Topological Data Analysis
Homework 5

Problem 1. Recall that a *regular $n-$gon* is convex planar polygon with n equal sides.

a) Prove that every triangulation of a regular $n-$gon is lexicographic...

b) ...and regular.

c) The set of all *bracket sequences* is obtained by the following rules: (a) an empty sequence is a bracket sequence; (b) if S and T are bracket sequences, then so is ST (i.e. concatenation of the two); (c) if S is a bracket sequence, then so is (S). Prove that there are as many bracket sequences with $n-2$ pairs of brackets as there are triangulations of a regular $n-$gon. As an example, here is the list of bracket sequences with 3 pairs of brackets: ((())), (())(), ()(()), ()()(), (()()).

Problem 2. (Triangulating the cube.) Consider the $n-$cube $C_n := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \in [0,1]\}$ (alternatively defined as the convex hull of 2^n verticies $\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \in \{0,1\}\}$). Recall that a *permutation* of size n is a bijective map from $\{1, \ldots, n\}$ to itself. For every permutation σ of size n define $P_\sigma := \{(x_1, \ldots, x_n) \in C_n \mid x_{\sigma(1)} \leqslant x_{\sigma(2)} \leqslant \ldots \leqslant x_{\sigma(n)}\}$. Prove that P_σ is a simplex, describe explicitly its verticies (in terms of σ) and, finally, prove that the collection of all P_σ provides a triangulation of C_n.
Hint: Play with small values of n first.

Problem 3. Let $X \subset \mathbb{R}^n$ be finite and $\omega, \omega' \colon X \to \mathbb{R}$ be two height functions such that $\omega - \omega'$ is a restriction of an affine map $\mathbb{R}^n \to \mathbb{R}$ to X. Prove that if ω gives rise to a regular triangulation, then so does ω' and the two triangulations are the same. Conclude that for any regular triangulation the values of the corresponding height function may be assumed (without loss of generality) to be zero on any chosen affinely independent subset of X.

Problem 4. Prove that the triangulation on...

a) ...the figure 1 is not regular

b) ...the figure 2 is regular.

Remark. Exact coordinates are not needed for either part (but you can introduce them if you want). In figure 2 the outer triangle is rotated counterclockwise by a small angle.

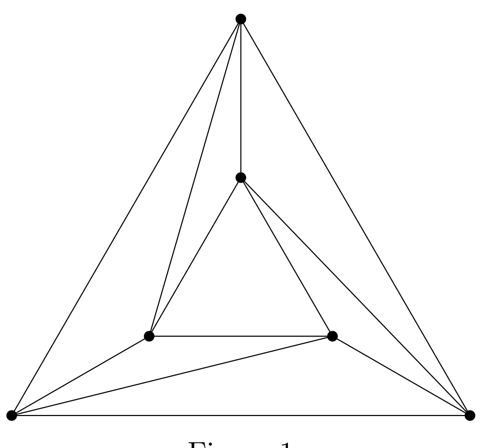

Figure 1

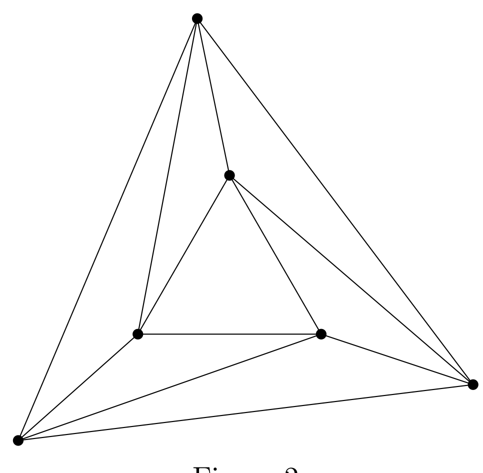

Figure 2