TROIS COULEURS: A NEW NON-EQUATIONAL THEORY

AMADOR MARTIN-PIZARRO AND MARTIN ZIEGLER

Abstract. A first-order theory is equational if every definable set is a Boolean combination of instances of equations, that is, of formulae such that the family of finite intersections of instances has the descending chain condition. Equationality is a strengthening of stability yet so far only two examples of non-equational stable theories are known. We construct non-equational ω-stable theories by a suitable colouring of the free pseudospace, based on Hrushovski and Srour's original example.

1. Introduction

Consider a first order complete theory T. A formula $\varphi(x; y)$ is an *equation* (for a given partition of the free variables into x and y) if, in every model of T, the family of finite intersections of instances $\varphi(x, a)$ has the descending chain condition. The theory T is *equational* if every formula $\psi(x; y)$ is equivalent modulo T to a Boolean combination of equations $\varphi(x; y)$.

Determining whether a particular stable theory is equational is not obvious. So far, the only known *natural* example of a stable non-equational theory is the free non-abelian finitely generated group [14, 10], though the first example of a non-equational stable theory is of combinatorial nature and appeared in unpublished notes of Hrushovski and Srour [7]. They coloured the free pseudospace [4] with two colours in order to obtain two types $r(x, y) \neq r'(x, y)$ which are not *equationally separated*, according to the terminology of [6, Section 2.1], that is, there are sequences $(a_i, b_i)_{i\in\mathbb{N}}$ and $(c_i, d_i)_{i\in\mathbb{N}}$, which can be assumed indiscernible over $\emptyset$, such that $r(a_i, b_i)$ and $r'(c_i, d_i)$ holds for all i, but $r'(a_i, b_j)$ and $r(c_i, d_j)$ holds for $i < j$. In an equational theory, any two distinct types are equationally separated.

All previously known examples of non-equational theories are so, due to the presence of two distinct non-equationally separated types $r(x, y) \neq r'(x, y)$ such that the length of x is 1. In this note, we will build on Hrushovski-Srour's example in order to construct new examples of non-equational theories, where all distinct real types $p \neq q$ in finitely many variables are equationally separated.

2. Equations and indiscernibly closed sets

Most of the results in this section come from [11, 9].

Consider a first order theory T. A formula $\varphi(x; y)$ is an *equation* (with respect to a given partition of the free variables into x and y) if, in every model of T, the

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family of finite intersections of instances $\varphi(x, b)$ has the descending chain condition. An easy compactness argument shows

Lemma 2.1. *The formula $\varphi(x; y)$ is an equation if there is no sequence $(a_i, b_i)_{i\in\mathbb{N}}$ in any model M such that $M \models \varphi(a_i, b_j)$ and $M \not\models \varphi(a_i, b_i)$ for all $i < j$.*

A Ramsey argument shows that, working in a sufficiently saturated model, the sequence (a_i, b_i) can be assumed to be indiscernible of any infinite order type. Thus, if $\varphi(x; y)$ is an equation, then so are $\varphi^{-1}(x; y) = \varphi(y, x)$ and $\varphi(f(x); y)$, whenever f is a $\emptyset$-definable function, which maps finite tuples to finite tuples. Finite conjunctions and disjunctions of equations are again equations. Note that equations are stable formulae.

In [9], an equivalent definition of equations was obtained in terms of indiscernibly closed sets: an element c lies in the *indiscernible closure* $\mathrm{icl}(X)$ of a set X if there is an indiscernible sequence $(a_i)_{i\in\mathbb{N}}$ such that a_i lies in X for $i > 0$ and $a_0 = c$. Note that $X \subset \mathrm{icl}(X)$. A set X is *indiscernibly closed* if $X = \mathrm{icl}(X)$.

Lemma 2.2. [9, Theorem 3.16] *A formula $\varphi(x; y)$ is an equation if and only if the set $\varphi(M, b)$ is indiscernibly closed in in every model M of T.*

Proof. Let us work inside a sufficiently saturated model M. If $\varphi(x; y)$ is not an equation, witnessed by the indiscernible sequence $(a_i, b_i)_{i\in\mathbb{Z}}$, as in Lemma 2.1, the set defined by $\varphi(x, b_0)$ is not indiscernibly closed, for it contains all a_i's with $i < 0$, but does not contain a_0. Conversely, if some instance $\varphi(x, b)$ is not indiscernibly closed, there is an indiscernible sequence $(a_i)_{i\in\mathbb{Z}}$ such that $M \models \varphi(a_i, b)$ for $i < 0$, but $M \not\models \varphi(a_0, b)$. For every j in $\mathbb{Z}$, there is an element b_j in M such that $M \models \varphi(a_i, b_j)$ for $i < j$, but $M \not\models \varphi(a_j, b_j)$. □

The theory T is *equational* if every formula $\psi(x; y)$ is equivalent modulo T to a Boolean combination of equations $\varphi(x; y)$. Since Boolean combinations of stable formulas are stable, equational theories are stable.

Typical examples of equational theories are the theory of an equivalence relation with infinite many infinite classes, the theory of R-modules for some ring R, or the theory of algebraically closed fields.

Equationality is preserved under unnaming parameters and bi-interpretability [8]. It is unknown whether equationality holds if every formula $\varphi(x; y)$, with x a single variable, is a boolean combination of equations.

It is easy to see that T is equational if and only if all completions of T are equational. So for the rest of this section we assume that T is complete and work in a sufficiently saturated model $\mathbb{U}$.

Notice that a theory T is equational if and only if every type p over A is implied by its *equational part* $\{\varphi(x, a) \in p \mid \varphi(x; y) \text{ is an equation}\}$.

Definition 2.3. Given two types $p(x, b)$ and $q(x, b)$, define $p(x, b) \to q(x, b)$ if $q(x, b) \subset \mathrm{icl}(p(x, b))$, or equivalently, if there is an indiscernible sequence $(a_i)_{i\in\mathbb{N}}$ such that all $\models p(a_i, b)$ for $i > 0$ and $\models q(a_0, b)$. If $p(x, y)$ and $q(x, y)$ are the the corresponding (complete) types over $\emptyset$, we write

$$p(x; y) \to q(x; y).$$

A standard argument as in Lemma 2.2 with p instead of φ and and q instead of $\neg\varphi$ yields the following:

Lemma 2.4. *We have $p(x;y) \to q(x;y)$ if an only if there is a sequence $(a_i, b_i)_{i\in\mathbb{N}}$ such that $\models p(a_i, b_j)$ for $i < j$, and $\models q(a_i, b_i)$ for all i. Furthermore, we may assume that the sequence is indiscernible and of any given infinite order type.*

The above characterisation provides an easy proof of the following remark:

Remark 2.5. Clearly $p \to p$. If $p(x;y) \to q(x;y)$, then $p^{-1} \to q^{-1}$, where $p^{-1}(x;y) = p(y;x)$.

Furthermore, if $\mathrm{tp}(a;b) \to \mathrm{tp}(a';b)$, then $a \overset{\mathrm{stp}}{\equiv} a'$. Thus, if $p(x;y)$ implies that x (or y) is algebraic, then $p \to q$ only when $q = p$.

