NOTES ON DIFFERENTIAL TOPOLOGY

1. Recap on smooth manifolds

We start with a brief overview of smooth manifolds.

Definition 1.1. A *smooth n-dimensional manifold* is a set M together with

(1) decomposition $M = \bigcup_i U_i$ into at most countable number of subsets (called *charts*),

(2) an injection $\varphi_i \colon U_i \hookrightarrow \mathbb{R}^n$, s.t. $\varphi_i(U_i)$ is open and for every intersection $U_i \cap U_j$ the map $\varphi_i \circ \varphi_j^{-1} \colon \varphi_j(U_i \cap U_j) \to \mathbb{R}^n$ is a smooth map from one open subset of $\mathbb{R}^n$ to the other (it is called a *transition map*).

The above data equips the set M with topology: the base consists of preimages of open subsets of $\mathbb{R}^n$. We will always assume that M is Hausdorff, i.e. every two points have disjoint neighborhoods. Every chart equips the corresponding open subset of M with local coordinates coming from $\mathbb{R}^n$ (and hence called *coordinate mapping*).

Example 1.2. An open set $U \subset \mathbb{R}^n$ is smooth, with a single chart.

Example 1.3. An *n-sphere* S^n is a manifold given by the equation $x_0^2 + \cdots + x_n^2 = 1$ in $\mathbb{R}^{n+1}$. Indeed, cover it with 2 charts $U_N = S^n \setminus N$ and $U_S = S^n \setminus S$, where $N = (0, \dots, 0, 1)$ and $S = (0, \dots, 0, -1)$. Maps $\varphi_{N,S}$ are given geometrically as sterographic projections to the hyperplane $\{x_n = 0\}$. To understand the transition maps, note first that for any $p \in U_N \cap U_S$ the points $\varphi_N(p)$ and $\varphi_S(p)$ are proportional as vectors in the subspace $\{x_n = 0\}$. The problem is thus reduced to the plane spanned by $0, N$ and $\varphi_N(p)$. Looking at right triangles in this plane one concludes $\|\varphi_N(p)\|\|\varphi_S(p)\| = 1$ (here $\|(x_0, \dots, x_{n-1})\| = \sqrt{x_0^2 + \cdots + x_{n-1}^2}$). The transition map is therefore $(y_0^N, \dots, y_{n-1}^N) = \frac{1}{\|\varphi_N(p)\|^2}(y_0^S, \dots, y_{n-1}^S)$, where y^N and y^S are local coordinates on the sphere arising from two charts. This map is indeed smooth.

Example 1.4. Consider $\{xy = c\} \subset \mathbb{R}^2$. It $c \neq 0$ it can be covered by two charts $U_+ = \{x > 0\}$ and $U_- = \{x < 0\}$ with coordinate mapping $\varphi_\pm(x, y) = x$. Since $U_+ \cup U_- = \varnothing$, there is nothing to check. If $c = 0$ the point $(0, 0)$ doesn't have a neighborhood homeomorphic to open subset of $\mathbb{R}$ since its complement has 4 connected components. Thus $\{xy = 0\}$ is not a 1-manifold and, as can be easily checked, not a manifold at all.

Definition 1.5. A map $F \colon M \to N$ is called *smooth at* $p \in M$ if for some open $U \ni p$ s.t. $U \subset U_i \subset M$ (for some chart $U_i \subset M$) and $f(U) \subset V_j \subset N$ (for some chart $V_j \subset N$) the composite $\psi_j \circ F \circ \varphi_i \colon \varphi_i(U) \to \mathbb{R}^n$ is smooth. A map F is called *smooth* if it's smooth at every point $p \in M$.

Remark 1.6. This definition should be checked for correctness in a sense that it doesn't depend on the choice of U_i and V_j.

Definition 1.7. A *path* in M is a smooth map $\gamma \colon (-1, 1) \to M$. Two paths γ_1 and γ_2 s.t. $\gamma_1(0) = \gamma_2(0)$ are called *equivalent* if for some chart $U \ni \gamma_1(0)$ one has $\frac{d(\varphi\gamma_1)}{dt}(0) = \frac{d(\varphi\gamma_2)}{dt}(0)$[1].

Remark 1.8. Again, independence of choice of a particular chart should be checked.

[1]For a path in $\mathbb{R}^n$ by $\frac{d\gamma}{dt}$ we mean $(\frac{d\gamma^1}{dt}, \dots, \frac{d\gamma^n}{dt})$, where $\gamma^i(t)$ are coordinates of $\gamma(t)$.

