EQUATIONAL THEORIES OF FIELDS

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Abstract. A first-order theory is equational if every definable set is a Boolean combination of instances of equations, that is, of formulae such that the family of finite intersections of instances has the descending chain condition. Equationality is a strengthening of stability. We show the equationality of the theory of proper extensions of algebraically closed fields and of the theory of separably closed fields of arbitrary imperfection degree.

1. Introduction

Consider a first order theory T. A formula $\varphi(x;y)$ is an *equation* (for a given partition of the free variables into x and y) if, in every model of T, the family of finite intersections of instances $\varphi(x,a)$ has the descending chain condition. The theory T is *equational* if every formula $\psi(x;y)$ is equivalent modulo T to a Boolean combination of equations $\varphi(x;y)$.

Quantifier elimination implies that the theory of algebraically closed fields is equational. Separably closed fields of positive characteristic have quantifier elimination after adding λ-*functions* to the ring language [2]. The *imperfection degree* of a separably closed field K of positive characteristic p encodes the linear dimension of K over K^p. If the imperfection degree is finite, restricting the λ-functions to a fixed p-*basis* yields again equationality. A similar manipulation yields elimination of imaginaries for separably closed field K of positive characteristic and finite imperfection degree, in terms of the field of definition of the corresponding defining ideals. However, there is not an explicit description of imaginaries for separably closed fields K of infinite imperfection degree, that is, when K has infinite linear dimension over the definable subfield K^p.

Another important (expansion of a) theory of fields having infinite linear dimension over a definable subfield is the theory of an algebraically closed field with a predicate for a distinguished algebraically closed proper subfield. Any two such pairs are elementarily equivalent if and only if they have the same characteristic. They are exactly the models of the theory of Poizat's *belles paires* [15] of algebraically closed fields.

It can be far from obvious to determine whether a particular theory is equational. So far, the only known *natural* example of a stable non-equational theory is the free non-abelian finitely generated group [16, 12]. In this paper, we will prove the equationality of two theories of fields: the theory of belles paires of algebraically

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closed fields, as well as the theory of separably closed fields of arbitrary imperfection degree. In [5] an alternative proof for belles paires of characteristic 0 was obtained, by showing that definable sets are Boolean combination of certain definable sets, which are Kolchin-closed in the corresponding expansion DCF_0. We generalise this approach to arbitrary characteristic in Section 8.

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2. Equations and indiscernible sequences

Most of the results in this section come from [14, 6, 7]. We refer the avid reader to [10] for a gentle introduction to equationality.

Consider a first order theory T. A formula $\varphi(x;y)$, with respect to a given partition of the free variables into x and y, is an *equation* if, in every model of T, the family of finite intersections of instances $\varphi(x,b)$ has the descending chain condition. If $\varphi(x;y)$ is an equation, then so are $\varphi^{-1}(y;x)=\varphi(x,y)$ and $\varphi(f(x);y)$, whenever f is a $\emptyset$-definable map. Finite conjunctions and disjunctions of equations are again equations.

The theory T is *equational* if every formula $\psi(x;y)$ is equivalent modulo T to a Boolean combination of equations $\varphi(x;y)$.

Typical examples of equational theories are the theory of an equivalence relation with infinite many infinite classes or the theory of R-modules.

{E:hilbert}

Example 2.1. In any field K, for every polynomial $p(X,Y)$ with integer coefficients, the equation $p(x;y)=0$ is an equation in the model-theoretic sense.

Proof. This follows immediately from Hilbert's Basis Theorem, which implies that the Zariski topology on K^n is noetherian, i.e. the system of all algebraic sets

$$\Big\{a\in K^n \;\Big|\; \bigwedge_{i=1}^{m} q_i(a)=0\Big\},$$

where $q_i\in K[X_1,\dots,X_n]$, has the descending chain condition. However, there is a simpler proof, without using Hilbert's Basis Theorem: Observe first that if p is linear in the tuple x, then $p(x;y)=0$ is an equation, since its instances define subspaces of K^n. Now,

$$p(x,y)=q(M_1(x),\dots,M_m(x),y),$$

for some monomials $M_1,\dots,M_m$ in x and a polynomial $q(u_1,\dots,u_m;y)$ linear in the u_i's. □

Quantifier elimination for the theory ACF of algebraically closed fields and the above example yield that ACF is equational.

{E:raudenbush}

Example 2.2. In any differential field (K,δ), given a differential polynomial $p(X,Y)$ with integer coefficients, the differential equation $p(x;y)=0$ is an equation in the model-theoretic sense.

Proof. Note that $p(x;y)$ can be written as $q(M_1,\dots,M_m;y)$, for some differential monomials $M_1,\dots,M_m$ in x and a polynomial $q(u_1,\dots,u_m;y)$, which is linear in the u_i. □

Equationality is preserved under unnaming parameters and bi-interpretability [6]. It is unknown whether equationality holds if every formula $\varphi(x;y)$, with x a single variable, is a boolean combination of equations.

It is not hard to see that T is equational if an only if all completions of T are equational. So for the rest of this section we assume that T is complete and work in a sufficiently saturated model $\mathbb{U}$.

By compactness, a formula $\varphi(x;y)$ is an equation if there is no sequence $(a_i,b_i)_{i\in\mathbb{N}}$ such that $\varphi(a_i,b_j)$ holds for $i<j$, but $\not\models\varphi(a_i,b_i)$. We may assume that the sequence is indiscernible. Thus, equationality implies stability [14]. In stable theories, *non-forking* provides a natural notion of independence. We say that two sets A and B are independent over a common subset C, denoted by $A\mathop{\smile\!\!\!\!\!\!|}_C B$, if, for every finite tuple a in A, the type $\mathrm{tp}(a/B)$ does not fork over C.