Corollary 2.6. *Let f and g be $\emptyset$-definable functions and a, a', b, b' finite tuples, with $\mathrm{tp}(a;b) \to \mathrm{tp}(a';b')$. Then $\mathrm{tp}(f(a);g(b)) \to \mathrm{tp}(f(a');g(b'))$.*

Corollary 2.7. *A formula $\varphi(x;y)$ is an equation if and only if, whenever a type $p(x,y)$ contains $\varphi(x,y)$ and $p(x;y) \to q(x;y)$, then $\varphi(x,y)$ lies in $q(x;y)$.*

Proof. One direction follows clearly from Lemma 2.2. For the converse, assume that $\varphi(x;y)$ is not an equation and choose an indiscernible sequence $(a_i, b_i)_{i\in\mathbb{N}}$ as in Lemma 2.1. Let p be the common type of the pairs (a_i, b_j), with $i < j$ and q be the common type of the pairs (a_i, b_i). Then $p \to q$ and φ belongs to p, but not to q. □

Definition 2.8. A *cycle of types* is a sequence

$$p_0(x;y) \to p_1(x;y) \to \cdots \to p_{n-1}(x;y) \to p_0(x;y).$$

The cycle is *proper* if all the p_i's are different. The theory T is *indiscernibly acyclic* if there is no proper cycle of types of length $n \geq 2$.

Following the terminology of [6, Section 2.1], two distinct types $p(x;y)$ and $q(x;y)$ are not *equationally separated* if and only $p \to q \to p$.

Remark 2.9. Every indiscernibly acyclic theory is stable.

Proof. If there is a formula $\varphi(x;y)$ in T with the order property, find an indiscernible sequence $(a_i)_{i\in\mathbb{Z}}$ in $\mathbb{U}$ such that $\models \varphi(a_i, a_j)$ if and only if $i < j$. Set $p = \mathrm{tp}(a_1;a_0)$ and $q = \mathrm{tp}(a_{-1};a_0)$. Then $p \neq q$, and since the sequence $(a_i)_{i\neq 0}$ is indiscernible, we have that $p \to q \to p$, so there is a proper cycle of types of length 2. □

Remark 2.10. Every equational theory is indiscernibly acyclic.

Proof. Consider a cycle

$$p_0 \to p_1 \to \cdots \to p_{n-1} \to p_0.$$

By Corollary 2.7, all the types p_i contain the same equations, so they all agree, by equationality of T. □

Definition 2.11. The theory T *satisfies the MS-criterion* if there is some formula $\varphi(x,y)$ and a matrix $(a_{ij}, b_{ij})_{i,j\in\mathbb{N}}$ such that:

(1) $\models \varphi(a_{ij}, b_{il})$ if and only if $j = l$.
(2) $a_{ij}, b_{ij} \equiv a_{ij}, b_{kl}$, whenever $i < k$ and $j < l$.

Lemma 2.12. *If a theory T satisfies the MS-criterion, then there is a proper cycle of types $p \to q \to p$. In particular, the theory is not equational (cf.* [10, Proposition 2.6]*).*

Proof. We may assume that the matrix $(a_{ij}, b_{ij})_{i,j\in\mathbb{N}}$ is indiscernible, that is, the type $\mathrm{tp}(a_{ij}, b_{ij})_{i\in I, j\in J}$ only depends on $|I|$ and $|J|$. Set $p = \mathrm{tp}(a_{00}; b_{00})$, $q = \mathrm{tp}(a_{00}; b_{01})$ and $r = \mathrm{tp}(a_{00}; b_{11})$. Since $(a_{0j}b_{0j})_{j\in\mathbb{N}}$ is indiscernible, we have $q \to p$. Since $(a_{i0}b_{i1})_{i\in\mathbb{N}}$ is indiscernible, we have $r \to q$.

Now, by Definition 2.11 (1), the formula $\varphi(x; y)$ belongs to $p(x; y)$ but not to $q(x; y)$, so $p \neq q$. By Definition 2.11 (2) we have $p = r$, as desired. □

Since p and q contain the same equations, it follows that φ cannot be a boolean combination of equations (cf. [10, Proposition 2.6]).

Let us assume for the rest of this section that T is stable.

Lemma 2.13. *Let $p_0(x, b) \to \cdots \to p_{n-1}(x, b) \to p_0(x, b)$ be a proper cycle of types and b' be some tuple such that $p_0(x, b)$ has only finitely many distinct non-forking extensions to bb'. Then there is a proper cycle of types starting with some nonforking extension $p'_0(x; b, b')$ of $p_0(x, b)$ whose length is a multiple of n.*

Proof. First notice that, whenever $p(x, b) \to q(x, b)$ and $q'(x, b, b')$ is a non-forking extension of $q(x, b)$, then $p(x, b)$ has a nonforking extension $p'(x, b, b')$ with

$$p'(x; b, b') \to q'(x; b, b').$$

Indeed, consider an indiscernible sequence $(a_i)_{i\in\mathbb{N}}$ such that $\models p(a_i, b)$, for $i > 0$, and $q(a_0, b)$. We may assume that a_0 realises $q'(x, b, b')$ and that the sequence $(a_i)_{i\in\mathbb{N}}$ is independent from b' over b. By a Ramsey argument, we may assume that the sequence $(a_i)_{i>0}$ is indiscernible over a_0bb'. Set now $p'(x, b, b')$ to be the type of a_1 over bb', so $p'(x, b, b') \to q'(x, b, b')$, as desired.

Let k now be the number of distinct nonforking extensions of $p_0(x, b)$ to bb'. Working backwards in the cycle of types, we deduce from the above that there is a sequence $r_0(x; b, b') \to \cdots \to r_{n\cdot k}(x; b, b')$, where $r_{n\cdot i+j}(x, b, b')$ is a non-forking extension of $p_j(x, b)$ for each $i \leq k$. Since p_0 has only finitely many distinct nonforking extensions to bb', there are two indices $i < i'$ such that $r_{n\cdot i}(x, b, b') = r_{n\cdot i'}(x, b, b')$. Choose i and i' such that $0 < i' - i$ is least possible. Then

$$r_{n\cdot i}(x; b, b') \to \cdots \to r_{n\cdot i'}(x; b, b')$$

is a proper cycle of types. □

Corollary 2.14. *If T is totally transcendental, then it is indiscernibly acyclic if and only if so is T^{eq}.*

Proof. We need only show that T^{eq} is indiscernibly acyclic, provided that T is indiscernibly acyclic. Assume first that the type $p(x, e)$ starts a proper cycle of types, where e is an imaginary element. Choose a real tuple b such that $\pi_E(b) = e$ for some 0-definable equivalence relation E. Since T is totally transcendental, the type $p(x, b)$ has only finitely many nonforking extensions to $\{b, e\}$, so there is a proper cycle starting with some nonforking extension $p'(x, b, e)$, by the Lemma 2.13. By the Corollary 2.6, if we restrict the types in the cycle to b, we have a cycle of types which must be proper, because e is definable from b.

Since the relation $\to$ is symmetric in x and y, we can now replace x by some real tuple, so T is not indiscernibly acyclic. □

Notation. Given two stationary types $p_1(x, b)$ and $p_2(x', b)$, we denote by $p_1(x, b) \otimes p_2(x', b)$ the type of the pair (a_1, a_2) over b, where $\models p_i(a_i, b)$, for $i = 1, 2$, and $a_1 \mathop{\smile\!\!\!\!\!\!|}_b a_2$.

Observe that

$$p_1(x,b)\otimes\Big(p_2(x',b)\otimes p_3(x'',b)\Big)=\Big(p_1(x,b)\otimes p_2(x',b)\Big)\otimes p_3(x'',b).$$

Lemma 2.15. *Given stationary types $p^j(x^j,y^j,c)$ and $q^j(x^j,y^j,c)$ over a tuple c in $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$ such that*

$$p^j(x^j;y^j,c)\to q^j(x^j;y^j,c),\ \textit{for } j=1,2,$$

then

$$p^1(x^1;y^1,c)\otimes p^2(x^2;y^2,c)\to q^1(x^1;y^1,c)\otimes q^2(x^2;y^2,c).$$

By the above, the lemma generalises to an arbitrary finite product of types.