Definition 1.9. The *tangent space at p to M* is the set T_pM of equivalence classes of paths γ s.t. $\gamma(0) = p$ (notation: $[\gamma]$). The *tangent space to M, TM,* is $\bigcup_{p\in M} T_pM$.

Example 1.10. $M = \mathbb{R}^2, \gamma_1(t) = (\cos(t), \sin(t)), \gamma_2(t) = (0, t)$. We have $\frac{d\gamma_1}{dt}(0) = \frac{d\gamma_2}{dt}(0) = (0, 1)$, so the two paths are indeed equivalent, as suggested by the fact that one is the tangent line to the other.

Example 1.11. If $M \subset \mathbb{R}^n$ is given (locally, if necessary) by some equations, then T_pM can be canonically identified with the tangent space in the "usual" sense. Say, for $S^{n-1} \subset \mathbb{R}^n$ one has $T_pS^{n-1} = \{v \in \mathbb{R}^n \mid v \perp p\}$

Proposition 1.12. *TM is a smooth $2n$-manifold.*

Proposition 1.13. *T_pM is a vector space w.r.t. the following operations:*

1. $c \cdot [\gamma] := \varphi^{-1} \circ (c\varphi) \circ \gamma$,
2. $[\gamma_1] + [\gamma_2] := \varphi^{-1} \circ ((\varphi \circ \gamma_2) + (\varphi \circ \gamma_1))$.

Remark 1.14. Independence of representatives of equivalence classes should be checked.

For a chart $U \ni p$ s.t. $\varphi(p) = 0$ we denote by $\frac{\partial}{\partial x_i} \in T_pM$ the tangent vector $[\gamma(t)]$, where $\gamma(t) = \varphi^{-1}(0, \ldots, t, \ldots, 0)$. Local coordinates give rise to coordinates in the tangent space:

Proposition 1.15. (1) *$\{\frac{\partial}{\partial x_i}\}_i$ is basis of T_pM.*

(2) $[\gamma(t)] = \frac{d\varphi_1(t)}{dt}\frac{\partial}{\partial x_1} + \cdots + \frac{d\varphi_n(t)}{dt}\frac{\partial}{\partial x_n}$

Definition 1.16. For a smooth map $F\colon M \to N$ the *differential of F at $p \in M$* is the map $dF_p\colon T_pM \to T_{F(p)}N$ defined as $dF_p([\gamma]) := [F\circ\gamma]$. *Differential of F* is the map $dF\colon TM \to TN$ defined as $dF(p, v) = dF_p(v)$, where $v \in T_pM$.

Remark 1.17. Differential is sometimes called derivative.

Proposition 1.18. *The map dF_p is linear.*

Proposition 1.19. *Choose local coordinates $(x_1, \ldots, x_m)$ near p and $(y_1, \ldots, y_n)$ near $F(p)$. The map $F\colon M \to N$ is given by n functions of m variables, i.e. $y_i = f_i(x_1, \ldots, x_m), 1 \leqslant i \leqslant n$. In the above-mentioned basis, the map dF_p is given by*

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_m} \end{pmatrix}.$$

This matrix is called *Jacobi matrix.*

2. Critical and regular values

One way to study maps is by looking at preimages of points (or, more generally, subsets). It turns out that the behavior of a preimage (as point varies) can be read off from the differential.

Definition 2.1. The point $p \in M$ is called *critical point* if $dF_p\colon T_pM \to T_{F(p)}N$ is not surjective; $F(p)$ is then called a *critical value.* If $x \in N$ is not a critical value, it's called a *regular value.*

Example 2.2. For $F\colon \mathbb{R}^n \to \mathbb{R}^{n-1}$, $F(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-1})$ all the points in $\mathbb{R}^{n-1}$ are regular and $\ker dF_p = \langle\frac{\partial}{\partial x_n}\rangle$ (for any p). For $F|_{S^{n-1}}\colon S^{n-1} \to \mathbb{R}^{n-1}$, the differential is not surjective whenever it has a kernel. As follows from the above, this happens precisely when $p \in S^{n-2} = \{(x_1, \ldots, x_n) \in S^{n-1} \mid x_n = 0\}$. This equatorial S^{n-2} is the set of both critical points and values of $F|_{S^{n-1}}$.

