Topological Data Analysis
Homework 3

Problem 1. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be an affine map.

a) Prove that if $C \subset \mathbb{R}^n$ is convex, then $f(C)$ is convex as well. Is the preimage of a convex set always convex?

b) For $X \subset \mathbb{R}^n$ arbitrary, prove that $\operatorname{conv}(f(X)) = f(\operatorname{conv}(X))$.

Problem 2. Let $X \subset \mathbb{R}^n$ be a compact (i.e. bounded and closed) set. Prove that $\operatorname{diam}(\operatorname{conv}(X)) = \operatorname{diam}(X)$, where the *diameter* $\operatorname{diam}(Y)$ of a set Y is $\max\{\|x-y\| : x, y \in Y\}$.

Problem 3. Let $K \subset \mathbb{R}^n$ be a convex set and let $C_1, \ldots, C_k \subset \mathbb{R}^n$, $k \geqslant n+1$, be convex sets such that the intersection of every $n+1$ of them contains a translated copy of K. Prove that then the intersection of all the sets C_i also contains a translated copy of K.

Problem 4. A *strip of width* w is a part of the plane bounded by two parallel lines at distance w. The *width* of a set $X \subset \mathbb{R}^2$ is the smallest width of a strip containing X.

a) Prove that a compact convex set of width 1 contains a segment of length 1 of every direction.

b) Let $\{C_1, C_2, \ldots, C_k\}$ be compact convex sets in the plane, $k \geqslant 3$, such that the intersection of every 3 of them has width at least 1. Prove that $\bigcap_{i=1}^{n} C_i$ has width at least 1.

Remark to problem 4: for simplicity, you may assume that every line intersects the boundary of any compact set in question along a point.