{D:heir_def}

Definition 2.3. A type q over B is an *heir* of its restriction $q\restriction M$ to the elementary substructure M if, whenever the formula $\varphi(x,m,b)$ belongs to q, with m in M and b in B, then there is some m' in M such that $\varphi(x,m,m')$ belongs to $q\restriction M$.

A type q over B is definable over M if, for each formula $\varphi(x,y)$, there is a formula $\theta(y)$ with parameters in M such that for every b in B,

$$\varphi(x,b)\in q \text{ if and only if } \models\theta(b).$$

Observe that if q is definable over M, for any formula $\varphi(x,y)$, any two such formulae $\theta(y)$ are equivalent, so call it the *φ-definition* of q.

Whenever φ is an equation, the φ-definition of a type q over B is particularly simple. The intersection

$$\bigcap_{\varphi(x,b)\in q}\varphi(\mathbb{U},b)$$

is a definable set given by a formula $\psi(x)$ over B contained in q. For the φ-definition θ of q, it suffices to set

$$\theta(y)=\forall x\,(\psi(x)\to\varphi(x,y))\,.$$

In a stable theory, whenever the type q over B does not fork over the elementary substructure M, then q is definable over M and a heir of its restriction $q\restriction M$.

By the above characterisation, a formula $\varphi(x;y)$ is an equation if and only if every instance $\varphi(a,y)$ is an indiscernibly closed definable set [7, Theorem 3.16]. A definable set X is *indiscernibly closed* if, whenever $(b_i)_{i\leq\omega}$ is an indiscernible sequence such that b_i lies in X for $i<\omega$, then so does b_ω.

Extending the indiscernible sequence so that it becomes a Morley sequence over an initial segment, we conclude the following:

{L:icl_model}

Lemma 2.4. *In a complete stable theory T, a set defined by the instance $\varphi(a,y)$ is indiscernibly closed if, for every elementary substructure M and every Morley sequence $(b_i)_{i\leq\omega}$ over M such that*

$$a\mathop{\smile\!\!\!\!\!\!|}_M b_i \text{ with } \models\varphi(a,b_i) \text{ for } i<\omega,$$

then b_ω realises $\varphi(a,y)$ as well.
We may take the sequence of length $\kappa+1$, for every infinite cardinal κ, and assume that $a\mathop{\smile\!\!\!\!\!\!|}_M\{b_i\}_{i<\kappa}$.

In [19, Theorem 2.5], Srour stated a different criterion for the equationality of a formula. We refer to an extended version of this work [11, Section 2] for another proof of his result.

We will finish this section with an observation on imaginaries in equational theories.

{L:eqEI}

Lemma 2.5. *Assume that there is a collection $\mathcal{F}$ of equations, closed under finite conjunctions, such that every formula with parameters is a boolean combination of instances of formulae in $\mathcal{F}$. If every instance of an equation in $\mathcal{F}$ has a real canonical parameter, then the theory has weak elimination of imaginaries.*

Proof. Since the theory is stable, it suffices to show that every global type q has a real canonical base. By assumption, it suffices to show that the φ-definition of q (see Definition 2.3) has a real canonical parameter for every formula φ in $\mathcal{F}$. As φ is an equation, the canonical parameter of the φ-definition of q is interdefinable with the canonical parameter of the formula

$$\psi(x) = \bigcap_{\varphi(x,b)\in q} \varphi(\mathbb{U}, b).$$

By hypothesis, the formula ψ is an instance of a formula in $\mathcal{F}$ and thus has a real canonical parameter. $\square$

{S:fields}

3. Basics on fields

In this section, we will include some basic notions of field theory and commutative algebra needed in order to prove the equationality of the theories of fields we will consider later on. We will work inside some sufficiently large algebraically closed field $\mathbb{U}$.

Two subfields L_1 and L_2 are *linearly disjoint* over a common subfield F, denoted by

$$L_1 \mathop{\downarrow}^{\mathrm{ld}}_{F} L_2,$$

if, whenever the elements $a_1, \ldots, a_n$ of L_1 are linearly independent over F, then they remain so over L_2, or, equivalently, if L_1 has a linear basis over F which is linearly independent over L_2.

Linear disjointness implies algebraic independence and agrees with the latter whenever the base field F is algebraically closed. Let us note that linear disjointness is symmetric, and a transitive relation: If $F \subset D_2 \subset L_2$ is a subfield, denote by $D_2 \cdot L_1$ the field generated by D_2 and L_1. Then

$$L_1 \mathop{\downarrow}^{\mathrm{ld}}_{F} L_2$$

if and only if

$$L_1 \mathop{\downarrow}^{\mathrm{ld}}_{F} D_2 \qquad \text{and} \qquad D_2 \cdot L_1 \mathop{\downarrow}^{\mathrm{ld}}_{D_2} L_2.$$

{D:spec}

Definition 3.1. Consider a theory T of fields in the language $\mathcal{L}$ extending the language of rings $\mathcal{L}_{rings} = \{+, -, \cdot, 0, 1\}$ such that there is a predicate $\mathcal{P}$ which is interpreted in every model of T as a subfield. A subfield A of a sufficiently saturated model K of T is *$\mathcal{P}$-special* if

$$A \mathop{\downarrow}^{\mathrm{ld}}_{\mathcal{P}(A)} \mathcal{P}(K),$$

where $\mathcal{P}(A)$ equals $\mathcal{P}(K) \cap A$.

It is easy to see that elementary substructures of K are $\mathcal{P}$-special.