Proof. For $j=1,2$, choose a tuple b^j and an indiscernible sequence $(a^j_i)_{i\in\mathbb{N}}$ such that $\models p^j(a^j_i,b^j,c)$, for $i>0$, and $\models q^j(a^j_0,b^j,c)$. We may assume that

$$b^1\cup\{a^1_i\}_{i\in\mathbb{N}}\mathop{\smile\!\!\!\!\!\!\;\;|}_{c} b^2\cup\{a^2_i\}_{i\in\mathbb{N}}.$$

Since c is algebraic over $\emptyset$, the sequences $\{a^1_i\}_{i\in\mathbb{N}}$ and $\{a^2_i\}_{i\in\mathbb{N}}$ are both indiscernible over c and therefore mutually indiscernible, by stationarity of strong types, so $\{a^1_i,a^2_i\}_{i\in\mathbb{N}}$ is indiscernible. Notice that (a^1_i,a^2_i) realises $p^1(x^1;b^1,c)\otimes p^2(x^2;b^2,c)$ for $i>0$, and (a^1_0,a^2_0) realises $q^1(x^1;b^1,c)\otimes q^2(x^2;b^2,c)$, as desired. □

Proposition 2.16. *If T is totally transcendental, then it is indiscernibly acyclic if and only if there is no proper cycle of types in T^{eq} of length 2.*

Proof. By Corollary 2.14, we need only prove one direction, so suppose

$$p_0(x,y)\to\cdots\to p_{n-1}(x,y)\to p_0(x,y)$$

is a proper cycle of types with real variables. Since T is totally transcendental, there is a finite tuple c in $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$ such that all nonforking extensions of all p_i's to c are stationary. Lemma 2.13 gives a proper cycle of stationary types

$$p_0(x;y,c)\to\cdots\to p_{k-1}(x;y,c)\to p_0(x;y,c)$$

for some k in $\mathbb{N}$.

Denote by $\bar{x}=(x^0,\dots,x^{k-2})$ and $\bar{y}=(y^0,\dots,y^{k-2})$ and consider the types

$$r_1(\bar{x};\bar{y},c)=p_0(x^0,y^0,c)\otimes p_1(x^1,y^1,c)\otimes\ldots\otimes p_{k-2}(x^{k-2},y^{k-2},c)$$
$$r_2(\bar{x};\bar{y},c)=p_1(x^0,y^0,c)\otimes p_2(x^1,y^1,c)\otimes\ldots\otimes p_{k-1}(x^{k-2},y^{k-2},c)$$
$$r_3(\bar{x};\bar{y},c)=p_1(x^0,y^0,c)\otimes p_2(x^1,y^1,c)\otimes\ldots\otimes p_{k-2}(x^{k-3},y^{k-3},c)\otimes p_0(x^{k-2},y^{k-2},c)$$

The Lemma 2.15 yields the cycle of types

$$r_1(\bar{x};\bar{y},c)\to r_2(\bar{x};\bar{y},c)\to r_3(\bar{x};\bar{y},c).$$

Given $(\bar{a},\bar{b})$ realising $r_1(\bar{x};\bar{y},c)$ and $(\bar{a}',\bar{b}')$ realising $r_2(\bar{x};\bar{y},c)$, notice that

$$(a^1,a^2,\dots,a^{k-2},a^0,b^1,b^2,\dots,b^{k-2},b^0)$$

realise $r_3(\bar{x};\bar{y},c)$. If f denotes the function which maps a $k-1$-tuple $(f^0,\dots,f^{k-1})$ to the imaginary coding the set $\{f^1,\dots,f^{k-1}\}$, Corollary 2.6 implies that

$$\mathrm{tp}(\{a^0,\dots,a^{k-2}\};\{b^0,\dots,b^{k-2}\},c)\to\mathrm{tp}(\{a^1,\dots,a^{k-1}\};\{b^1,\dots,b^{k-1}\},c)\to$$
$$\to\mathrm{tp}(\{a^0,\dots,a^{k-2}\};\{b^0,\dots,b^{k-2}\},c)$$

In order to conclude, we need only show that the above two imaginary types are different. Otherwise, if the two types are equal, we have for each $1 \leq i \leq k-1$, two values $0 \leq \rho(i), \tau(i) \leq k-2$ such that $(a^{\rho(i)}, b^{\tau(i)}) \models p_i(x, y, c)$. Observe that no two elements a^i and a^j, with $i \neq j$, can be equal since the independence $a^i \mathop{\smile\!\!\!\!\!\!\mid}_c a^j$ would imply that a_i is algebraic, and thus $p_{i+1}(x, y, c) = p_i(x, y, c)$, by the Remark 2.5. Likewise, no two elements b^i and b^j can be equal, for $i \neq j$. Thus, each of the maps $i \mapsto \rho(i)$ and $i \mapsto \tau(i)$ is a bijection.

If $\rho(k-1) = \tau(k-1) = j$, then (a^j, b^j) realises both $p_j(x; y, c)$ and $p_{k-1}(x; y, c)$, which contradicts that the cycle of types is proper. Hence, the values $\rho(k-1)$ and $\tau(k-1)$ are different, so there must be some $1 \leq i \leq k-2$ such that $\rho(i) \neq \tau(i)$. The independences

$$a^{\rho(i)} \mathop{\smile\!\!\!\!\!\!\mid}_c b^{\tau(i)} \text{ and } a^{\rho(k)} \mathop{\smile\!\!\!\!\!\!\mid}_c b^{\tau(k)}$$

imply that $p_i(x, y, c) = p_{k-1}(x, y, c)$, by the Remark 2.5 and stationarity of strong types, which yields the desired contradiction.

□

We do not know whether Corollary 2.14 and Proposition 2.16 are true for arbitrary stable theories.

All known examples of non-equational stable theories have a proper cycle of real types of length 2. Indeed, in Hrushovski and Srour's primordial example [7], the type of a white point and the type of a red point in a plane indiscernibly converge to each other, whereas the non-abelian free group satisfies the MS-criterion [10, Lemmata 3.4 & 3.6]. In this note, we will provide new examples of non-equational totally transcendental theories, one for each natural number k, having proper cycles of length k but no proper cycles of real types of length strictly smaller than k. We will do so by suitable colouring the free pseudospace, mimicking the construction of Hrushovski and Srour. The following question seems hence natural, though we do not have a solid guess what the answer will be.

Question. Is there a non-equational indiscernibly acyclic theory?

Related to the above, we wonder whether there is a local characterisation of equationality in terms of cycles of types:

Question. Is a formula $\varphi(x, y)$ a Boolean combination of equations if and only if whenever

$$\varphi \in p_0(x, y) \to p_1(x, y) \to \ldots \to p_{n-1}(x, y) \to p_0(x, y),$$

then φ belongs to p_i for every $i > 0$?

Do two types p and q contain the exact same equations if and only if p and q both occur in a (proper) cycle of types?

Observe that a positive answer to the second question would positively answer the first one.

3. Indiscernible Kernels

To our knowledge, the results in this section only appeared in print form in Adler's Master's Thesis [1] (in German). Therefore, we will include their proofs, even if the results are most likely well-known among the community.

As before, work inside a sufficiently saturated model $\mathbb{U}$ of the complete theory T.

Notation. Given two subsets I_0 and I_1 of a linearly ordered infinite index set with no endpoints, we write $I_0 \ll I_1$ if $i_0 < i_1$ for all i_0 in I_0 and i_1 in I_1. If $(a_i)_{i\in I}$ is a sequence indexed by I, set $\mathrm{acl}^{\mathrm{eq}}(a_{I_0}) = \mathrm{acl}(\{a_i\}_{i\in I_0})$.

