

Topological Data Analysis

Homework 4

Problem 1. (Kirchberger's theorem in 2D.) Two finite subsets of $\mathbb{R}^2$ are called *line-separable* if there is a line that doesn't meet either of them and splits $\mathbb{R}^2$ into two half-planes both containing one of the mentioned subsets.

Let $X, Y \subset \mathbb{R}^2$ be two finite subsets. Suppose that for every subset $S \subset X \cup Y$ of cardinality at most 4, $S \cap X$ and $S \cap Y$ are line-separable. Prove that X and Y are line-separable.

Problem 2. For a finite $A \subset \mathbb{R}^n$ call $x \in \mathbb{R}^n$ a *Radon point* of A if x is contained in the convex hulls of two disjoint subsets of A^1 .

Suppose $|A| = n + 2$ and points of A are in general position; in other words, no $n + 1$ of them are affinely dependent. Prove that Radon point is unique.

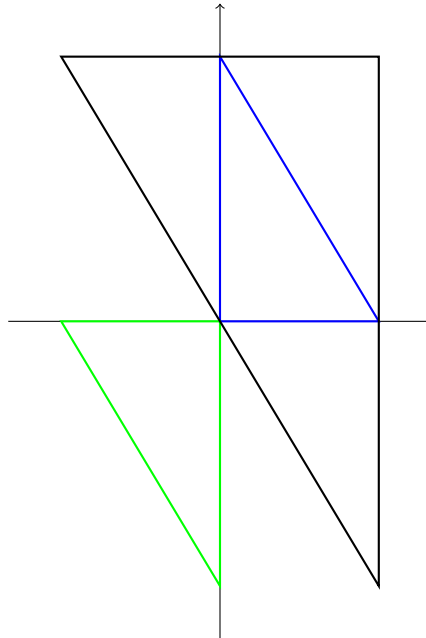
Problem 3. This exercise presents a geometrical derivation of Caratheodory's theorem from Helly's (for convex polytopes). Consider a finite $X \subset \mathbb{R}^n$ and $y \in \text{conv}(X) \setminus X$.

- For any $x \in X$ let H_x be the closed half-space bounded by the hyperplane through x perpendicular to xy s.t. $y \notin H_x$. Prove that $\bigcap_{x \in X} H_x = \emptyset$.
- Strengthen this result by proving that $\bigcap_{i=1}^s H_{x_i} = \emptyset$ for some $s \leq n + 1$ and $x_i \in X$.
- Prove that $y \in \text{conv}\{x_i\}_{i=1}^s$.

Hint (to all parts). Argue by contradiction. Understand the case $n = 2$ first.

Problem 4. Let $c_1, \dots, c_k$ be k points in $\mathbb{R}^n$ and define $C := \text{conv}\{c_i\}_i$. Prove that there exists $x \in \mathbb{R}^n$ s.t. $C \subset -nC + x$. In words, C can be covered by its antipode $-C$ (defined as $\{-p : p \in C\}$) scaled by a factor of n . (See the picture below.)

Hint: consider $A_c := \{v \in \mathbb{R}^n \mid c \in -nC + v\}$ for any $c \in C$ and prove that $\bigcap_{c \in C} A_c \neq \emptyset$.



¹Radon's theorem states that if $|A| \geq n + 2$ then Radon point exists.