{L:coh}

Lemma 3.2. *Inside a sufficiently saturated model K of a stable theory T of fields in the language $\mathcal{L} \supset \mathcal{L}_{rings}$ equipped with a definable subfield $\mathcal{P}(K)$, consider a $\mathcal{P}$-special field A and a field B, both containing an elementary substructure M of K such that $A \mathop{\smile\!\!\!\!\!\!|}_M B$. The fields $\mathcal{P}(K) \cdot A$ and $\mathcal{P}(K) \cdot B$ are linearly disjoint over $\mathcal{P}(K) \cdot M$.*

Note that we write $L \cdot L'$ for the field generated by L and L'.

Proof. It suffices to show that elements $a_1, \ldots, a_n$ of A which are linearly dependent over $\mathcal{P}(K) \cdot B$ are also linearly dependent over $\mathcal{P}(K) \cdot M$. Thus, let $z_1, \ldots, z_n$ in $\mathcal{P}(K) \cdot B$, not all zero, such that

$$\sum_{i=1}^{n} a_i \cdot z_i = 0.$$

Multiplying by a suitable denominator, we may assume that all the z_i's lie in the subring generated by $\mathcal{P}(K)$ and B, so

$$z_i = \sum_{j=1}^{m} \zeta_{ij} b_j,$$

for some ζ_{ij}'s in $\mathcal{P}(K)$ and $b_1, \ldots, b_m$ in B, which we may assume to be linearly independent over $\mathcal{P}(K)$.

The type $\mathrm{tp}(a_1, \ldots, a_n/Mb_1, \ldots b_m)$ is a nonforking extension of $\mathrm{tp}(a_1, \ldots, a_n/M)$, so it is in particular an heir of its restriction to M. Thus, there are some η_{ij}'s in $\mathcal{P}(K)$, not all zero, and $c_1, \ldots, c_m$ in M linearly independent over $\mathcal{P}(K)$, such that

$$\sum_{i=1}^{n} a_i \sum_{j=1}^{m} \eta_{ij} c_j = 0.$$

Since A is $\mathcal{P}$-special, we may assume all the η_{ij}'s lie in $\mathcal{P}(A)$. As the $\{c_j\}_{1 \leq j \leq m}$ are $\mathcal{P}$-linearly independent, at least one of the elements in

$$\{ \sum_{1 \leq j \leq m} \eta_{1j} c_j, \ldots, \sum_{1 \leq j \leq m} \eta_{nj} c_j \}$$

is different from 0, as desired.

□

A natural example of a definable subfield is the field of p^{th} powers K^p, whenever K has positive characteristic $p > 0$. The corresponding notion of K^p-special is *separability*: A non-zero polynomial $f(T)$ over a subfield K is separable if every root (in the algebraic closure of K) has multiplicity 1, or equivalently, if f and its formal derivative $\frac{\partial f}{\partial T}$ are coprime. Whenever f is irreducible, the latter is equivalent to $\frac{\partial f}{\partial T} \neq 0$. In particular, every irreducible polynomial in characteristic 0 is separable. In positive characteristic p, an irreducible polynomial f is separable if and only if f is not a polynomial in T^p.

An algebraic extension $K \subset L$ is *separable* if the minimal polynomial over K of every element in L is separable. Algebraic field extensions in characteristic 0 are always separable. In positive characteristic p, the finite extension $K \subset L$ is separable if and only if the fields K and L^p are linearly disjoint over K^p. This explains the following definition:

{D:sep_ext}

Definition 3.3. An arbitrary (possibly not algebraic) field extension $K \subset L$ is *separable* if, either the characteristic is 0 or, in case the characteristic is $p > 0$, the fields K and L^p are linearly disjoint over K^p.

A field K is *perfect* if either it has characteristic 0 or if $K = K^p$, for $p = \mathrm{char}(K)$. Any field extension of a perfect field is separable. Given a field K, we define its *imperfection degree* as follows:

- If the characteristic of K is 0, its imperfection degree is 0.
- If K has positive characteristic p and $[K : K^p]$ is infinite, then its imperfection degree is infinite.
- If K has positive characteristic p and $[K : K^p]$ is finite, then $[K : K^p] = p^e$ for some natural number e. The value e is the degree of imperfection.

Thus, a field is perfect if and only if its imperfection degree is 0

{S:MTSCF}

4. Model Theory of separably closed fields

Recall that the class of separably closed fields is axiomatisable, since we need only write for each degree $d \geq 1$ a sentence in the language of rings expressing that every non-constant separable polynomial over the field K of degree d has a root in K. Separably closed fields of characteristic zero are algebraically closed. Let SCF denote the theory of separably closed fields and, for a prime p, denote by SCF_p, resp. $\mathsf{SCF}_{p,e}$, the theory of separably closed fields of characteristic p, resp. of characteristic p and imperfection degree e, where e is either a natural number or ∞. Note that $\mathsf{SCF}_{p,0}$ is the theory ACF_p of algebraically closed fields of characteristic p.

{F:Delon}

Fact 4.1. (cf. [2, Proposition 27]) The theory $\mathsf{SCF}_{p,e}$ is complete and stable, but not superstable if $e > 0$. Given a model K and a subfield k such that the field extension $k \subset K$ is separable, the type of k in K is completely determined by its quantifier-free type. In particular, the theory has quantifier elimination in the language

$$\mathcal{L}_\lambda = \mathcal{L}_{rings} \cup \{\lambda_n^i \mid 1 \leq i \leq n < \omega\},$$

where the value $\lambda_n^i(a_0, \ldots, a_n)$ is defined as follows in K. If there is a *unique* sequence $\zeta_1, \ldots, \zeta_n$ in K with $a_0 = \zeta_1^p\, a_1 + \cdots + \zeta_n^p\, a_n$, we set $\lambda_n^i(a_0, \ldots, a_n) = \zeta_i$. Otherwise, we set $\lambda_n^i(a_0, \ldots, a_n) = 0$.