Example 2.3. (Fold.) $F\colon \mathbb{R}^n \to \mathbb{R}^n, F(x_1, \dots, x_n) = (x_1^2, \dots, x_n)$. Again, both critical sets are $\{x_1 = 0\} \subset \mathbb{R}^n$ since $dF_p = \operatorname{diag}(2x_1, 1, \dots, 1)$ (for any p).

Example 2.4. (Pleat.) $F\colon \mathbb{R}^2 \to \mathbb{R}^2, F(x_1, x_2) = (x_1^3 + x_1x_2, x_2)$ (target space has coordinates (y_1, y_2)). Jacobi matrix is

$$J = \begin{pmatrix} 3x_1^2 + x_2 & x_1 \\ 0 & 1 \end{pmatrix},$$

its determinant is $3x_1^2 + x_2 = \frac{\partial y_1}{\partial x_1}$. To visualize critical points/values, consider $\mathbb{R}^3$ with coordinates (x_1, x_2, y_1) [2] and the graph of $y_1 = x_1^3 + x_1x_2$ in it. Equip this graph itself with coordinates (x_1, x_2). Then the critical points of its projection to the plane $(x_2 = y_2, y_1)$ are precisely the critical points of F, i.e. (x_1, x_2) s.t. $\frac{\partial y_1}{\partial x_1} = 0$; the same goes for the set of critical values. To understand this graph, look at crossections at various x_2. If $x_2 > 0$, we get a cubic curve in the plane (x_1, y_1) with non-vanishing derivative. If $x_2 < 0$, we get a cubic curve with two critical points (that go apart as x_2 decreases further). The set of critical points is a smooth curve, that expectedly projects to a parabola $3x_1^2 + x_2 = 0$ on the (x_1, x_2) plane.

To get the parametric description of the critical values, express both coordinates as functions of x_1: $y_2 = x_2 = -3x_1^2$, $y_1 = x_1^3 + x_1x_2 = -2x_1^3$. This parametric curve is given by algebraic equation $y_1^2 = \frac{4}{27}y_2^3$ and is called a *semicubical parabola*.

Recall that $B \subset \mathbb{R}^n$ is said to have Lebesgue measure 0 if $\forall \epsilon > 0$ in can be covered by a countable number of balls of total volume ϵ. It follows from this condition that $\mathbb{R}^n \setminus B$ is everywhere dense.

Definition 2.5. A subset $B \subset M$ is said to have *measure 0* if $\varphi(U \cap B) \subset \mathbb{R}^n$ has measure 0 for any chart $U \subset M$.

Remark 2.6. Note that we don't speak of measures other than zero. Diffeomorphism of $\mathbb{R}^n$ preserves the property of being measure 0, but in general distorts measure.

Theorem 2.7. (Sard) The set of critical values of a smooth map has measure 0.

Remark 2.8. We omit the proof, since it is of analytical nature and the ideas it is based on don't show up later.

Corollary 2.9. Regular values exist.

Remark 2.10. Even though the corrolary is much weaker, there seem to be no way to derive it by simpler means.

Definition 2.11. The map $F\colon M \to N$ is called a *smooth embedding* if

(1) F is a topological embedding [3] and

(2) dF_p is injective for any $p \in M$.

Definition 2.12. The map $F\colon M \to N$ is called a *diffeomorphism* if $\exists G\colon N \to M$ s.t. $F \circ G = Id_N$ and $G \circ F = Id_M$.

Theorem 2.13. (Inverse function theorem.) If for $F\colon M \to N$ the differential dF_p is isomorphism for some p, then it is a local diffeomorphism [4].

Proof. Find U s.t. both U and $F(U)$ belong to some charts. Now apply the classical inverse function theorem to $\psi \circ F \circ \varphi^{-1}\colon \varphi(U) \to \mathbb{R}^n$. $\square$

[2] The order of these 3 coordinates is irrelevant, since we orient axes non-standardly anyway.

[3] i.e. it is a homeomorphism onto its image. The standard injective non-example is $[0, 1) \hookrightarrow S^1$.

[4] i.e. $\exists U \ni p$ s.t. $F|_U\colon U \to F(U)$ is a diffeomorphism.

Theorem 2.14. (Regular value theorem.) Let $F\colon M \to N$ be a smooth map, $m \geqslant n$ and $y \in N$ a regular value. Then $f^{-1}(y)$ is a smooth submanifold of M of dimension $m-n$.

