TOPOLOGICAL DATA ANALYSIS

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Foreword. Topological data analysis is a pipeline of several notions from pure mathematics. It is impossible to cover all of them in a rigourous way in one term (algebraic topology alone requires two). The chosen approach is to blackbox some of the definitions (and, more conventionally, theorems), providing a supply of examples, presumably sufficient to acquire an intuition.

1. Simplicial complexes

Basic definitions. We start with pure combinatorics. By 2^M one denotes the power set of M, i.e. the set of all subsets of M.

Definition 1.1. Let M be a set. Then the subset $K \subset 2^M$ is called a *simplicial complex on* M if

(1) if $I \in K$ and $J \subset I$, then $J \in K$;

(2) $\varnothing \in K$.

The set M is then called the *vertex set of* M and denoted by $V(M)$. The element $I \in K$ is called a *simplex in* K.

Remark 1.2. Thanks to the first condition, the second one is equivalent to $K \neq \varnothing$. Since the empty set is always there anyways, we will omit it in writing when defining a concrete K.

In what follows we will always assume that M is finite.

Example 1.3. $M = \{1, 2, 3\},\ K = \{\{1\}, \{3\}, \{1, 3\}\}$. Note that the vertex 2 doesn't belong to K (more formally, $\{2\} \notin K$). We'll call such vertices *illusory.*

Non-example 1.4. $M = \{1, 2, 3\},\ K = \{\{1\}, \{2\}, \{3\}, \{1, 2, 3\}\}$.

Example 1.5. $K = 2^M$. This simplicial complex is called a *simplex on* M and denoted by Δ_M. Its *boundary* is defined as $\partial\Delta_M \coloneqq \Delta_M \setminus M$.

By $[n]$ one denotes the set $\{0, 1, \ldots, n\}$; one writes Δ_n for $\Delta_{[n]}$. The number $|I| - 1$ is called the *dimension* of the simplex I; one then says that I is an $(|I| - 1)-$simplex. Such a notion is explained by the following geometrical standpoint on simplicial complexes.

Definition 1.6. *Geometrical n-simplex* is the subset $\{(x_0, \ldots, x_n) \mid \sum_{i=0}^n x_i = 1, \forall i\ x_i \geqslant 0\} \subset \mathbb{R}^{n+1}$. Its *verticies* are points with $x_i = 1$ for some $i \in [n]$.

Example 1.7. Cases $n = 0, 1, 2$ evidently give a point, a segment and a triangle respectively. One can see that the case $n = 3$ gives a (solid) tetrahedron, by say, projecting it onto $\{x_3 = 0\} = \mathbb{R}^3 \subset \mathbb{R}^4$.

Remark 1.8. For any subset $J \subset [n]$ the set of points of geometrical $n-$simplex satisfying the condition $x_j = 0\ \forall j \in J$ is a geometrical $(n-|J|)-$simplex. Its verticies are $(n-|J|+1)$ points with $x_i = 1$ for some $i \in [n] \setminus J$ and $x_j = 0\ \forall j \in J$.

Definition 1.9. Let K be simplicial complex on M. Choose a bijection $M \xrightarrow{\sim} [|M|-1]$[1]. A *geometrical realization* of a simplicial complex K is the union of geometrical simplicies in $\mathbb{R}^{|M|}$ corresponding (under the chosen bijection) to simplicies of K.

We often drop the words "geometrical realization" and think of a simplicial complex both combinatorially and geometrically.

Definition 1.10. A simplex $I \subset K$ is called *maximal w.r.t. inclusion*[2] if $\nexists J \in K$ s.t. $I \subsetneq J$.

Since Definition 1.1 obliges all the subsimplicies of I to belong to K as soon as I does, it is enough to specify simplicial complex by its maximal simplicies.

Example 1.11. Although Definition 1.9 gives a direct way to view K as a subset of $\mathbb{R}^{|M|}$, it is rarely of practical use. Say, consider a simplicial complex given by its two maximal simplicies: $\{1,2,3\}$ and $\{3,4\}$. Then Definition 1.9 embeds it into $\mathbb{R}^4$. However, it is much more illustrative to draw it on the plane as a solid triangle with a segment sticking out of its vertex "3".