Definition 3.1. The *kernel* of the indiscernible sequence $(a_i)_{i\in I}$ is defined as

$$\mathrm{Ker}((a_i)_{i\in I}) = \bigcup_{\substack{I_0, I_1 \subset I \\ I_0 \ll I_1}} \mathrm{acl}^{\mathrm{eq}}(a_{I_0}) \cap \mathrm{acl}^{\mathrm{eq}}(a_{I_1}).$$

Note that we may assume that both I_0 and I_1 are finite subsets of I. Furthermore, the set $\mathrm{acl}^{\mathrm{eq}}(a_{I_0}) \cap \mathrm{acl}^{\mathrm{eq}}(a_{I_1})$ only depends on $|I_0|$ and $|I_1|$ (possibly after enlarging I), since $(a_i)_{i\in I\setminus I_0}$ is indiscernible over a_{I_0}. If the sequence is indiscernible as a set (which is always the case in stable theories), then we may define the kernel by considering all the intersections given by pairs (I_0, I_1) with $I_0 \cap I_1 = \emptyset$.
Observe that (if I is large enough),

$$\mathrm{Ker}((a_i)_{i\in I}) = \mathrm{acl}^{\mathrm{eq}}(a_{I_0}) \cap \mathrm{acl}^{\mathrm{eq}}(a_{I_1}), \text{ for any } I_0 < I_1 \text{ both infinite.}$$

Lemma 3.2. *The kernel K of an indiscernible sequence $(a_i)_{i\in I}$ is the largest subset of $\mathrm{acl}^{\mathrm{eq}}((a_i)_{i\in I})$ over which the sequence is indiscernible.*

Proof. We may assume that I has no endpoints. Clearly, the sequence is indiscernible over K. Given a tuple b in $\mathrm{acl}^{\mathrm{eq}}(a_{I_0})$, for $I_0 \subset I$ finite, such that the sequence is indiscernible over b, the tuple b lies in $\mathrm{acl}^{\mathrm{eq}}(a_{I_1})$, whenever $I_0 < I_1$, so b lies in K. □

Lemma 3.3. *If T is stable, then the kernel K of an indiscernible sequence $(a_i)_{i\in I}$ is the smallest algebraically closed subset (in T^{eq}) over which the sequence is independent.*

Proof. Let E be an algebraically closed subset (in T^{eq}) such that $(a_i)_{i\in I}$ is E-independent. In particular, for each $I_0 < I_1$, we have that

$$a_{I_0} \underset{E}{\downarrow} a_{I_1},$$

so $K \subset E$.
Let now $\mathfrak{p} = \mathrm{Av}((a_i)_{i\in I})$ be the average type, that is,

$$\mathfrak{p} = \{\varphi(x,b)\ \mathcal{L}_{\mathbb{U}}\text{-formula} \mid \varphi(a_i, b) \text{ for all but finitely many } i \in I\}.$$

Since $\mathfrak{p}$ is invariant over every infinite subsequence of $(a_i)_{i\in I}$, its canonical base C is contained in K. Thus, the sequence is C-indiscernible and $\mathfrak{p}$ is a nonforking extension of the stationary type $\mathfrak{p}\restriction_K$.
It suffices to show that $a_i \models \mathfrak{p}\restriction_{K\cup(a_j)_{j<i}}$, since any Morley sequence of $\mathfrak{p}\restriction_K$ has this property and its type over K is unique. Thus, let $\varphi(x, (a_j)_{j<i})$ be a formula in $\mathfrak{p}\restriction_{K\cup(a_j)_{j<i}}$. We may clearly assume that I has no last element. By definition of the average type, there is some $a_t \models \mathfrak{p}\restriction_{K\cup(a_j)_{j<i}}$ with $t \geq i$. By indiscernibility,

$$a_i \models \mathfrak{p}\restriction_{K\cup(a_j)_{j<i}},$$

as desired. □

Corollary 3.4. *In a stable theory T, every indiscernible sequence is a Morley sequence over its kernel.*

Using kernels, we can provide a different characterisation of the relation $\to$ in a stable theory.

Corollary 3.5. *Given types $p(x;y)$ and $q(x;y)$ in a stable theory T, we have that $p \to q$ if and only if there is a set C and tuples a, a' and b such that:*

- $\models p(a,b)$ *and* $\models q(a',b)$*;*
- $a \overset{\text{stp}}{\equiv}_C a'$*, and*
- $a \mathop{\smile\!\!\!\!\!\mid}_C b$*.*

In particular, given a cycle

$$p_0(x,y) \to p_1(x,y) \to \ldots \to p_{n-1}(x,y) \to p_0(x,y),$$

there are tuples b, $a_0, \ldots, a_n$ and subsets $C_0, \ldots, C_{n-1}$ such that:

- $\models p_r(a_r, b)$*, for* $0 \le r \le n-1$*, and* $\models p_0(a_n, b)$*.*
- $a_r \overset{\text{stp}}{\equiv}_{C_r} a_{r+1}$ *for* $0 \le r \le n-1$*.*
- $a_r \mathop{\smile\!\!\!\!\!\mid}_{C_r} b$ *for* $0 \le r \le n-1$*.*

Proof. If $p \to q$, choose some tuple b and an indiscernible sequence $(a_i)_{i<|T|^+}$ such that $q(a_0, b)$ and $p(a_i, b)$ for each $i > 0$. Consider the kernel K of the sequence, which is algebraically closed in T^{eq}, so $a_i \overset{\text{stp}}{\equiv}_K a_j$ for all i, j. In particular, the subsequence $(a_i)_{0<i<|T|^+}$ is Morley sequence over K, so there is some $i_0 < |T|^+$ such that $a_{i_0} \mathop{\smile\!\!\!\!\!\mid}_K b$. Set $C = K$, $a' = a_0$ and $a = a_{i_0}$.
For the other direction, set $a_0 = a'$ and choose for each $0 \neq i$ in $\mathbb{N}$ a realisation $a_i \overset{\text{stp}}{\equiv}_C a$ such that $a_i \mathop{\smile\!\!\!\!\!\mid}_C b, \{a_j\}_{j<i}$. Since strong types are stationary, we have that $p(a_i, b)$, for $i \neq 0$. Furthermore, the sequence $\{a_i\}_{i \ge 0}$ is indiscernible over C, by construction. □

The above provides a simpler characterisation of equations in stable theories (cf. [6, Remark 2.4]).

Remark 3.6. In a stable theory T, a formula $\varphi(x;y)$ is an equation if and only if for every set set C and tuples a, a' and b such that $\varphi(a,b)$ holds with $a \mathop{\smile\!\!\!\!\!\mid}_C b$, then so does $\varphi(a', b)$ hold, whenever $a' \overset{\text{stp}}{\equiv}_C a$.

Proof. Given C, a, a' and b as in the statement, Corollary 3.5 yields that $\text{tp}(a,b) \to \text{tp}(a',b)$. As φ belongs to $\text{tp}(a,b)$, it must lie in $\text{tp}(a',b)$, by Corollary 2.7.

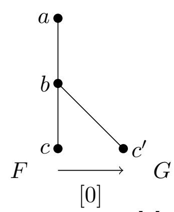
For the other direction, it suffices to show that φ lies in q, whenever φ belongs to p and $p \to q$, by Corollary 2.7. By Corollary 3.5, there are C, a, a' and b such that $p(a,b)$, $q(a',b)$, $a \mathop{\smile\!\!\!\!\!\mid}_C b$ and $a' \overset{\text{stp}}{\equiv}_C a$. Since $\varphi(a,b)$ holds, we conclude that so does $\varphi(a',b)$, that is, the formula φ belongs to q, as desired. □

4. A blank pseudospace

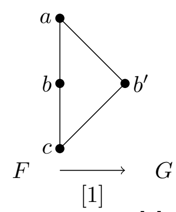
Hrushovski and Srour produced the first example [7] of a non-equational stable theory by adding two colours to an underlying (2-dimensional) free pseudospace, a structure later studied by Baudisch and Pillay [4]. Subsequently, the free (n-dimensional) pseudospace has been considered from different perspectives, either as a lattice [12, 13] or as a right-angled building [2, 3], in order to show that the ample hierarchy is strict. In this section, we will recall the basic properties of the free 2-dimensional pseudospace.

