On a theorem of Lascar

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We denote by $\mathbb{C}$ be a big saturated model (the monster model). If R, S are binary relations on $\mathbb{C}$ the product RS is the class of all pairs (a, c) for which there is a $b \in \mathbb{C}$ such that aRb and bSc. The smallest type–definable relation which contains R is denoted by by $\overline{R}$. The smallest invariant equivalence relation whith a bounded number of classes is $\mathrm{E_L}$, the relation of having the same strong Lascar type. The smallest bounded type–definable equivalence relation, $\mathrm{E_{KP}}$, was introduced by Kim and Pillay.

The following theorem was proved by Lascar using the Lascar galois group:

Theorem 1 (D. Lascar)

$$\overline{\mathrm{E_L}}\,\mathrm{E_L} = \mathrm{E_{KP}}$$

I will give another of this theorem.

A formula $\theta(x, y)$ is called thick if there is no infinite sequence of (a_i) such that $\neg\theta(a_i, a_j)$ for $i < j$. We denote by $\Theta(x, y)$ the relation which is defined by the set of all thick formulas.

It is well known that $\mathrm{E_L}$ is the transitive closure of Θ:

$$\mathrm{E_L} = \Theta \cup \Theta^2 \cup \Theta^3 \cup \dots.$$

We will prove Theorem 1 in the following slightly stronger form.

Theorem 2

$$\overline{\mathrm{E_L}}\,\Theta = \mathrm{E_{KP}}$$

Lemma 3 (Open mapping) *Let A be a set of parameters, a an element and $\theta(x, y)$ a thick formula, possibly with parameters from A. Then there is an L_A–formula ϕ in $\mathrm{tp}(a/A)$ such that every type $p \in \mathrm{S}(A)$ which contains ϕ can be realized by an element b such that $\mathbb{C} \models \theta(a, b)$.*

PROOF: We can assume that A is empty. Otherwise we name the elements of A.

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Let D be the class of all conjugates of a and D_0 a finite subset of D such that D is contained in the definable class

$$B = \{b \in \mathbb{C} \mid \mathbb{C} \models \theta(a_0, b) \ \text{ for some } a_0 \in D_0\}.$$

The set of all types $p \in \mathrm{S}(\emptyset)$ which can be realized by an element of $\mathbb{C} \setminus B$ is closed. Therefore the set Φ of all p with only realizations in B is open. Since $D \subset B$, Φ contains $\mathrm{tp}(a)$. Thus we find a $\phi \in \mathrm{tp}(a)$ such that every p which contains ϕ can only be realized by elements of B.

Fix a $p \in \mathrm{S}(\emptyset)$ which contains ϕ. Choose a realization $b \in B$ and $a_0 \in D_0$ such that $\mathbb{C} \models \theta(a_0, b)$. Since a_0 has the same type as a we find a b' such that ab' has the same type as $a_0 b$. Then b' realizes p and $\mathbb{C} \models \theta(a, b')$. This proves the Lemma.

Let R and S be two invariant relations on C. It is easy to see[1] that $\overline{RS}$ is always contained in $\overline{R}\,\overline{S}$. The converse inclusion is not generally true: Take as a model a set with a sequence of named elements $0, 1, 2, \ldots$. Take for R the set of all pairs $(0,1), (0,3), (0,5), \ldots$ and for S the set of all pairs $(2,0), (4,0), (6,0), \ldots$. Then $\overline{R}\,\overline{S}$ contains $(0,0)$ and $\overline{RS}$ is empty.

Of course, R and S are not connected in the following sense:

Definition 4 *Two invariant relation R and S are called connected if there is a complete type p over $\emptyset$ such that both, $R(x,y)$ and $S(y,z)$, imply $p(y)$.*

Example: Look at the group $G = \mathbb{R} \times \mathbb{R}$ with the lexicographical ordering. Forget everything except the ordering, addition with $(1,0)$ and addition with $(0,1)$. Define

$$R(x,y) \Leftrightarrow \bigvee_{n \in \omega} y < x + (0,n).$$

Since there is only one type over the empty set, R and R are connected. Also R is transitive, while

$$\overline{R}(x,y) \Leftrightarrow \bigwedge_{n \in \omega} y < x + (1,-n)$$

is not. Whence $\overline{R}\,\overline{R} \neq \overline{R} = \overline{RR}$.