Definition 1.12. Simplicial complex is K called *pure* if all its maximal simplicies are of the same dimension (say, d). In this case d is called the *dimension* of K.

Topological digression. Eventually, we'll by studying simplicial complexes from a topological point of view. As explained in the foreword, we're not able to be precise here. Armed with this disclaimer, we say that topology studies nice subsets of $\mathbb{R}^n$, called *topological spaces*.

Example ■ 1.13. [3] Geometrical realization of a simplicial complex K is a topological space (in $\mathbb{R}^{|M|}$).

Example ■ 1.14. An *n-sphere* $S^n := \{\sum_{i=0}^n x_i^2 = 1\} \subset \mathbb{R}^{n+1}$ is a topological space (in $\mathbb{R}^{n+1}$). Cases $n=1$ and $n=2$ give a circle and a surface of a tennis ball.

Example ■ 1.15. An *$(n+1)$-ball* $B^{n+1} := \{\sum_{i=0}^n x_i^2 \leqslant 1\} \subset \mathbb{R}^{n+1}$ is a topological space.

To make this notion useful, we better define which maps between topological spaces we are interested in[4]: those are called *continuous* maps. The name suggests informal definition: a map $f\colon A \to B$ between topological spaces A and B is called continuous if continous movement of a point $a \in A$ gives rise to a continuous movement of a point $f(a) \in B$.

One approach to a rigourous definition is as follows. Euclidean distance in $\mathbb{R}^n$ allows one to speak of distances in $A \subset \mathbb{R}^n$. A sequence $(a_n)_n$ of points in A is said to *converge* to $a \in A$ if the sequence of distances $(d(a_n,a))_n$ converges to zero. The map $f\colon A \to B$ is called continuous whenever if $(a_n)_n$ converges to a, then $(f(a_n))_n$ converges to $f(a)$. This should be compared with Heine's defintion of continuity of a function studied in analysis.

Finally, topology studies topological spaces considered up to a *homeomorphism*: A and B are called homeomorphic whenever there is a continuous bijection between them s.t. its inverse is also continuous[5]. Notation: $A \cong B$. One common informal example of a homeomorphism is bending and twisting of a space inside $\mathbb{R}^n$ without tearing it apart. Note that, however, the dimension of ambient space, n, is not important when it comes to homeomorphism. Say, in Example 1.11 one of the two homeomorphic spaces is embedded in $\mathbb{R}^4$ and another one in $\mathbb{R}^2$.

Fact ■ 1.16. $S^{n-1} \cong \partial\Delta_n$.

Idea of a proof. We will construct a bijective map $f\colon \partial\Delta_n \to S^{n-1}$. To this aim, recall that $\partial\Delta_n \subset \mathbb{R}^n \subset \mathbb{R}^{n+1}$ (the latter embedding is that of an affine hyperplane $\sum_{i=0}^n x_i = 1$). Note

[1] In plain words, enumerate points of M from 0 to $|M|-1$.

[2] Colloqially, just maximal.

[3] Statements whose proofs use notions that were not defined rigorously, but rather blackboxed, end with a blackbox.

[4] Just as in linear algebra the notion of a linear map is neccessary to talk about vector spaces.

[5] This is analogous to a notion of isomorphism of vector spaces.

that $\{\sum_{i=0}^{n} x_i = 1\} \cap \{\sum_{i=0}^{n}(x_i - 1/n)^2 = 1\}$[6] is homeomorphic to S^{n-1} (excercise: construct explicitly a map that proves it). Now that both S^{n-1} and $\partial\Delta_n$ are embedded in the same $\mathbb{R}^n = \{\sum_{i=0}^{n} x_i = 1\}$, we define $f(x)$ to be $r(x) \cap S^{n-1}$, where $r(x)$ is a ray emanating from $(1/n, \ldots, 1/n)$ that passes through $x \in \partial\Delta_n$. Bijectivity is easily seen by considering a similar map in the opposite direction. (Continuity of both maps is out of our scope.) □

Remark 1.17. One similarly proves a homeomorphism $B^n \cong \Delta_n$.