A *geometry* is a graph whose vertices have levels 0, 1 and 2. Vertices of level 0 are called *points* (usually denoted by the letter c), whereas vertices of level 1 are *lines* (denoted by b) and vertices of level 2 are *planes* (denoted by a). By an abuse of notation, we say that *the point c lies in the plane a* if there is a line b contained in a passing through c, though there are no edges between points and planes. We refer to a subgraph of the form $a - b - c$ as a *flag*.

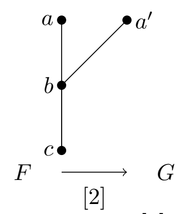
A *letter* s is a non-empty subinterval of $[0, 2]$. Given a flag F in a geometry A, a new geometry B is obtained from A, F via the *operation* s by freely adding a new flag G which coincides with F on the levels in $[0, 2] \setminus s$:



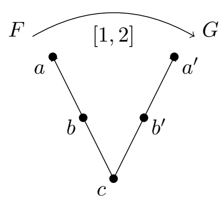
Operation [0]



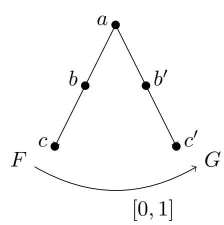
Operation [1]



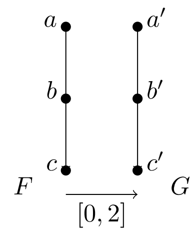
Operation [2]



Operation $[1, 2]$



Operation $[0, 1]$



Operation $[0, 2]$

The *free pseudospace* M_∞ is obtained by successively applying countably many times all of the above operations starting from a flag. The geometry M_∞ is independent, up to isomorphism, of the order in which the operations are applied. It is denoted by M^2_∞ in [2, Definition 4.6]. Observe that the geometry obtained by only considering the operations 0, 1 and 2 is an elementary substructure of M_∞ (namely, the prime model).

We will now exhibit the axioms for the theory PS of M_∞. Let us first fix some notation. A *word* is a sequence of letters. A *permutation* of the word u is obtained by successively replacing an occurrence of the subword $0 \cdot 2$ by the subword $2 \cdot 0$; similarly the subword $2 \cdot 0$ is permuted to $0 \cdot 2$. The word u is *reduced* if it does not contain, up to permutation, a subword of the form $s \cdot t$, where $s \subset t$ or $t \subset s$ (please note that our notation $s \subset t$ does not imply $s \subsetneq t$).

A *flag path*

$$F_0 \underset{s_1}{\longrightarrow} F_1 \cdots F_{n-1} \underset{s_n}{\longrightarrow} F_n$$

with word $u = s_1 \cdots s_n$ is a sequence of flags such that, for each $1 \leq i \leq n$, the flag F_i differs from F_{i-1} exactly in the levels in $[0, 2] \setminus s_i$. The above flag path is *reduced* if its word is reduced and for each i, the flags F_{i-1} and F_i cannot be connected by a *splitting*, that is, a flag subpath whose word consists of proper subletters of s_i. It is not hard to show that every two flags are connected by a reduced path [3, Corollary 3.13].

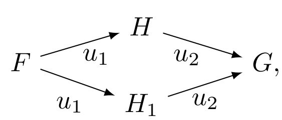
Fact 4.1. [3, Theorem 4.12] The theory PS is axiomatised by the following properties:

(1) The universe is a geometry such that every vertex lies in a flag.
(2) For every level i in $[0,2]$ and every flag F, there are infinitely many flags G with $F \underset{i}{\rightarrow} G$.
(3) Every closed reduced flag path $F_0 \underset{s_1}{\longrightarrow} F_1 \cdots F_{n-1} \underset{s_n}{\longrightarrow} F_0$ has length $n = 0$.

It was proved in [3, Theorem 3.26] that property (3) can be expressed by a set of elementary sentences.

We will now describe types and the geometry of forking in the pseudospace. We refer the reader to [3, Sections 3–7] for the corresponding proofs. Since there are no non-trivial reduced closed paths of flags, the word u connecting two flags F and G by a reduced path $F \underset{u}{\rightarrow} G$ is unique, up to permutation, and will be denoted by $\mathrm{d}(F,G)$. The flags F and G agree modulo a subset S of $[0,2]$, that is, they have the same vertices in all levels off S, if and only if the letters in $\mathrm{d}(F,G)$ are all contained in S. In particular, the collection of points and lines, resp. lines and planes, form a pseudoplane, so every two lines intersect in at most one point, resp. lie in at most one plane. Furthermore, the intersection of two distinct planes is either empty, a unique point or a unique line [4]. Actually, the geometry forms a lattice, once a smallest element **0** and a largest element **1** are added [12].

If $u = \mathrm{d}(F,G) = u_1 \cdot u_2$, given two reduced flag paths



and a vertex p in H of level i which does not *wobble*, that is, such that $u_1 \cdot [i]$ or $[i] \cdot u_2$ is reduced, then p is also a vertex of H_1. In particular, the vertex p is definable over F, G.

A non-empty subset A of M_∞ is *nice* if:

- every vertex in A lies in a flag fully contained in A; and
- every two flags in A are connected by a reduced path of flags in A.

algebraic closure and the definable closure of a set X agree [13, Corollary 5.4] and coincide with the intersection of all nice sets $A \supset X$. If X is finite, then so is the algebraic closure. The quantifier-free type of a nice subset determines its type. More generally:

Fact 4.2. [13, Corollary 3.12] The quantifier-free type of an algebraically closed subset determines its type in PS.

Observe that if we apply one of the operations [0], [1] or [2] to a flag in a nice set A, the resulting geometry is again nice.

Given a flag F and a nice subset A, there is a flag G in A (called a *base-point* of F over A) such that, for any flag G' in A, the word $\mathrm{d}(F,G')$ is the *non-splitting reduction* of $\mathrm{d}(F,G) \cdot \mathrm{d}(F,G)$, that is, whenever a subword $s \cdot t$ or $t \cdot s$ occurs in a permutation of the product $\mathrm{d}(F,G) \cdot \mathrm{d}(F,G)$, with $s \subset t$, we cancel s. If we consider a reduced flag path P connecting F to some base-point G over A with word $\mathrm{d}(F,G)$, the set $A \cup P$ is again nice. Any flag occurring in the nice set P appears in a permutation of the path P.

The theory PS of M_∞ is ω-stable of rank ω^2, equational with perfectly trivial forking and has weak elimination of imaginaries. Forking can be easily described: Given nice sets A and B containing a common algebraically closed subset C, we have that $A \mathop{\smile\!\!\!\!\!|}_C B$ if and only if for every nice set $D \supset C$ and flags F in A and H in B we have that $\mathrm{d}(F,G)$ is the non-splitting reduction of $\mathrm{d}(F,G)\cdot\mathrm{d}(G,H)$, where G is a base-point of F over D. In particular,

$$F \mathop{\smile\!\!\!\!\!|}_G D.$$

Remark 4.3. [13, Proposition 4.3 & Theorem 4.13] Assume that A, B and $C = A\cap B$ are algebraically closed and $A \mathop{\smile\!\!\!\!\!|}_C B$. Then

(1) $A\cup B$ is algebraically closed,
(2) if a vertex x in A is directly connected to a vertex y in B, then x or y must lie in C,
(3) if a point in A lies in a plane of B, then there is a line in C connecting them,
(4) a point c, which belongs to both a line in $A\setminus C$ and to a line in $B\setminus C$, lies in C.