Notation. For the elements $a_0, \ldots, a_n$ of K, if there is a unique sequence $\zeta_1, \ldots, \zeta_n$ in K with $a_0 = \zeta_1^p\, a_1 + \cdots + \zeta_n^p\, a_n$, we write $\overline{\lambda}_n(a_0, a_1, \ldots, a_n)\downarrow$.

Note that $\overline{\lambda}_n(a_0, a_1, \ldots, a_n)\downarrow$ if and only if

$$K \models \neg\, p\text{-}\mathrm{Dep}_n(a_1, \ldots, a_n) \wedge p\text{-}\mathrm{Dep}_{n+1}(a_0, a_1, \ldots, a_n),$$

where $p\text{-}\mathrm{Dep}_n(a_1, \ldots, a_n)$ means that $a_1, \ldots, a_n$ are K^p-linearly dependent.

Given a subfield k of a model K of SCF_p, the field extension $k \subset K$ is separable if and only if k is closed under λ-functions.

{R:fte_e_EI}

Remark 4.2. If the imperfection degree e is finite, we can fix a p-basis $\mathbf{b} = (b_1, \ldots, b_e)$ of K, that is, a tuple such that the collection of monomials

$$\bar{\mathbf{b}} = (b_1^{\nu_1} \cdots b_e^{\nu_e} \;\mid\; 0 \leq \nu_1, \ldots, \nu_e < p)$$

is a linear basis of K over K^p. All p-bases have the same type. If we replace the λ-functions by the functions $\Lambda^\nu(a) = \lambda^\nu_{p^e}(a, \bar{\mathbf{b}})$, then the theory $\mathsf{SCF}_{p,e}(\mathbf{b})$, in the language of rings with constants for $\mathbf{b}$ and equipped with the functions $\Lambda^\nu(x)$, has again quantifier elimination. Furthermore, the Λ-values of a sum or a product can be easily computed in terms of the values of each factor. In particular, the canonical base of the type $\operatorname{tp}(a/K)$ in $\mathsf{SCF}_{p,e}(\mathbf{b})$ is the field of definition of the vanishing ideal of the infinite tuple

$$(a, \overline{\Lambda}(a), \overline{\Lambda}(\overline{\Lambda}(a)), \ldots).$$

Thus, the theory $\mathsf{SCF}_{p,e}(\mathbf{b})$ has elimination of imaginaries.

By separating the variables x and y, it follows that the formula $t(x; y) = 0$ is a model-theoretic equation, for every $\mathcal{L}_\Lambda$-term $t(x, y)$. This implies that $\mathsf{SCF}_{p,e}(\mathbf{b})$, and therefore $\mathsf{SCF}_{p,e}$, is equational, whenever the imperfection degree e is finite, as shown by Srour [18, Proposition 9].

Whether there is an explicit expansion of the language of rings in which $\mathsf{SCF}_{p,\infty}$ has elimination of imaginaries is not yet known.

From now on, work inside a sufficiently saturated model K of the (incomplete) theory SCF_p. The imperfection degree of K may be either finite or infinite.

Since an $\mathcal{L}_\lambda$-substructure determines a separable field extension, Lemma 3.2 implies the following result:

{C:coh_SCFinfty}

Corollary 4.3. *Consider two subfields A and B of K containing an elementary substructure M of K. Whenever*

$$A \underset{M}{\overset{\mathsf{SCF}_p}{\mathop{\smile\!\!\!\!\!\!|}}} B,$$

the fields $K^p \cdot A$ and $K^p \cdot B$ are linearly disjoint over $K^p \cdot M$.

Note that the field $K^p \cdot A$ is actually the ring generated by K^p and A, since A is algebraic over K^p.

Proof. The $\mathcal{L}_\lambda$-structure A' generated by A is a subfield, since $a^{-1} = \lambda^1_1(1, a^p)$ for $a \neq 0$. Since $A' \mathop{\smile\!\!\!\!\!\!|}^{\mathsf{SCF}_{p,e}}_M B$, and A' is K^p-special, we have that $K^P \cdot A'$ and $K^p \cdot B$ are linearly disjoint over M. Whence $K^P \cdot A$ and $K^P \cdot B$ are also linearly disjoint over M □

We will now exhibit our candidate formulae for the equationality of SCF_p, uniformly on the imperfection degree.

{D:lambda_tame}

Definition 4.4. The collection of λ-tame formulae is the smallest collection of formulae in the language $\mathcal{L}_\lambda$, containing all polynomial equations and closed under conjunctions, such that, for any natural number n and polynomials $q_0, \ldots, q_n$ in $\mathbb{Z}[x]$, given a λ-tame formula $\psi(x, z_1, \ldots, z_n)$, the formula

$$\varphi(x) = p\text{-}\mathrm{Dep}_n(q_1(x), \ldots, q_n(x)) \ \vee \\ \big(\,\overline{\lambda}_n(q_0(x), \ldots, q_n(x))\!\downarrow \ \wedge\ \psi(x, \overline{\lambda}_n(q_0(x), \ldots, q_n(x)))\,\big)$$

is λ-tame.