So, we found a simplicial complex which is homeomorphic to an $n-$sphere. We say colloquially that $\partial\Delta_n$ provides a *simplicial model* for S^n.

Platonic solids. In this subsection we work exclusively in $\mathbb{R}^3$. Introduce the following aliases:

- 0-polytope[7] a.k.a. vertex: simply a point,
- 1-polytope a.k.a. edge: a straight segment,
- 2-polytope a.k.a. face: a solid planar polygon (i.e. a triangle, a square, a pentagon, ...).

Finally, a $k-$polytope which is a part of some bigger polytope is called a $k-$*face* of this bigger one. For example, a square has four 1-faces and four 0-faces. Note that a $k-$face is uniquely determined by its corner verticies; we say that a $k-$face is *spanned* by its corner verticies. You can view how the main heroes of this subsection are rotating here.

Consider now a standard solid cube in $\mathbb{R}^3$. It is a 3-dimensional subset of $\mathbb{R}^3$, whose boundary is a union of 2-polytopes. Such a subset is called 3-polytope[8]. It has 8 vertices, 12 edges and 6 faces. Now pick a point p_i in the interior of each face (for concreteness and nicety of pictures one can take the middle). Define the boundary of the *dual polytope* as follows[9]:

- the verticies are p_i,
- verticies $p_{i_1}, \ldots, p_{i_l}$ span a k-face precisely when the intersection of the corresponding faces of the initial polytope (i.e. a cube in our case) is non-empty.

This construction gives a 3-polytope with 6 verticies, 12 edges and 8 faces, called *octahedron*. Applying this construction to octahedron gives one a cube back (check it). This justifies the name: duality applied twice usually outputs the object one started with[10]. Note that the boundary of an octahedron is a pure 2-dimensional simplicial complex, since each of its $k-$face is actually a $k-$simplex.

The dual of a tetrahedron is a tetrahedron itself (check it). Surely, as discussed above, it is also a simplicial complex.

Finally, consider the *dodecahedron* (a Wikipedia picture where all of its verticies are seen is here). Unlike cube, it is not so easy define rigoursly; say, one can list explicitly the coordinates of all its vertices, but this is not illuminating. Some of the illuminating constructions require geometry and are out of our scope. At any rate, as seen from the picture, it has 20 vertices, 30 edges and 12 pentagonal 2-faces. Careful construction of the dual polytope described above gives an *icosahedron* (picture is here). Note that its boundary is a pure 2-dimensional simplical complex. Check that the dual of icosahedron is expectedly a dodecahedron.

To summarize, we have 5 3-polytopes, called Platonic solids: one self-dual, 4 others split into two dual pairs. The boundaries of three of the five (tetrahedron, octahedron and icosahedron) are 2-dimensional simplicial complexes. They provide 3 simplicial models for S^2.

[6] The second set is a (shifted) $n-$sphere.

[7] "k-polytope" is a shortening of "$k-$dimensional polytope".

[8] We don't give the formal definition, since it's not needed for a handful of examples of polytopes that we will consider.

[9] Dual polytope itself is the shape enclosed by this boundary.

[10] Compare with the notion of dual of a finite-dimensional vector space in linear algebra.

Higher-dimensional analogs. Here we'll be working with $n-$polytopes in $\mathbb{R}^n = (x_1, \dots, x_n)$. The terminology is consistent with the previous subsection if one adopts the convention that "face" is a shortening of "$(n-1)$-face". In what follows it is important and instructive to first substitute $n = 3$ to recover the results of the previous subsection.

First off, note that $n-$simplex is an example of an $n-$polytope. It has $\binom{n+1}{k+1}$ $k-$faces which are themselves $k-$simplicies (for any k). These $k-$faces are enumerated by $(k+1)-$element subsets $I \subset [n]$, hence the binomial coefficient. It is obviously simplicial and self-dual. It's a generalization of a tetrahedron.