Before introducing the k-colored pseudospace in section 5, we will prove several auxiliary results about the free pseudospace. We hope that this will allow the reader to become more familiar with the theory PS.

Lemma 4.4. *Let X and Y be algebraically closed sets independent over their common intersection Z. Given a point c not contained in $Y\setminus Z$ lying in the line b of X, then*

$$X\cup\{c\} \mathop{\smile\!\!\!\!\!|}_Z Y.$$

Proof. By the transitivity of non-forking, we may assume that $Z = X$. If c belongs to X, then there is nothing to prove. Otherwise, the type of c over X has Morley rank 1 (it is actually strongly minimal), by [4, Remark 6.2] (cf. [2, Corollary 7.13]). Since the extension $\mathrm{tp}(c/Y)$ is not algebraic, it does not fork over X. □

Lemma 4.5. *The type of a set X is determined by the collection of types $\mathrm{tp}(x,x')$, with x and x' in X.*

In particular, if $X\equiv_Z X'$ and $X\equiv_Y X'$, then $X\equiv_{YZ} X'$.

Proof. Choose an enumeration of $X=\{x_\alpha\}_{\alpha<\kappa}$ and flags F_α containing x_α, for $\alpha<\kappa$, such that $F_\alpha \mathop{\smile\!\!\!\!\!|}_{x_\alpha} X\cup\{F_\beta\}_{\beta<\alpha}$. In particular, for $\alpha\neq\beta$, we have that

$$F_\alpha \mathop{\smile\!\!\!\!\!|}_{x_\alpha} x_\beta \quad\text{and}\quad F_\alpha \mathop{\smile\!\!\!\!\!|}_{x_\alpha,x_\beta} F_\beta.$$

Since the type of F_α over x_α is stationary, the type of the pair (x_α,x_β) determines the type of F_α,F_β. By [3, Theorem 7.24], the type of $(F_\alpha)_{\alpha<\kappa}$, hence the type of X, is uniquely determined by the collection of types $\mathrm{tp}(x_\alpha,x_\beta)$, for $\alpha,\beta<\kappa$. □

5. A colored pseudospace

Work inside a sufficiently saturated model $\mathbb{U}$ of the theory PS of the free pseudospace and consider a natural number $k\geq 2$. For $0\leq i<k$, we use the notation $i+1$ instead of $i+1 \bmod k$, and likewise $i-1$ for $i-1 \bmod k$.

We colour the lines in $\mathbb{U}$, as well as the pairs (a, c), where the point c lies in the plane a, with k many colours. Formally, we partition the set of lines into subsets $C_0, \ldots, C_{k-1}$, and the set of pairs (a, c), where c lies in the plane a, into $I_0, \ldots, I_k$. Given a plane a and an index $0 \leq i < k$, we denote by the *section* $I_i(a)$ the collection of points c with $I_i(a, c)$.

Consider the theory CPS_k of k-colored pseudospaces with following axioms:

- The axioms of PS.

Universal Axioms

- For each $0 \leq i < k$, given a line b with colour i in a plane a, all the points c in b lie in the section $I_i(a)$ except at most one point, which lies in $I_{i+1}(a)$ (if it exists, we call it the *exceptional* point of b in a).

Inductive Axioms

- Every line b in a plane a contains an *exceptional* point, denoted by $\mathrm{ep}(a, b)$.
- For each $0 \leq i < k$, given a point c and a plane a with $I_i(a, c)$, there are infinitely many lines in a passing through c with colour i.
- For each $0 \leq i < k$, given a point c and a plane a with $I_i(a, c)$, there are infinitely many lines in a passing through c with colour $i - 1$.
- For every point c in a line b, there are infinitely many planes a containing b such that c is exceptional for b in a.

We can construct a model of CPS_k as follows: We start with a flag $A_0 = \{a\text{–}b\text{–}c\}$ with any colouring, eg. $b \in C_0$ and $I_0(a, c)$ and construct an ascending sequence $A_0 \subset A_1 \subset \cdots$ of coloured geometries by applying one the operations [0], [1] and [2] to a flag $a - b - c$ in A_j obtain A_{j+1}, extending the colouring to A_{j+1} in an arbitrary way whilst preserving the Universal Axioms. For example, do as follows:

- Operation [0] adds a new point c' to b. If b has colour i, then for all a'' in A_j containing b, paint the pair (a'', c') with the colour i, if $\mathrm{ep}(a'', b)$ already exists in A_j. Otherwise, paint (a'', c') with the colour $i + 1$ otherwise.
- Operation [1] adds a new line b' between a and c. If (a, c) has colour i, then paint b' with the colour i or the colour $i - 1$, and see to it that each choice occurs infinitely often in the sequence.
- Operation [2] adds a new plane a' which contains b. If b has colour i, then for all c'' in A_j which lie in b, we give the pair (a', c'') one of the colours i or $i - 1$. Each choice should occur infinitely often.

It is easy to see that the structure obtained in this fashion satisfies all axioms of CPS_k, so the theory CPS_k is consistent.

Notation. Given a subset X of a model of CPS_k, we will denote by $\langle X \rangle$ the algebraic closure of X in the reduct PS, and by $\mathrm{EP}(X) = \{\mathrm{ep}(a, b)\,,\, (a, b) \in X \times X\}$ the exceptional points of lines and planes from X.

Remark 5.1. If the point c is directly connected to a line in X, then $\langle X, c \rangle = \langle X \rangle \cup \{c\}$.

In particular, if $X = \langle X \rangle$, given c in $\mathrm{EP}(X)$, then $X \cup \{c\}$ is algebraically closed in the reduct PS.

Proof. In order to show that $\langle X, c\rangle = \langle X\rangle \cup \{c\}$, it suffices to consider the case when X is nice. The geometry $X \cup \{c\}$ is either X or obtained from X by applying the operation [0], so it is nice again, and thus algebraically closed. □

Similar to [13, Proposition 3.10] working inside two $\aleph_0$-saturated models of CPS_k, it is easy to see that the collection of partial isomorphisms between PS-algebraically closed finite sets which are closed under exceptional points is non-empty and has the back-and-forth property, so we deduce the following:

Theorem 5.2. *The theory CPS_k is complete. Given a set X in a model of CPS_k with $X = \langle X\rangle$ and $\mathrm{EP}(X) \subset X$, then the quantifier-free type of X determines its type.*

The back-and-forth system yields an explicit description of the algebraic closure, as well as showing that the theory CPS_k is ω-stable, by a standard counting types argument.

Corollary 5.3. *The theory CPS_k is ω-stable. The algebraic closure $\mathrm{acl}(X)$ of a set X is obtained by closing $\langle X\rangle$ under exceptional points:*

$$\mathrm{acl}(X) = \langle X\rangle \cup \mathrm{EP}(\langle X\rangle).$$

We deduce the following characterisation of forking over (colored) algebraically closed sets.

Corollary 5.4. *Let X and Y two supersets of an algebraically closed set $Z = \mathrm{acl}(Z)$ in CPS_k. We have that*

$$X \mathop{\downarrow}_{Z}^{\mathrm{CPS}_k} Y$$

if and only if

- $X \mathop{\downarrow}_{Z}^{\mathrm{PS}} Y$, *and*
- $\mathrm{EP}(\langle X\rangle) \cap \mathrm{EP}(\langle Y\rangle) \subset Z$.

Types over algebraically closed sets are stationary, that is, the theory CPS_k has weak elimination of imaginaries.

