topological space is called an *H-space* is there is 1) a continuous map $\mu\colon X \times X \to X$ (called multiplication) and 2) an element $e \in X$ (called identity) s.t. two maps $x \mapsto \mu(x, e)$ and $x \mapsto \mu(e, x)$ from X to itself are homotopic to id_X (through maps that preserve e).

Prove that multiplication in $\pi_n(X, e)$ can be defined as $(f + g)(x) := \mu(f(x), g(x))$.