Next, define the *$n-$cube* as $C_n \coloneqq \{x_i \in [-1,1], i = 1, \dots, n\} \subset \mathbb{R}^n$. It is an $n-$polytope on 2^n verticies, that have coordinates $(\pm 1, \dots, \pm 1)$. It has $2n$ faces, which are $(n-1)-$cubes inside affine hyperplanes $\{x_i = \pm 1\}, i = 1, \dots, n$. More generally, it has $\binom{n}{k} 2^{n-k}$ $k-$faces, which are $k-$cubes. To see this, argue as follows. First, one has to choose a linear subspace that is parallel to the affine one containg a $k-$face. There are $\binom{n}{k} = \binom{n}{n-k}$ ways to do this, i.e. subsets $I \subset \{1, \dots, n\}$; a linear subspace is then given by $(n-k)$ equations $\{x_{i \in I} = 0\}$. Second, to get the affine subspace, one has to fill each of the chosen $n-k$ coordinates with 1 or -1. This introduces a factor of 2^{n-k}.

The dual of the $n-$cube is called *n-octahedron*[11]. It has $2n$ verticies (that geometrically can be thought of as the middles of the corresponding faces of $n-$cube). More generally, $k-$faces of the dual polytope correspond to $(n-k-1)-$faces of the initial one. So, $n-$octahedron has $\binom{n}{n-k-1} 2^{n-(n-k-1)} = \binom{n}{k+1} 2^{k+1}$ $k-$faces. These $k-$faces are $k-$simplicies. To see this, note that any $(n+1)$ $(n-1)-$faces of the $n-$cube don't intersect. One of the exercises provides explicit construction of the corresponding simplicial complex.

To summarize, we have three series of $n-$polytopes: $n-$simplex, $n-$cube and $n-$octahedron. The first one is self-dual, the other two are dual to each other. The boundaries of $n-$simplex and $n-$octahedron are pure $(n-1)-$dimensional simplicial complexes and provide simplical models for S^{n-1}. Icosahedron and dodecahedron don't have similar generalizations; they are exceptional Platonic solids.

Simplicial maps. (To be filled.)

Operations on simplicial complexes. One can construct new simplicial complexes out of old. In what follows we denote the vertex set of a simplicial complex K by $V(K)$ (this is what used to be M before).

Definition 1.18. For two simplicial complexes K and L their *disjoint union* $K \sqcup L$ is defined as follows: the vertex set is $V(K) \sqcup V(L)$, the set of simplicies is $K \sqcup L$.

Geometrically this corresponds to placing two complexes side by side.

Example 1.19. $K = \Delta_{\{1,2\}}$, $L = \Delta_{\{3,4\}}$, $K \sqcup L = \{\{1\}, \{2\}, \{1,2\}, \{3\}, \{4\}, \{3,4\}\}$.

Definition 1.20. For two simplicial complexes K and L their *join* $K * L$ is defined as follows: the vertex set is $V(K) \sqcup V(L)$, the set of simplicies is comprised of subsets $I \sqcup J \subset V(K) \sqcup V(L)$, for all possible choices of $I \in K$ and $J \in L$.

Geometrically, this corresponds to considering the union of all the segments starting at K and ending at L, provided that K and L are embedded in $\mathbb{R}^n$ is such a way that the afore-mentioned segments don't intersect each other (except at the endpoints).

Example 1.21. $K = \Delta_n = \Delta_{\{0,\dots,n\}}$, $L = \Delta_m = \Delta_{\{n+1,\dots,n+m+1\}}$. Note that every subset $I \subset \{0, \dots, n+m+1\}$ is a disjoint union of some $I_1 \subset \{0, \dots, n\}$ and some $I_2 \subset \{n+1, \dots, n+m+1\}$. We conclude that $\Delta_n * \Delta_m = \Delta_{n+m+1}$.

[11] More conventional terms are hyperoctahedron and cross-polytope.