Proof. Since PS has weak elimination of imaginaries, we have that non-forking in CPS_k implies nonforking in the reduct PS over algebraically closed sets, by [5, Lemme 2.1]. Clearly $\mathrm{EP}(\langle X\rangle) \cap \mathrm{EP}(\langle Y\rangle) \subset Z$.
For the other direction, we may assume that $X = \langle X\rangle$ and $Y = \langle Y\rangle$. Lemma 4.4 yields that

$$X \cup \mathrm{EP}(X) \mathop{\downarrow}_{Z}^{\mathrm{PS}} Y \cup \mathrm{EP}(Y).$$

Since $\mathrm{acl}(X) = X \cup \mathrm{EP}(X)$, Remark 4.3 implies that the set $\mathrm{acl}(X) \cup \mathrm{acl}(Y)$ is algebraically closed in PS. We need only show that it contains all exceptional points, so it determines a unique type in the stable theory CPS_k. If c is an exceptional point of a plane a and a line b in $\mathrm{acl}(X) \cup \mathrm{acl}(Y)$, we may assume that a lies in X and b lies in Y. Since a and b are directly connected and $X \mathop{\downarrow}_{Z}^{\mathrm{PS}} Y$, Remark 4.3 implies that a or b lies in Z. Therefore c lies in $\mathrm{EP}(X) \cup \mathrm{EP}(Y)$ and hence is contained in $\mathrm{acl}(X) \cup \mathrm{acl}(Y)$, as desired. □

Corollary 5.5. *Let X, Y and $Z = \mathrm{acl}(Z)$ be sets such that*

$$X \underset{Z}{\downarrow} Y.$$

Then $\langle X, Y\rangle \cap \mathrm{acl}(X, Z) = \langle X, Y\rangle \cap \langle X, Z\rangle$.

Proof. Let ξ be in $\langle X, Y\rangle \cap \mathrm{acl}(X, Z)$. The independence

$$X \underset{Z}{\downarrow} Y$$

yields that

$$\xi, X \underset{Z}{\downarrow} Y.$$

It follows from Corollary 5.4 that

$$\xi, X \overset{\mathrm{PS}}{\underset{Z}{\downarrow}} Y,$$

and thus

$$\xi \overset{\mathrm{PS}}{\underset{X,Z}{\downarrow}} X, Y.$$

Since ξ lies in $\langle X, Y\rangle$, the above independence implies that ξ lies in $\langle X, Z\rangle$, as desired. □

Proposition 5.6. *Let $X = \langle X\rangle$ and $Y = \langle Y\rangle$ be two subsets of a model of CPS_k. A map $F : X \to Y$ is elementary with respect to the theory CPS_k if and only if it satisfies the following conditions:*

1. *The map F is a partial isomorphism with respect to the reduct PS.*
2. *The function F preserves colours of lines and sections.*
3. *For all a, a' and b in X, we have that $\mathrm{ep}(a, b) = \mathrm{ep}(a', b)$ if and only if $\mathrm{ep}(F(a), F(b)) = \mathrm{ep}(F(a'), F(b))$.*

Proof. We need only show that F is elementary, if it satisfies all three conditions. By Theorem 5.2, it suffices to show that F extends to a partial isomorphism $\tilde{F}$ preserving colours between $\mathrm{acl}(X) = X \cup \mathrm{EP}(X)$ and $\mathrm{acl}(Y) = Y \cup \mathrm{EP}(Y)$.

For each line b in X contained in a plane a of X, set $\tilde{F}(\mathrm{ep}(a, b)) = \mathrm{ep}(F(a), F(b))$. Let us first show that $\tilde{F}$ is well-defined, which analogously yields that $\tilde{F}$ is a bijection. Suppose that $\mathrm{ep}(a, b) = \mathrm{ep}(a_1, b_1)$, for a line b_1 contained in the plane a_1, both in X. If $b \neq b_1$, then $\mathrm{ep}(a, b)$ is the unique intersection of b and b_1, both lines in X, so $\mathrm{ep}(a, b)$ lies in X and hence its image is determined by F. Otherwise, we conclude that $b = b_1$, and thus $\tilde{F}$ is bijective, by Condition (3).

Similarly, the map $\tilde{F}$ defined above is a partial isomorphism with respect to the reduct PS. We need only show that $\tilde{F}$ preserves the colours of sections. Choose a new point $\mathrm{ep}(a, b)$ not in X and an arbitrary plane $a_1 \neq a$ in X containing $\mathrm{ep}(a, b)$. Since $\mathrm{ep}(a, b)$ does not lie in X, the intersection of a and a_1 cannot solely consist of the point $\mathrm{ep}(a, b)$. Hence, the intersection of a and a_1 is given by a unique line b_1, which lies in X and contains $\mathrm{ep}(a, b)$. We conclude as before that $b = b_1$. The colour of $\mathrm{ep}(a, b)$ in a_1 is uniquely determined according to whether $\mathrm{ep}(a, b) = \mathrm{ep}(a_1, b)$, and thus so is the colour of its image in $F(a_1)$ by $\tilde{F}$, by Condition (3).

□

6. Colored paths

We will now show that the theory CPS_k is not indiscernibly acyclic, and hence it is not equational, yet every proper cycle of types has length at least k (cf. Theorem 6.2), so we expect the complexity of these theories to increase as k grows. However, we do not know whether two of these theories are bi-interpretable.

Theorem 6.1. *In CPS_k there is a proper cycle of types*

$$p_0(x;y) \to p_1(x;y) \to \ldots \to p_{k-1}(x;y) \to p_0(x;y),$$

where both the variables x and y have length 1. In particular, the theory CPS_k is not equational.

Proof. For each $0 \leq r < k$, a pair (a,c) with colour I_r has a unique type $p_r = \mathrm{tp}(c,a)$, for the set $\{a,c\}$ is algebraically closed, since it is the intersection of all the flags containing a and c, and it is closed under exceptional points, for it contains no line. Clearly $p_r \neq p_{r+1}$, for each $0 \leq r < k$.

It suffices to show that $p_r \to p_{r+1}$: Let (a,c) with colour I_r, and choose a line b connecting them with colour r. Let c' be the exceptional point of b in a, so $(c',a) \models p_{r+1}$. Now, the set $\{b\}$ is algebraically closed in CPS_k. By Corollary 5.4, the points c and c' have the same strong type over b, and

$$c \mathop{\smile\!\!\!\!\!\!\;\mid}_b a.$$

Corollary 3.5 implies that $p_r = \mathrm{tp}(c,a) \to \mathrm{tp}(c',a) = p_{r+1}$, as desired. □

Theorem 6.2. *Let x and y be finite tuples of variables. In CPS_k, every proper cycle of types*

$$p_0(x;y) \to p_1(x;y) \to \ldots \to p_{n-1}(x;y) \to p_0(x;y),$$

has length $n \geq k$.

Proof. A proper cycle of types $p_0(x;y) \to p_1(x;y) \to \ldots \to p_{n-1}(x;y) \to p_0(x;y)$ as above induces a cycle in the reduct PS, which is equational. Therefore, the colourless reducts of p_r and p_s agree, for all r, s.

Corollary 3.5 implies that there are tuples f, $e_0, \ldots, e_n$ and algebraically closed subsets $Z_0, \ldots, Z_{n-1}$ such that:

- $\models p_r(e_r, f)$, for $0 \leq r \leq n-1$, and $\models p_0(e_n, f)$.
- $e_r \equiv_{Z_r} e_{r+1}$ for $0 \leq r \leq n-1$.
- $e_r \mathop{\smile\!\!\!\!\!\!\;\mid}_{Z_r} f$ for $0 \leq r \leq n-1$.