Note that the formula φ above is equivalent to

$$p\text{-}\mathrm{Dep}_n(q_1, \ldots, q_n) \ \vee\ \big(p\text{-}\mathrm{Dep}_{n+1}(q_0, \ldots, q_n) \ \wedge\ \psi(x, \overline{\lambda}_n(\overline{q}(x)))\big).$$

In particular, the formula $p\text{-Dep}_n(q_1(x),\ldots,q_n(x))$ is a tame λ-formula, since it is equivalent to

$$p\text{-Dep}_n(q_1(x),\ldots,q_n(x)) \;\vee\; \big(\overline{\lambda}_n(0,q_1(x),\ldots,q_n(x))\!\downarrow\;\wedge\; 0=1\big).$$

There is a natural degree associated to a λ-tame formula, in terms of the amount of nested λ-tame formulae it contains, where polynomial equations have degree 0. The degree of a conjunction is the maximum of the degrees of the corresponding formulae.

The next remark is easily proved by induction on the degree of the formula:

{R:lambda_transf}

Remark 4.5. Given a λ-tame formula φ in m many free variables and polynomials $r_1(X),\ldots,r_m(X)$ in several variables with integer coefficients, the formula $\varphi(r_1(x),\ldots,r_m(x))$ is equivalent in SCF_p to a λ-tame formula of the same degree.

{P:lambda_BK}

Proposition 4.6. *Modulo SCF_p, every formula is equivalent to a Boolean combination of λ-tame formulae.*

Proof. By Fact 4.1, it suffices to show that the equation $t(x)=0$ is equivalent to a Boolean combination of λ-tame formulae, for every $\mathcal{L}_\lambda$-term $t(x)$. Proceed by induction on the number of occurrences of λ-functions in t. If no λ-functions occur in t, the result follows, since polynomial equations are λ-tame. Otherwise, write $t(x)=r(x,\lambda_n^i(q_0(x),\ldots,q_n(x)))$, for some $\mathcal{L}_\lambda$-term $r(x,z_1,\ldots,z_n)$, polynomials $q_1,\ldots,q_n$ and $1\le i\le n$. More generally, adding dummy variables (if needed),

$$t(x)=r(x,\overline{\lambda}_n(q_0(x),\ldots,q_n(x))).$$

By our induction hypothesis, the term $r(x,\bar z)=0$ is equivalent to a Boolean combination $BK(\psi_1(x,\bar z),\ldots,\psi_m(x,\bar z))$ of λ-tame formulae $\psi_1(x,\bar z),\ldots,\psi_m(x,\bar z)$. Consider now the λ-tame formulae

$$\varphi_i(x)=p\text{-Dep}_n(q_1(x),\ldots,q_n(x)) \;\vee\; \big(\overline{\lambda}_n(\overline{q}(x))\!\downarrow\;\wedge\;\psi_i(x,\overline{\lambda}_n(\overline{q}(x)))\big).$$

Note that

$$\mathsf{SCF}_p\models\Big((\overline{\lambda}_n(\overline{q}(x))\!\downarrow)\longrightarrow\Big(\psi_i(x,\overline{\lambda}_n(\overline{q}(x)))\;\leftrightarrow\;\varphi_i(x)\Big)\Big).$$

Therefore $t(x)=0$ is equivalent to

$$\Big(\neg\overline{\lambda}_n(\overline{q}(x))\!\downarrow\wedge\, r(x,0)=0\Big)\;\;\vee\;\;\Big(\overline{\lambda}_n(\overline{q}(x))\!\downarrow\wedge\, BK(\varphi_1(x),\ldots,\varphi_m(x))\Big),$$

which is, by induction, a Boolean combination of λ-tame formulae. □

We conclude this section with a homogenisation result for λ-tame formulae, which will be used in the proof of the equationality of SCF_p.

{P:lambda_hom}

Proposition 4.7. *For every λ-tame $\varphi(x,y_1,\ldots,y_n)$ there exists a* homogenisation *of φ with respect to $y_0,\ldots,y_n$, that is, a λ-tame formula $\varphi'(x,y_0,y_1,\ldots,y_n)$ of same degree such that*

$$\mathsf{SCF}_p\models\forall x,y_0\ldots y_n\left(\varphi'(x,y_0,\ldots,y_n)\longleftrightarrow\Big(y_0=0\vee\varphi(x,\frac{y_1}{y_0},\ldots,\frac{y_n}{y_0})\Big)\right).$$

Proof. Let y denote the tuple $(y_1,\ldots,y_n)$. By induction on the degree, we need only consider basic λ-tame formulae, since the result is preserved by taking conjunctions.

For degree 0, suppose that $\varphi(x, y)$ is the formula $q(x, y) = 0$, for some polynomial q. Write

$$q(x, \frac{y}{y_0}) = \frac{q'(x, y_0, y)}{y_0^N}.$$

Then $\varphi'(x, y_0, y) = y_0 \cdot q'(x, y_0, y) = 0$ is a homogenisation.

If $\varphi(x, y)$ has the form

$$p\text{-Dep}_m(q_1(x, y), \ldots, q_m(x, y)) \vee \left(\overline{\lambda}_m(q_0, \ldots, q_m)\downarrow \wedge \psi(x, y, \overline{\lambda}_m(q_0, \ldots, q_m))\right),$$

let $\psi'(x, y_0, y, z)$ be a homogenisation of $\psi(x, y, z)$ with respect to y_0, y. There is a natural number N such that for each $0 \leq j \leq m$,

$$q_j(x, \frac{y}{y_0}) = \frac{q'_j(x, y_0, y)}{y_0^N}$$

for polynomials q'_j. Set now $q''_j = y_0 \cdot q'_j$. Note that

$$\overline{\lambda}_m(q''_0(x, y, y_0), \ldots, q''_m(x, y, y_0)) = \overline{\lambda}_m(q_0(x, \frac{y}{y_0}), \ldots, q_m(x, \frac{y}{y_0}))$$

whenever $y_0 \neq 0$, since generally $\lambda^i_m(a_0, \ldots, a_m) = \lambda^i_m(b \cdot a_0, \ldots, b \cdot a_m)$, for $b \neq 0$. The formula

$$\varphi'(x, y_0, y) = p\text{-Dep}_m(q''_1, \ldots, q''_m) \vee \left(\overline{\lambda}_m(q''_0, \ldots, q''_m)\downarrow \wedge \psi'(x, y_0, y, \overline{\lambda}_m(q''_0, \ldots, q''_m))\right)$$

is the desired homogenisation of $\varphi(x, y)$. $\square$

5. Equationality of SCF

{S:EqSCFp_alt}

By Proposition 4.6, in order to show that the theory SCF is equational, we need only show that, for a fixed p, each λ-tame formula is an equation in every SCF_p. As before, work inside a sufficiently saturated model K.