Example 1.22. As an example of an example, take $n = m = 1$. We get that solid tetrahedron is a join of two of its opposite edges. (The whole picture can be drawn in $\mathbb{R}^3$.)

Definition 1.23. For a simplicial complex K, its *cone* CK is defined as $K * \mathrm{pt}$; its *suspension* ΣK is defined as $K * S^0$[12].

Geometrically, cone is formed by taking a point above K and considering all the segments that start at this point and end at K. Suspension is the union of two cones. Note the canonical embeddings $K \hookrightarrow CK \hookrightarrow \Sigma K$.

Posets.

Definition 1.24. A *partial order* on a set X is a relation $\preccurlyeq$ which is

(1) reflexive, i.e. $x \preccurlyeq x$,

(2) antisymmetric, i.e. if $x \preccurlyeq y$ and $y \preccurlyeq x$, then $x = y$,

(3) transitive, i.e. if $a \preccurlyeq b$ and $b \preccurlyeq c$, then $a \preccurlyeq c$.

A set with a chosen partial order is called a *poset* and denoted by $(X, \preccurlyeq)$.

Example 1.25. If S is a set, then the power set 2^S is a poset under inclusion: $\preccurlyeq = \subset$.

Example 1.26. Natural numbers is a poset under divisibility: $a \preccurlyeq b \Leftrightarrow b \,\vdots\, a$.

Example 1.27. Let K be a simplicial complex. The set of all its simplicies is a poset under inclusion. It's called a *face poset* of K[13] (notation: $P(K)$).

Example 1.28. If K has $\{1, 2, 3\}$ and $\{2, 4\}$ as maximal simplicies, then $P(K)$ has 9 elements, arranged into 3 "layers" of size 4, 4 and 1 according to the dimension of the correspoindng simplex. (Pic.)

Definition 1.29. Let $(X, \preccurlyeq)$ be a poset. A *chain* in X is a sequence $x_1 \prec \cdots \prec x_n$[14] with $x_i \in X$. The number k is then called a *length* of a chain. A chain is called *maximal* if it can't be extended to a longer one.

Definition 1.30. Let $(X, \preccurlyeq)$ be a poset. Simplicial complex $\Delta(X)$, called *order complex*, is defined as follows: (1) $V(\Delta(X)) = X$, (2) simplicies of $\Delta(X)$ are chains in X.

Remark 1.31. Note that the verticies of each simplex in $\Delta(X)$ are naturally linearly ordered. Simplicial complex structure on $\Delta(X)$, however, doesn't keep track of this linear order.

Example 1.32. Applying $P(\cdot)$ to Example 1.26 (for the first 8 numbers) gives a simplicial complex with maximal simplicies $\{1, 5\}, \{1, 7\}, \{1, 2, 6\}, \{1, 3, 6\}, \{1, 2, 4, 8\}$. (Pic.)

We now have:

$$\text{Simplicial complexes} \quad \underset{\Delta(\cdot)}{\overset{P(\cdot)}{\rightleftarrows}} \quad \text{Posets}$$

Definition 1.33. For a simplicial complex K, the complex $\operatorname{sd} K := \Delta(P(K))$ is called a (first) *barycentric subdivision* of K.

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[12] Note that $S^0 = \mathrm{pt} \sqcup \mathrm{pt}$.

[13] The words 'simplex' and 'face' are sometimes used interchangebly. At other times, the former is an example of the latter.

[14] $x_1 \prec x_2$ means that $x_1 \preccurlyeq x_2$ and $x_1 \neq x_2$.

Fact ■ 1.34. $K \cong \operatorname{sd} K$.

To see this, note, first, that it is enough to prove the statement for a simplex, since barycentric subdivision of K is effectively that of all its simplicies. This is done by induction on dimension. The step of this induction relies on barycentric coordinates.

This fact is meant to encourage to think of posets and simplicial complexes as of two incarnations of the same entity.

Example 1.35. The case $K = \Delta_2$ is a dissection of a triangle into 6 smaller ones by three (say) medians. (Pic.)