Proof. For $x \in f^{-1}(y)$ we will find a coordinate neighborhood. Checking that transition maps are smooth is an exercise. Denote $K = \ker dF_x$, choose a chart $U \ni x$ and define $G\colon U \to N \times K$ by $G(p) := (F(p), \mathrm{proj}_K \circ \varphi(p))$. Calculate $dG_x = (dF_x, \mathrm{proj}_K \circ d\varphi_x)$ and observe that this map is an isomorphism — indeed, $K = \ker dF_x$ is mapped isomorphically onto the second component. Apply inverse function theorem to get a local inverse $G^{-1}\colon V \xrightarrow{\sim} U$, where V is a neighborhood of y (replace U with a smaller subset if necessary.)

Now consider a subset $\{y\} \times K \cap V \subset V$. On the one hand, it is mapped homeomorphically by $pr_2\colon V \times K \to K$ onto an open subset of $K \simeq \mathbb{R}^{m-n}$. On the other hand, it is mapped homeomorphically by G^{-1} to $U \cap F^{-1}(y) \ni x$. Combining these, we get that $U \cap F^{-1}(y)$ is the desired neighborhood. □

Example 2.15. For $f\colon \mathbb{R}^n \to \mathbb{R}, f(x_1, \dots, x_n) = x_1^1 + \dots + x_n^2$ any $y \neq 0$ is regular, so we get another proof that $S^{n-1} = f^{-1}(1)$ is a manifold.

Definition 2.16. The definition of a *smooth manifold with boundary* $(M, \partial M)$ mimics that of a smooth manifold, except for the fact that the notion of a coordinate neighborhood depends on where exactly $p \in M$ lies:

(1) If $p \in M \setminus \partial M$, the neighborhood U is an open subset of $\mathbb{R}^n$ (as before),

(2) If $p \in \partial M$, the neighborhood U is an open subset of $\{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geqslant 0\}$ and, moreover, $\varphi(p) \in \{x_n = 0\}$.

Proposition 2.17. In the notations above, ∂M is an $(\dim M - 1)$ manifold.

Likewise, all the notions from the beginning are translated to into the setting of manifolds with boundary. If a manifold has non-empty boundary, it will be assumed to be compact. Compact manifold without boundary is called *closed*.

Example 2.18. The disk $D^n = \{x_1^2 + \dots + x_n^2 \leqslant 1\}$ is manifold with boundary (D^n, S^{n-1}).

Remark 2.19. Manifolds with boundary appear much less often in practice, but are of equal theoretical importance as boundaryless manifolds.

The proof of the following statement goes along the same lines as in the boundaryless case.

Proposition 2.20. Let $F\colon (M, \partial M) \to N, m \geqslant n$ be a smooth map s.t. $y \in N$ is a regular value for both f and $f|_{\partial M}$. Then $(f^{-1}(y), f^{-1}(y) \cap \partial M) \subset (M, \partial M)$ is a smooth submanifold with boundary.

Theorem 2.21. There is no map $F\colon M \to \partial M$ that leaves ∂M fixed.

Proof. (Due to Hirsch.) Suppose the contrary. Choose a regular value $y \in \partial M$. By the above, $(F^{-1}(y), F^{-1}(y) \cap \partial M)$ is a smooth 1-manifold with boundary. Moreover, since $F|_{\partial M} = id_{\partial M}$, it has a single boundary point. A contradicion.[5] □

Theorem 2.22. (Smooth Brouwer fixed point theorem.) Any smooth map $g\colon D^n \to D^n$ has a fixed point.

Proof. Suppose it doesn't. Consider a ray starting at $g(x)$ and passing through x. Let $f(x)$ be a point where this ray intersects S^{n-1}. We have obtained a map $f\colon D^n \to S^{n-1}$ (its smoothness is an exercise) s.t. $f|_{S^{n-1}} = id$, a contradiction. □

[5]Here we used the classifcation of complact 1-manifolds: they are either S^1 (the boundary is empty) or $([0,1], 0 \sqcup 1)$.

Definition 2.23. The map $F\colon M \to N$ is called *transversal* to a submanifold $P \subset N$ if $\operatorname{rank}(T_xM \xrightarrow{dF_x} T_{F(x)}N \to {}^{T_{F(x)}N}/_{T_{F(x)}P}) = n - p$. Notation: $F \pitchfork P$.

Example 2.24. If P is a point we recover the definition of a regular point.

