

Topological Data Analysis

Homework 2

Letters K and L denote simplicial complexes; $(Q, \preceq_Q)$ and $(R, \preceq_R)$ denote posets.

Problem 1. (Join via posets.) For two elements $x, y \in Q \sqcup R$, we say that $x \preceq y$ if 1) $x, y \in Q$ and $x \preceq_Q y$, or 2) $x, y \in R$ and $x \preceq_R y$, or 3) $x \in Q, y \in R$.

- Prove that this defines a poset structure on $Q \sqcup R$. It's called a *join* of posets $(Q, \preceq_Q)$ and $(R, \preceq_R)$ and denoted by $Q * R$.
- Explain how $\Delta(P(\Delta_1) * P(\Delta_1))$ is (expectedly) homeomorphic to $\Delta_1 * \Delta_1 = \Delta_3$.

Remark. More generally, for any K and L one has $K * L \cong \Delta(P(K) * P(L))$.

Problem 2. (Product of simplicial complexes.) For two elements $(q_1, r_1), (q_2, r_2) \in Q \times R$, we say that $(q_1, r_1) \preceq (q_2, r_2)$ if $q_1 \preceq_Q q_2$ and $r_1 \preceq_R r_2$.

- Prove that this defines a poset structure on $Q \times R$. It's called a *product* of posets $(Q, \preceq_Q)$ and $(R, \preceq_R)$ and denoted by $Q \times R$.
- Define the *product* of K and L as a simplicial complex $K \times L := \Delta(P(K) \times P(L))$. Explain how $\Delta_1 \times \Delta_1$ is homeomorphic to a solid square.

Problem 3. Let M be the vertex set of K and $I \in K$ be a simplex (so $I \subset M$).

- Define the *link* of I as a simplicial complex $\text{link}_K I := \{J \subset M \setminus I \mid J \sqcup I \in K\}$ (on the vertex set $M \setminus I$). Prove that if K is pure, then so is $\text{link}_K I$ and $\dim \text{link}_K I = \dim K - \dim I - 1$ (for any I).
- Define the *star* of I as a simplicial complex $\text{star}_K I := \{J \subset M \mid I \cup J \in K\}$ (on the vertex set M). Prove that $\text{star}_K I = \Delta_I * \text{link}_K I$.

Remark. Consult examples on the picture (courtesy Anton Ayzenberg).

Problem 4.

- (Explicit construction of n -octahedron.) Prove that $S^0 * \dots * S^0$ (n times) is the boundary of a solid n -octahedron.
- (Spheres are closed under joins.) Prove that $S^n * S^m = S^{n+m+1}$.

Hint (to both parts). Play with small values of n (and m) first.