Iterating barycentric subdivision sufficiently many times makes simplicies arbitrarily small. More formally, recall from "Topological digression" subsection that one can speak of a distance between two points in K. The diameter of a simplex is defined as maximal distance between any two of its points. The statement is now that the diameter of any simplex in $\operatorname{sd} K$ is at most $\frac{n}{n+1}$ times the diameter of the biggest simplex in K (here n is the maximal dimension of all the simplicies). Since $\left(\frac{n}{n+1}\right)^l \xrightarrow{l\to\infty} 0$, the initial statement follows (here l is the number of iterations of subdivision).

Example 1.36. Take any simplicial model of an $n-$sphere (boundary of a simplex or hyperoctahedron). Subdivide sufficiently many times so as to find k top simplicies that don't intersect. (Say, for the boundary of a simplex, without doing any subdivision, one can only find one such simplex, since any two intersect.) Remove them from simplicial complex. The result is a simplicial model for an *$n-$sphere with k holes*.

2. Convex geometry

Recall that the notions of linear subspace, combination, hull and dependence all have affine analogs.

Definition 2.1. An *affine subspace* of $\mathbb{R}^n$ is a subset $L + v$, where L is a linear subspace and $v \in \mathbb{R}^n$. An *affine combination* of points $x_1, \dots, x_k$ is $\sum_{i=1}^k \alpha_i x_i$, s.t. $\sum_i \alpha_i = 1$. An *affine hull* of a subset $X \subset \mathbb{R}^n$ is $\{\sum_{i=1}^k \alpha_i x_i \mid k \in \mathbb{N},\, x_i \in X,\, \sum_i \alpha_i = 1\}$. Finally, points $x_1, \dots, x_k$ are called *affinely dependent* if $\sum_i \alpha_i x_i = 0$ for some $\alpha_i \in \mathbb{R}$ s.t. $\sum_i \alpha_i = 0$ and at least one of the α_i's is non-zero.

Remark 2.2. Given k points $x_1, \dots, x_k$ one consider $(k-1)$ vectors $x_1 - x_k, \dots, x_{k-1} - x_k$. Under this operation, affine notions correspond precisely to linear ones. For example, $x_1, \dots, x_k$ are affinely dependent if and only if $x_1 - x_k, \dots, x_{k-1} - x_k$ are linearly dependent (check this). This explains the choice of conditions in the above definitions.

Example 2.3. An example of an affine subspace is an *affine hyperplane*, defined as $L + v$ for $L \subset \mathbb{R}^n$ of codimension 1[15] and some $v \in \mathbb{R}^n$. It is given by equation $\langle a, x \rangle = b$, where $a \in \mathbb{R}^n$, $b \in \mathbb{R}$.

Definition 2.4. A subset $C \subset \mathbb{R}^n$ is called *convex* if $tx + (1-t)y \in C$ for any $x, y \in C$ and $t \in [0,1]$. A *convex combination* of points $x_1, \dots, x_k$ is $\sum_{i=1}^k \alpha_i x_i$, s.t. $\sum_i \alpha_i = 1$ and $\alpha_i \geqslant 0$. An *convex hull* $\operatorname{conv} X$ of a subset $X \subset \mathbb{R}^n$ is $\{\sum_{i=1}^k \alpha_i x_i \mid k \in \mathbb{N},\, x_i \in X,\, \sum_i \alpha_i = 1,\, \alpha_i \geqslant 0\}$. Finally, points $x_1, \dots, x_k$ are called *convex dependent* if $\sum_i \alpha_i x_i = 0$ for some $\alpha_i \in \mathbb{R}$ s.t. $\sum_i \alpha_i = 0$, $\alpha_i \geqslant 0$ and at least one of the α_i's is non-zero.

It is straightforward to check that convex hull of any set is a convex subset.

Example 2.5. If $X \subset \mathbb{R}^n$ is finite then $\operatorname{conv} X$ is called a *convex polytope*. Its *vertices* are points of X that are in the boundary of $\operatorname{conv} X$.

[15]That is to say, $\dim L = n - 1$.