Lemma 5 *Assume the invariant relations R and S to be connected*[2]. *Then*

$$R\overline{S} \subset \overline{RS}\,\Theta \tag{1}$$
$$\overline{R}S \subset \Theta\,\overline{RS} \tag{2}$$
$$\overline{R}\,\overline{S} \subset \Theta\,\overline{RS}\,\Theta \tag{3}$$

[1] Note that the product of two type–definable relations is again type–definable.

[2] For (1) (and similarly for (2)) we only need that the first components of all pairs in S realize the same type.

Proof: We prove first (1). Assume $(R\overline{S})(a,c)$. Since $(\overline{RS}\,\Theta)(x,z')$ can be axiomatized

$$\{\exists z\,(\psi(x,z)\wedge\theta(z,z'))\mid \psi\in\overline{RS},\,\theta\text{ thick}\}$$

we have to show that for all $\psi(x,z)\in\overline{RS}$ and all thick θ there is a c' such that $\overline{RS}(a,c')$ and $\theta(c',c)$. Let b be such that $R(a,b)$ and $\overline{S}(b,c)$. If we apply Lemma 3 to $\mathrm{tp}(c/b)$ we obtain a formula $\phi(z,b)$ such that every type over b which realizes ϕ can be realized by an element c' which satisfies $\theta(c',c)$. Since $\overline{S}(b,c)$, and R and S are connected, there is a c' which realizes ϕ and satisfies $S(b,c')$. By the choice of ϕ we can choose c' in such a way that $\theta(c',c)$.

The proof of (2) is symmetrical.

(3) follows from (1) and (2):

$$\overline{R}\,\overline{S}\subset\overline{\overline{R}S}\Theta\subset\overline{\Theta\overline{RS}}\Theta=\Theta\,\overline{RS}\,\Theta$$

Proof of Theorem 2: Fix a complete type. **First we prove the theorem restricted to the type p.** We restrict the the meaning of $\mathrm{E_L}$, $\overline{\mathrm{E_L}}$, Θ and $\mathrm{E_{KP}}$ the the realization set of p. We can then apply the last lemma. Since $\overline{\mathrm{E_L}}\Theta$ is type–definable it suffices to prove that $\overline{\mathrm{E_L}}\Theta$ is transitive.

We have by the lemma $\mathrm{E_L}\overline{\mathrm{E_L}}\subset\overline{\mathrm{E_L}}\Theta$ and $\overline{\mathrm{E_L}}\mathrm{E_L}\subset\Theta\overline{\mathrm{E_L}}$. This gives

$$\overline{\mathrm{E_L}}\Theta\subset\overline{\mathrm{E_L}}\mathrm{E_L}\subset\Theta\overline{\mathrm{E_L}}\subset\mathrm{E_L}\overline{\mathrm{E_L}}\subset\overline{\mathrm{E_L}}\Theta$$

and all four terms are equal. Part (3) of the Lemma gives

$$\overline{\mathrm{E_L}}\,\overline{\mathrm{E_L}}\subset\Theta\overline{\mathrm{E_L}}\Theta=\overline{\mathrm{E_L}}\mathrm{E_L}.$$

Whence $\overline{\mathrm{E_L}}\Theta$ is transitive:

$$\overline{\mathrm{E_L}}\Theta\overline{\mathrm{E_L}}\Theta=\overline{\mathrm{E_L}}\,\overline{\mathrm{E_L}}\mathrm{E_L}=\overline{\mathrm{E_L}}\mathrm{E_L}{}^2=\overline{\mathrm{E_L}}\mathrm{E_L}=\overline{\mathrm{E_L}}\Theta.$$

Now the general case: Let for every complete type $\overline{\mathrm{E_L}(p)}$ be the closure of $\mathrm{E_L}\cap p^2$, $\Theta(p)=\Theta\cap p^2$ and $\mathrm{E_{KP}}(p)$ be the finest bounded type–definable equivalence relation on p. We have proved that

$$\overline{\mathrm{E_L}(p)}\Theta(p)=\mathrm{E_{KP}}(p).$$

Since $\mathrm{E_{KP}}(p)=\mathrm{E_{KP}}\cap p^2$ this implies

$$\bigcup_p\overline{\mathrm{E_L}(p)}\,\Theta=\mathrm{E_{KP}}.$$

But $\bigcup_p\overline{\mathrm{E_L}(p)}\subset\overline{\mathrm{E_L}}$ and the theorem is proved.