For the proof, we require *generalised λ-functions*: If the vectors $\bar{a}_1, \ldots, \bar{a}_n$ in K^N are linearly independent over the field K^p and the system

$$\bar{a}_0 = \sum_{i=1}^{n} \zeta_i^p \, \bar{a}_i$$

has a solution, then the solution is unique and denoted by $\lambda^i_{N,n}(\bar{a}_0, \ldots, \bar{a}_n)$. As in the previous section, we will denote this by $\overline{\lambda}_{N,n}(\bar{a}_0, \ldots, \bar{a}_n)\downarrow$. Otherwise, the λ-functions $\overline{\lambda}_{N,n}$ are *undefined*. Observe that $\lambda^i_{1,n} = \lambda^i_n$. We denote by $p\text{-Dep}_{N,n}(\bar{x}_1, \ldots, \bar{x}_n)$ the formula stating that the vectors $\bar{x}_1, \ldots, \bar{x}_n$ are linearly dependent over K^p.

{T:eq_SCF}

Theorem 5.1. *Given any partition of the variables, every λ-tame formula $\varphi(x; y)$ is an equation in SCF_p.*

Proof. We proceed by induction on the degree D of the λ-tame formula. For $D = 0$, it is clear. Let $\varphi(x; y)$ be a λ-tame formula of degree $D \geq 1$ and assume that the theorem is true for all λ-tame formulae of degree smaller than D.

Claim. *If*

$$\begin{aligned}\varphi(x; y) = {} & p\text{-Dep}_{N,n}(\bar{q}_1(x^p, y), \ldots, \bar{q}_n(x^p, y)) \ \vee \\ & \left(\overline{\lambda}_{N,n}(\bar{q}_0(x^p, y), \ldots, \bar{q}_n(x^p, y))\downarrow \ \wedge \ \psi(x, y, \overline{\lambda}_{N,n}(\bar{q}_0(x^p, y), \ldots, \bar{q}_n(x^p, y)))\right),\end{aligned}$$

where $\psi(x, y, z_1, \ldots, z_n)$ is a λ-tame formula of degree $D-1$, then $\varphi(x; y)$ is an equation.

Proof of Claim. It suffices to show that every instance $\varphi(x, b)$ is equivalent to a formula $\psi'(x, b', b)$, where $\psi'(x, y', y)$ is a λ-tame formula of degree $D-1$, for some tuple b'. Indeed, our induction hypothesis will imply that $\psi'(x, b', b)$ is an instance of an equation, and thus it is indiscernibly closed (see the remark after Definition 2.3). In particular, so is $\varphi(x, b)$, and hence $\varphi(x; y)$ is an equation. Actually, it follows from the proof below that one can choose $\psi'(x, y', y)$ independently of b.

Choose a K^p-basis $b_1, \ldots, b_{N'}$ of all monomials in b occurring in the $\bar{q}_k(x^p, b)$'s and write $\bar{q}_k(x^p, b) = \sum_{j=1}^{N'} \bar{q}_{j,k}(x, b')^p \cdot b_j$. We use the notation $\mathbf{q}_k(x, b')$ for the vector of length NN' which consists of the concatenation of the vectors $\bar{q}_{j,k}(x, b')$. Let $\mathbf{Q}(x, b')$ be the $(NN' \times n)$-matrix with columns $\mathbf{q}_1(x, b'), \ldots, \mathbf{q}_n(x, b')$. The vectors $\bar{q}_1(x^p, b), \ldots, \bar{q}_n(x^p, b)$ are linearly dependent over K^p if and only if the columns of $\mathbf{Q}(x, b')$ are linearly dependent over K. Let J range over all n-element subsets of $\{1, \ldots, NN'\}$ and let $\mathbf{Q}^J(x, b')$ be the corresponding $n \times n$-submatrix. Thus

$$\mathsf{SCF}_p \models \Big(p\text{-Dep}_{N,n}(\bar{q}_1(x^p, b), \ldots, \bar{q}_n(x^p, b)) \longleftrightarrow \bigwedge_J \det(\mathbf{Q}^J(x, b')) = 0 \Big). \qquad (\star)$$