Example 2.25. If F is an embedding, the condition can be reformulated as $T_xM + T_xP = T_xN$ (the sum doesn't have to be a direct sum, i.e. $m + p$ can be greater than n). One then writes $M \pitchfork P$. It is an exercise to check that $M \pitchfork P \Leftrightarrow P \pitchfork M$.

The following is in the exercise sheet and can proven analogously to Theorem 2.14.

Theorem 2.26. For $F\colon M \to N$ and $P \subset N$ suppose that $F \pitchfork P$. Then $F^{-1}(P)$ is a submanifold of M of dimension $m + p - n$.

3. Approximations

Our goal here is to make sense of and prove statements of the following type: a map from a wider class (e.g. a continuous one) can be approximated by a that from a smaller class (e.g. by a smooth one) with any given error.

We start by recalling notions and facts from analysis.

Theorem 3.1. (Existence of bump functions.) Consider $K \subset U \subset \mathbb{R}^n$, where K is compact and U is open. Then there exists a smooth $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ s.t. $\varphi(K) = 1$ and $\varphi(\mathbb{R}^n \setminus U) = 0$.

Remark 3.2. Such a function φ is almost never analytical.

Definition 3.3. Let $\{U_i\}$ be an open conver of M. A *partition of unity subordinate to* $\{U_i\}$ is a set of smooth maps $\lambda_i\colon M \to [0, 1]$ s.t.

(1) $\operatorname{supp} \lambda_i \subset U_i$,

(2) $\operatorname{supp} \lambda_i$ is locally finite,

(3) $\sum_i \lambda_i = 1$.

Theorem 3.4. For any open cover of M there exists a subordinate partion of unity.

From now on till the end of this section we assume the source manifold M to be compact for simplicity[6]; many statements hold in greater level generality after the necessary minor adjustments.

Let $C^k(M, N)$ denote the set of maps of differentiability class k ($0 \leqslant k \leqslant \infty$); we write $C(M, N)$ for $C^0(M, N)$. Eventually we will only care about $k = 0, \infty$.

The following steps define topology on $C^k(M, N)$.

(1) Assume first that $k < \infty$. Choose $f \in C^\infty(M, N)$, a chart (U, φ) on M, a chart (V, ψ) on N, a compact $K \subset U$ s.t. $f(K) \in V$ and $\varepsilon > 0$.

(2) Define

$$\mathcal{N}(f, (U, \varphi), (V, \psi), K, \varepsilon) := \{g \in C^k(M, N) \mid g(K) \subset V,\ \mathbb{D}^k(\psi f \varphi^{-1} - \psi g \varphi^{-1}) < \varepsilon\},$$

where the notation $\mathbb{D}^k(\alpha) < \varepsilon$ means that all the partial derivatives of α up to order k are less than ε at every point x in the domain of α. We say such g is *ε-close* to f (w.r.t. all the choices).

(3) Declare the sets $\mathcal{N}$ for all the possible choices to be the prebase[7] for the topology on $C^k(M, N)$.

[6]For the target manifold N this assumption is not needed since the image of a compact is compact.

[7]Also known as subbase.

(4) For the remaining case $k = \infty$ define a topology as the union of the induced topologies w.r.t. all the injections $C^\infty(M, N) \hookrightarrow C^k(M, N)$.

Remark 3.5. The definition is abstract. We will only need it to prove that a certain subset $S \subset C^k(M, N)$ is dense, i.e. any open $W \ni w$ contains a point $s \in S$. Since any open set is the union of those from the base of the topology, it is safe to assume that W itself belongs to the base. Since an open set from the base is the finite intersection $\bigcap_i \mathcal{N}_i(f_i, \dots)$ of those from the prebase, it is safe to assume (by taking the minimum) that ε is the same for all i. All of this boils down the task to finding s that would be ε-close to each f_i (for any ε). In turn, this will be achieved by finding s that would be ε-close to w (for any ε) and then using the triangle inequality.

Our next goal is Theorem 3.8, for which the following partial case is the main step.

Theorem 3.6. The set $C^\infty(M, \mathbb{R}^n)$ is dense in $C(M, \mathbb{R}^n)$.

Proof. We are given a locally finite cover $\{V_\alpha\}_\alpha$ of M, a number $\varepsilon_\alpha > 0$[8] and $f \in C(M, \mathbb{R}^n)$. We want to find $g \in C^\infty(M, \mathbb{R}^n)$ s.t. $|f - g| < \varepsilon_\alpha$ on V_α.