Example 2.6. Given $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ one can consider a closed *half-space*, defined as a set ruled out by the inequality $\langle a, x\rangle \geqslant b$. It is convex.

Recall that a subset $X \in \mathbb{R}^n$ is called *compact* if it is closed (i.e. contains all its limit points) and bounded (i.e. contained in some $n-$ball).

Theorem 2.7. (Separation theorem.) Let $C, D \subset \mathbb{R}^n$ be two compact convex subsets s.t. $C \cap D = \varnothing$. Then there is an affine hyperplane $H \subset \mathbb{R}^n$ that separates C and D.

Proof. Consider the product $C \times D \subset \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$ and equip it with a function $f\colon C \times D \to \mathbb{R}$ defined as $f(x, y) := \|x - y\|$. Since $C \times D$ is compact it attains its minimum, say, on $(p, q) \in C \times D$. Define H as a hyperplane perpendicular to the segment pq and containing the midpoint $\frac{p+q}{2}$. Its equation is $\langle p - q, x\rangle = \langle p - q, \frac{p+q}{2}\rangle$ (check this). Note that $H \cap C = \varnothing$ since otherwise the segment pr (where $r \in H \cap C$) would contain a point closer to q than p. Similarly $H \cap D = \varnothing$. Since p and q are separated by H, we're done. □

Remark 2.8. Note that convex geometry by itself is unaware of distances — this is the territory of metric geometry. The interplay between the two is tight, however, as demonstrated by the above proof.

Theorem 2.9. (Radon's theorem.) Let $p_1, \dots, p_{n+2}$ be points in $\mathbb{R}^n$. Then there exist subsets $I_1, I_2 \subset \{1, \dots, n+2\}$ s.t. $I_1 \cap I_2 = \varnothing$ and $\operatorname{conv}(p_i)_{i\in I_1} \cap \operatorname{conv}(p_i)_{i\in I_2} \neq \varnothing$.

Proof. Given points are neccesarily affinely dependent, i.e. $\sum_{i=1}^{n+2} \alpha_i p_i$ for some α_i s.t. $\sum_{i=1}^{n+2} \alpha_i = 0$. We claim the sets $I_1 := \{i \mid \alpha_i > 0\}$, $I_2 := \{i \mid \alpha_i < 0\}$ are the desired ones. To this aim, we will construct explicitly $x \in \operatorname{conv}(p_i)_{i\in I_1} \cap \operatorname{conv}(p_i)_{i\in I_2}$. Define $x := \frac{1}{S}\sum_{i\in I_1} \alpha_i p_i$, where $S := \sum_{i\in I_1} \alpha_i$. Since $\alpha_i/S > 0$ and $\frac{1}{S}\sum_{i\in I_1} \alpha_i = 1$, the combination is (tailor-made to be) convex and $x \in \operatorname{conv}(p_i)_{i\in I_1}$. Next, since $\sum_{i\in I_1} \alpha_i + \sum_{i\in I_2} \alpha_i = 0$, we have $x = -\frac{1}{S}\sum_{i\in I_2} \alpha_i p_i$ and thus $x \in \operatorname{conv}(p_i)_{i\in I_2}$. We're done. □

Theorem 2.10. (Helly's theorem.) Let $C_1, \dots, C_k \subset \mathbb{R}^n$ be a collection convex subsets, $k \geqslant n + 1$. Suppose any $n + 1$ of them intersect. Then all of them intersect.