Set $Y = \mathrm{acl}(f)$, $X_r = \mathrm{acl}(e_r)$, for $0 \leq r \leq n$. Since the definable and algebraic closure coincide, and the colourless reducts of all p_r agree, all the types $\mathrm{tp}^{\mathrm{PS}}(X_r Y)$ are equal. Denote $\langle X_r Y \rangle$ by P_r. We find colourless isomorphisms

$$F_r : P_r \to P_{r+1},$$

which fix Y pointwise. Note that X_r and X_{r+1} have the same type over Z_r, for $r \leq n-1$. Lemma 4.5 yields that $\mathrm{tp}^{\mathrm{PS}}(X_r Y Z_r) = \mathrm{tp}^{\mathrm{PS}}(X_{r+1} Y Z_r)$, for $r \leq n-1$. The above map F_r extends to a colourless isomorphism between $\langle X_r Y Z_r \rangle$ and $\langle X_{r+1} Y Z_r \rangle$, which is the identity on $\langle Y Z_r \rangle$. We will still refer to this colourless isomorphism as F_r, keeping in mind that it is elementary in the sense of CPS_k on $\langle X_r Z_r \rangle$ and (clearly) on $\langle Y Z_r \rangle$ separately. Observe that

$$\langle X_r Y Z_r \rangle = \langle X_r Z_r \rangle \cup \langle Y Z_r \rangle,$$

by the Remark 4.3 (1).

If a set W is finite, so are the closures $\langle W\rangle$ and $\operatorname{acl}(W)$. Define its *defect* as the natural number

$$\operatorname{defect}(W) = |\operatorname{acl}(W) \setminus \langle W\rangle| = |\operatorname{EP}(\langle W\rangle) \setminus \langle W\rangle|.$$

Claim. *For each $r \leq n-1$, we have that* $\operatorname{defect}(P_r) \geq \operatorname{defect}(P_{r+1})$.

Proof of Claim. Whenever $\operatorname{ep}(a,b) = \operatorname{ep}(a_1,b_1)$, for b and b_1 in P_r, with $b \neq b_1$, then the point $\operatorname{ep}(a,b)$ lies in P_r by 4.3 (4). Thus, it suffices to show the following:

(1) Whenever the line b in P_r lies in the plane a in P_r, with $\operatorname{ep}(a,b)$ in P_r, then $\operatorname{ep}(F_r(a),F_r(b))$ lies in P_{r+1}.
(2) Whenever a, a_1 and b lie in P_r and $\operatorname{ep}(a,b) = \operatorname{ep}(a_1,b)$, then $\operatorname{ep}(F_r(a),F_r(b)) = \operatorname{ep}(F_r(a_1),F_r(b))$.

For (1), since $X_r \mathop{\smile\!\!\!\!\!\!\;|}_{Z_r} Y$, the plane a and the line b must both lie in the same set $P_r \cap \langle X_r Z_r\rangle$ or in $P_r \cap \langle Y Z_r\rangle$, by the independence

$$a \mathop{\smile\!\!\!\!\!\!\;|}_{Z_r}^{\mathrm{PS}} b$$

and Remark 4.3 (2). For example, let a and b lie in $P_r \cap \langle X_r Z_r\rangle$, so $\operatorname{ep}(a,b)$ lies in $P_r \cap \operatorname{acl}(X_r Z_r) = P_r \cap \langle X_r Z_r\rangle$, by Corollary 5.5. Since F_r is elementary on $P_r \cap \langle X_r Z_r\rangle$, we have that $\operatorname{ep}(F_r(a),F_r(b)) = F_r(\operatorname{ep}(a,b))$ lies in P_{r+1}, as desired. Observe that we have actually shown that

$$\operatorname{ep}(a,b) \in P_r \iff \operatorname{ep}(F_r(a),F_r(b)) \in P_{r+1}.$$

For (2), we need only consider the case when $a \neq a_1$ and the exceptional point $\operatorname{ep}(a,b) = \operatorname{ep}(a_1,b)$ does not lie in P, by (1). Again, if both a and a_1 lie in $\langle X_r Z_r\rangle$ or in $\langle Y Z_r\rangle$, then so does b, and we are done by Proposition 5.6, since F_r is elementary on each side. If this is not the case, and a lies in $P_r \cap \langle X_r Z_r\rangle$ and a_1 in $P_r \cap \langle Y Z_r\rangle$, then the line b lies in $P_r \cap Z_r$, by the Remark 4.3 (3). Thus the point $\operatorname{ep}(a,b) = \operatorname{ep}(a_1,b)$ lies in $\operatorname{acl}(X_r,Z_r) \cap \operatorname{acl}(Y,Z_r) = Z_r$, so we conclude as before since F_r is elementary on each side separately. □ Claim

As P_0 and P_n have the same type, their defect is the same, so $\operatorname{defect}(P_r) = \operatorname{defect}(P_{r+1})$, for all $0 \leq r \leq n-1$. Hence, for all a, a' and b in P_r, we have that

$$\operatorname{ep}(a,b) = \operatorname{ep}(a',b) \text{ if and only if } \operatorname{ep}(F_r(a),F_r(b)) = \operatorname{ep}(F_r(a'),F_r(b)).$$

Since P_r and P_{r+1} are closed in the reduct PS_k, but $\operatorname{tp}(P_r) \neq \operatorname{tp}(P_{r+1})$, Proposition 5.6 implies that F_r restricted to P_r cannot preserve colours. As F_r is elementary on each side separately, the colours of lines are preserved. Thus, there is a pair (a,c) in P_r whose colour j, with $0 \leq j < k$, is not preserved under F_r. We will show now that the colour of the pair $(F_r(a),F_r(c))$ is $j+1$.

Since F_r is elementary on $\langle X_r Z_r\rangle$ and on $\langle Y Z_r\rangle$ separately, neither a nor c lie in Z_r. The independence $X_r \mathop{\smile\!\!\!\!\!\!\;|}_{Z_r}^{\mathrm{PS}} Y$ and Remark 4.3 (3) yield that there is a line b in Z_r connecting a and c. The characterisation of the independence in Corollary 5.4 implies that $c \neq \operatorname{ep}(a,b)$. Hence the line b must have colour j. The map F_r is the identity on Z_r, and the plane $F_r(a)$ is connected to the point $F_r(c)$ by $b = F_r(b)$, so the only possible colours for the pair $(F_r(a),F_r(c))$ are j or $j+1$. As the colour of the pair (a,c) is not preserved, we deduce that $(F_r(a),F_r(c))$ has colour $j+1$, as desired.

Let F_n be the CPS_k-elementary map mapping P_n to P_0 (as both (e_0, f) and (e_n, f) realise the type p_0) and write $F^r = F_r \circ \ldots \circ F_0$. Notice that the map F^n is the identity of P_0. Let (a, c) be one of the pairs in P_0 whose colour j_0 changes under F_0. The colours of the pairs

$$(a, c), F^0(a, c), \ldots, F^{n-1}(a, c)$$

change at each step by at most adding 1 (modulo k), so the colour of $F^{n-1}(a, c)$ equals $j_0 + m$ modulo k, for some $1 \leq m \leq n$. Since F_n preserves colours and $F^n(a, c) = (a, c)$, we have that m is divisible by k, and thus $k \leq m \leq n$. We conclude that the original cycle had length at least k. □

Remark 6.3. Given a function $\pi : \{0, \ldots, k-1\} \to \{0, \ldots, k-1\}$ with no fix points, we could similarly consider the theory CPS_π of colored pseudospaces such that given a line b with colour i inside a plane a, all points in b lie in the section $I_i(a)$ except one unique exceptional point which lies in $I_{\pi(i)}(a)$.

The corresponding theory CPS_π is not equational. Every closed path of real types has length at least the length of the shortest π-cycle.

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Mathematisches Institut, Albert-Ludwigs-Universität Freiburg, D-79104 Freiburg, Germany

Email address: pizarro@math.uni-freiburg.de

Email address: ziegler@uni-freiburg.de