For any $x \in M$ set $\delta_x = \min_{V_\alpha \ni x} \varepsilon_\alpha$. Choose U_x so that $|f(y) - f(x)| < \delta_x$ for any $y \in U_x$. Define a constant function $g_x \colon M \to \mathbb{R}^n$ as $g_x(y) := f(x)$. Relabelling the indices, we have found a cover $\{U_i\}_i$ of M and maps $g_i \colon M \to \mathbb{R}^n$ s.t. if $y \in U_i \cap V_\alpha$, then $|g_i(y) - f(y)| < \varepsilon_\alpha$.

Let $\{\lambda_i\}_i$ is a partition of unity subordinate to $\{U_i\}_i$. Define $g \colon M \to \mathbb{R}^n$ as $g(y) := \Sigma_i \lambda_i(y) g_i$. Then

$$|f - g| = |\Sigma_i \lambda_i f - \Sigma_i \lambda_i g_i| = |\Sigma_i \lambda_i (f - g_i)| \leqslant \Sigma_i \lambda_i |f - g_i|.$$

Suppose $y \in V_\alpha$. If $\lambda_i(y) > 0$ for some i, then $y \in V_\alpha \cap U_i$ and we thus have $(\Sigma_i \lambda_i |f - g_i|)(y) < \Sigma_i \lambda_i \varepsilon_\alpha = \varepsilon_\alpha$. ☐

To derive the general case we will use the following fact. For a closed submanifold $K \subset M$ there is an open subset $T \supset K$ and a smooth surjective map $\pi \colon T \to K$ s.t. $\pi^{-1}(p) \simeq \mathbb{R}^{m-k}$ for any $p \in K$.

Remark 3.7. In fact T and π can be chosen so that they form an *$(m-k)$-dimensional real vector bundle* over K. In such a case T is called a *tubular neighborhood* of K in M. We will use this term for convenience (the only property that will actually be used is stated before the remark).

Theorem 3.8. (Continuous map can be approximated by a smooth one.) The set $C^\infty(M, N)$ is dense in $C(M, N)$.

Proof. Embed N in $\mathbb{R}^k$ and view the given $f \in C(M, N)$ as a map from M to $\mathbb{R}^k$. Let T be a tubular neighbborhood of N in $\mathbb{R}^k$ and replace it with a smaller one for which $|p - \pi(p)| < \varepsilon/2$ (distance is measured w.r.t. embedding in $\mathbb{R}^k$). Approximate f by a smooth g s.t. g is $\varepsilon/2$-close to f and $g(M) \subset T$. The map $\pi \circ g$ is now the desired one. ☐

Corollary 3.9. (Continuous Brouwer fixed point theorem.) Any continous map $G \colon D^n \to D^n$ has a fixed point.

Proof. Suppose not. Equip D^n with some coordinates and set $\mu := \min_{x \in D^n} |G(x) - x|$. Find a smooth $F \colon D^n \to D^n$ s.t. $|F - G| < \mu$. Then F also doesn't have a fixed point, which contradicts Theorem 2.22. ☐

For the corollary below we need the following definiton.

[8] Since for us M is compact, one could have chosen a single ε for all the covering sets, but this doesn't simplify the proof much.

Definition 3.10. A *simplex* in $\mathbb{R}^n$ is a topological subspace $\Delta^n = \{x_1 + \cdots + x_n = 1, x_i \geqslant 0\} \subset \mathbb{R}^n$.

We leave as an exercise that Δ^n is homeomorphic to D^n (note, however, that it is not a smooth submanifold of $\mathbb{R}^n$).

Corollary 3.11. (Perron-Frobenius theorem.) Let M be a real matrix with non-negative entries. Then M has an eigenvector with non-negative coordinates and non-negative eigenvalue.

Proof. Assume M is non-singular since otherwise the statement is trivial. The matrix M gives rise to a map $M\colon \mathbb{R}^n \to \mathbb{R}^n$. By assumption $M(\mathbb{R}^n_{\geqslant 0}) \subset \mathbb{R}^n_{\geqslant 0}$, where $\mathbb{R}^n_{\geqslant 0} = \{x_i \geqslant 0\}$. This allows to define a map $f\colon \Delta^n \to \Delta^n$ as $f(x) := \langle Mx \rangle \cap \Delta^n$[9]. By Corollary 3.9 the map f has a fixed point, which is the desired vector. $\square$

[9] By $\langle v \rangle$ we mean the one-dimensional subspace generated by v.