Proof. We use induction on k. (Note that the case $k = n + 1$ is trivial.)
<u>Base ($k = n + 2$).</u> By assumption, one can find a point $x_j \in \cap_{i\neq j} C_i$ for any j. Apply Radon's theorem to these $n + 2$ points to get subsets $J_1, J_2 \subset \{1, \dots, n+2\}$ s.t. $J_1 \cap J_2 = \varnothing$ and a point $x \in \operatorname{conv}(x_j)_{j\in J_1} \cap \operatorname{conv}(x_j)_{j\in J_2}$. We claim that $x \in \cap_{i=1}^{n+2} C_i$. Indeed, fix any i_0 from 0 to $n + 2$. Suppose that $i_0 \notin J_1$ (the other alternative, $i_0 \notin J_2$, is treated similarly). By definition of x_j's we have that $x_j \in C_{i_0}$ for any $j \in J_1$. By convexity of C_{i_0}, $\operatorname{conv}(x_j)_{j\in J_1} \subset C_{i_0}$. Finally, $x \in \operatorname{conv}(x_j)_{j\in J_1} \subset C_{i_0}$ (for any $i_0 \notin J_1$). We're done.
<u>Induction step.</u> Consider new collection of subsets $(C_i \cap C_k)_i$ for all $i = 1, \dots, k - 1$. Take any $(n + 1)-$element subcollection. It gives rise to $(n + 2)-$element subcollection of $(C_i)_i$ (by adding C_k). By assumption, any $(n + 1)$ sets from this subcollection intersect. By the induction base, all elements in this subcollection intersect. Therefore, the same holds for the initial subcollection of $(C_i \cap C_k)_i$. Now induction hypothesis implies that $\cap_{i=1}^{k-1}(C_i \cap C_k) \neq \varnothing$. Therefore, $\cap_{i=1}^{k} C_i \neq \varnothing$. □

Remark 2.11. Note how induction step uses the base, which is indeed unusual. In fact, the argument can be rephrased as induction with the (trivial) base $k = n + 1$ which is free of this quirk. (Radon's theorem is used in the step in this case.) The chosen approach is meant to highlight 1) the core case $k = n + 2$; 2) the set-theoretical trick to deduce the general statement from this core case.

Definition 2.12. Let $X \subset \mathbb{R}^n$ be a $k-$point set. Then a point $x \in \mathbb{R}^n$ is called a *centerpoint* is each closed half-space containing x contains at least $\frac{k}{n+1}$ point of X.

Example 2.13. In the case $n = 1$, centerpoint boils down to what is called the *median* of values $x_1, \dots, x_k \in \mathbb{R}$. It provides an alternative way to "average out" the given set of values. Unlike the mean value, it is doesn't have a disadvantage that a single non-representative point that is far away from the rest shifts the mean value.

Remark 2.14. An alternative definition of a centerpoint is provided by the fact that x is a centerpoint if and only if x belongs to any open half-space γ s.t. $|X \cap \gamma| > \frac{n}{n+1}k$. In the $\Rightarrow$ direction, suppose not, i.e. there is γ s.t. $|X \cap \gamma| < \frac{n}{n+1}k$, then $x \notin \mathbb{R}^n \setminus \gamma$ while $|(\mathbb{R}^n \setminus \gamma) \cap X| \geqslant \frac{1}{n+1}k$, a contradiction. The other direction is similar.

Theorem 2.15. (Centerpoint theorem.) For any finite set $X \subset \mathbb{R}^n$ centerpoint always exists.

Proof. Use the alternative definition. Let γ run through all the open half-spaces s.t. $|X \cap \gamma| > \frac{n}{n+1}k$. The aim is to prove that all of them intersect. To overcome the obstacle that there are infititely many of them, replace each γ with $\gamma \cap \operatorname{conv} X$. This gives finitely many distinct compact convex subsets with the same property, while the task is still to prove that all of them intersect; let $\mathcal{C}$ be this finite collection.

Every element in $\mathcal{C}$ intersects with X along at least $\frac{n}{n+1}k$ points. Applying the inclusion-exclusion formula $|A \cap B| = |A| + |B| - |A \cup B|$ to the collection of all intersections of members of $\mathcal{C}$ with X implies that intersecting with any element from this collection decreases the cardinality by at most $\frac{1}{n+1}k$. Therefore, the intersection of any $d+1$ members of $\mathcal{C}$ has cardinality strictly greater than 0. The proof is concluded by the Helly's theorem. $\square$

Remark 2.16. The centerpoint theorem can be considered as an application of Helly's theorem since it doesn't mention any convex sets in its formulation.

Theorem 2.17. (Caratheodory's theorem.) (To be filled.)