For a fixed n-element subset J of $\{1, \ldots, NN'\}$, if $\det(\mathbf{Q}^J(x, b'))$ is not zero, then the generalised λ-functions $\overline{\lambda}_{N,n}(\bar{q}_0(x^p, b), \ldots, \bar{q}_n(x^p, b))$ (see the beginning of this section) are defined. Furthermore, the equality $\mathbf{q}_0(x, b') = \mathbf{Q}(x, b') \cdot \overline{\zeta}$ holds if and only if $\overline{\zeta} = \overline{\lambda}_{N,n}(\bar{q}_0(x^p, b), \ldots, \bar{q}_n(x^p, b))$. In that case,

$$\overline{\zeta} = \det(\mathbf{Q}^J(x, b'))^{-1} \cdot B^J(x, b') \cdot \mathbf{q}_0^J(x, b'),$$

with $B^J(x, b')$ the adjoint matrix of $\mathbf{Q}^J(x, b')$. Set $r^J(x, b') = B^J(x, b') \cdot \mathbf{q}_0^J(x, b')$, so

$$\overline{\zeta} = \det(\mathbf{Q}^J(x, b'))^{-1} \cdot r^J(x, b').$$

Consider the λ-tame formula of degree $D-1$

$$\psi'(x, b', b, \overline{z}) \quad = \quad \big(\psi(x, b, \overline{z}) \ \wedge \ \mathbf{q}_0(x, b') = \mathbf{Q}(x, b') \cdot \overline{z}\big).$$

We shall see that $\varphi(x, b)$ is equivalent to

$$\bigwedge_J \ \big(\det(\mathbf{Q}^J(x, b')) = 0 \ \ \vee \ \ \psi'(x, b', b, \det(\mathbf{Q}^J(x, b'))^{-1} \cdot r^J(x, b'))\big).$$

Indeed, we consider two cases: either $p\text{-Dep}_{N,n}(\bar{q}_1(x^p, b), \ldots, \bar{q}_n(x^p, b))$ holds, in which case both formulae are true, by $(\star)$, or $\neg\, p\text{-Dep}_{N,n}(\bar{q}_1(x^p, b), \ldots, \bar{q}_n(x^p, b))$. Then there is some n-element subset J_0 such that $\det(\mathbf{Q}^{J_0}(x, b')) \neq 0$. If the above conjunction holds, the formula $\psi'(x, b', b, \det(\mathbf{Q}^{J_0}(x, b'))^{-1} \cdot r^{J_0}(x, b'))$ is true, and thus the vector $\overline{\lambda}_{N,n}(\bar{q}_0(x^p, b), \ldots, \bar{q}_n(x^p, b)$ is defined and equals the product $\det(\mathbf{Q}^{J_0}(x, b'))^{-1} \cdot r^{J_0}(x, b')$, so $\psi(x, b, \overline{\lambda}_{N,n}(\bar{q}_0(x^p, b), \ldots, \bar{q}_n(x^p, b)))$ holds. If $\varphi(x, b)$ is true, so is

$$\psi\Big(x, b, \overline{\lambda}_{N,n}(\bar{q}_0(x^p, b), \ldots, \bar{q}_n(x^p, b))\Big).$$

By the previous discussion, if there is a solution, then it is unique. Thus, for every n-element subset J with $\det(\mathbf{Q}^J(x, b')) \neq 0$, the formula $\psi'(x, b', b, \det(\mathbf{Q}^J(x, b'))^{-1} \cdot r^J(x, b'))$ must hold.

Since the above conjunction is a λ-tame formula of degree $D-1$, by Remark 4.5 and Proposition 4.7, we conclude the desired result.

□ Claim

For the proof of the theorem, since a conjunction of equations is again an equation, we may assume that

$$\varphi(x;y) = p\text{-Dep}_n(q_1(x,y),\dots,q_n(x,y)) \ \vee \\ \big(\overline{\lambda}_n(q_0(x,y),\dots,q_n(x,y))\!\downarrow\ \wedge\ \psi(x,y,\overline{\lambda}_n(q_0(x,y),\dots,q_n(x,y)))\big)$$

for some λ-tame formula $\psi(x,y,z_1,\dots,z_n)$ of degree $D-1$. It suffices to show that $\varphi(a,y)$ is indiscernibly closed. By Lemma 2.4, consider an elementary substructure M of K and a Morley sequence $(b_i)_{i\leq\omega}$ over M such that

$$a \mathop{\smile\!\!\!\!\!\!\;|}_{M} b_i \text{ with } \models \varphi(a,b_i) \text{ for } i<\omega.$$

We must show that $K \models \varphi(a,b_\omega)$.

Choose a $(K^p\cdot M)$-basis $a_1,\dots,a_N$ of the monomials in a which occur in the $q_k(a,y)$ and write $q_k(a,y)=\sum_{j=1}^N q_{j,k}(a'^p,m,y)\cdot a_j$, for some tuple m in M and a' in K. Let $\bar{q}_k(a'^p,m,y)$ be the vector $\big(q_{j,k}(a'^p,m,y)\big)_{1\leq j\leq N}$ and consider the formula

$$\varphi'(x,x';y',y) = p\text{-Dep}_{N,n}(\bar{q}_1(x'^p,y',y),\dots,\bar{q}_n(x'^p,y',y)) \ \vee \\ \big(\overline{\lambda}_{N,n}(\bar{q}_0(x'^p,y',y),\dots,\bar{q}_n(x'^p,y',y))\!\downarrow\ \wedge \\ \psi(x,y,\overline{\lambda}_{N,n}(\bar{q}_0(x'^p,y',y),\dots,\bar{q}_n(x'^p,y',y)))\big).$$

Observe that for any c in K we have the following:

- If the vectors $\bar{q}_1(a'^p,m,c),\dots,\bar{q}_n(a'^p,m,c)$ are K^p-linearly dependent, then so are the elements $q_1(a,c),\dots,q_n(a,c)$.
- If $q_1(a,c),\dots,q_n(a,c)$ are K^p-linearly independent and the functions

  $$\overline{\lambda}_{N,n}(\bar{q}_0(a'^p,m,c),\dots,\bar{q}_n(a'^p,m,c))$$

  are defined, then so are the functions $\overline{\lambda}_n(q_0(a,c),\dots,q_n(a,c))$ and furthermore

  $$\overline{\lambda}_n(q_0(a,c),\dots,q_n(a,c)) = \overline{\lambda}_{N,n}(\bar{q}_0(a'^p,m,c),\dots,\bar{q}_n(a'^p,m,c)),$$

  so

  $$\mathsf{SCF}_p \models \varphi'(a,a',m,c) \to \varphi(a,c).$$

- If $a_1,\dots,a_N$ remain linearly independent over $K^p\cdot M(c)$, then the vectors $\bar{q}_1(a'^p,m,c),\dots,\bar{q}_n(a'^p,m,c)$ are K^p-linearly dependent if and only if the elements $q_1(a,c),\dots,q_n(a,c)$ are. Therefore,

  $$\mathsf{SCF}_p \models \varphi'(a,a',m,c) \leftrightarrow \varphi(a,c).$$

Let us now show that $K\models\varphi(a,b_\omega)$. By Corollary 4.3, the elements $a_1,\dots,a_N$ are linearly independent over the field $K^p\cdot M(b_i)$, so $\varphi'(a,a',m,b_i)$ holds in K, since $K\models\varphi(a,b_i)$ for $i<\omega$. By the previous claim, the λ-tame formula $\varphi'(x,x';y',y)$ is an equation. Since the sequence $(m,b_0),\dots,(m,b_\omega)$ is indiscernible, we have that $\varphi'(a,a',m,b_\omega)$ holds in K, so $K\models\varphi(a,b_\omega)$, as desired. □

Together with Proposition 4.6, the above theorem yields the following:

{C:SCF_eq}

Corollary 5.2. *The theory* SCF *of separably closed field is equational.*

Lemma 2.5 and Theorem 5.1 yield a partial elimination of imaginaries for $\mathsf{SCF}_{p,e}$.

{C:EI_SCF}

Corollary 5.3. *The theory $\mathsf{SCF}_{p,e}$ of separably closed fields of characteristic $p > 0$ and imperfection degree e has weak elimination of imaginaries, after adding canonical parameters for all instances of λ-tame formulae.*

Question. Is there an explicit description of the canonical parameters of instances of λ-tame formulae, similar to the geometric sorts introduced in [13]?

In [11, Lemma 7.5 & Proposition 7.7], we provide an alternative proof to the equationality of $\mathsf{SCF}_{p,\infty}$, by showing inside a particular model of $\mathsf{SCF}_{p,\infty}$, namely a differentially closed field of positive characteristic, that every λ-tame formula is equivalent to an *S-formula*, as defined in [19, p.211]. Srour showed in [19] that S-formulae are equations. In the aforementioned extended version [11] of the present work , we give another proof of his result.

6. Model Theory of Pairs

{S:MTPairs}

The second theory of fields we will consider in this work is the theory ACFP of proper pairs of algebraically closed fields.

Work inside a sufficiently saturated model (K, E) of ACFP in the language $\mathcal{L}_P = \mathcal{L}_{rings} \cup \{P\}$, where $E = P(K)$ is the proper subfield. We will use the index P to refer to the expansion ACFP.

A subfield A of K is *tame* if A is algebraically independent from E over $E_A = E \cap A$, that is,

$$A \mathop{\smile\!\!\!\!\!\!\downarrow}^{\mathsf{ACF}}_{E_A} E.$$

Tameness was called *P-independence* in [1], but in order to avoid a possible confusion, we have decided to use a different terminology.

The following fact appears in this form in [15, 1]. It can be deduced from the proof of completeness in [8].

{F:Kiesler}

Fact 6.1. The completions of the theory ACFP of proper pairs of algebraically closed fields are obtained by fixing the characteristic. Each of these completions is ω-stable of Morley rank ω. The $\mathcal{L}_P$-type of a tame subfield of K is uniquely determined by its $\mathcal{L}_P$-quantifier-free type.

Every subfield of E is automatically tame, so the induced structure on E agrees with the field structure. The subfield E is a pure algebraically closed field and has Morley rank 1.

If A is a tame subfield, then its $\mathcal{L}_P$-definable closure coincides with the inseparable closure of A and its $\mathcal{L}_P$-algebraic closure is the field algebraic closure $\mathrm{acl}(A)$ of A, with $E_{\mathrm{acl}_P(A)} = \mathrm{acl}(E_A)$.

Based on the above fact, Delon [3] considered the following expansion of the language $\mathcal{L}_P$:

$$\mathcal{L}_D = \mathcal{L}_P \cup \{\mathrm{Dep}_n, \lambda_n^i\}_{1 \leq i \leq n \in \mathbb{N}},$$

where the relation Dep_n is defined as follows:

$$K \models \mathrm{Dep}_n(a_1, \dots, a_n) \iff a_1, \dots, a_n \text{ are } E\text{-linearly dependent.}$$

The λ-functions take values in E and are defined by the equation

$$a_0 = \sum_{i=1}^{n} \lambda_n^i(a_0, a_1 \ldots, a_n)\, a_i,$$

if $K \models \neg\,\mathrm{Dep}_n(a_1, \ldots, a_n) \wedge \mathrm{Dep}_{n+1}(a_0, a_1, \ldots, a_n)$, and are 0 otherwise. Clearly, a field A is closed under the λ-functions if and only if it is linearly disjoint from E over E_A, that is, if it is P-special, as in Definition 3.1. Note that the fraction field of an $\mathcal{L}_D$-substructure is again closed under λ-functions and thus tame. The theory ACFP has therefore quantifier elimination [3] in the language $\mathcal{L}_D$. Note that the formula $P(x)$ is equivalent to $\mathrm{Dep}_2(1, x)$. Likewise, the predicate $\mathrm{Dep}_n(a_1, \ldots, a_n)$ is equivalent to $\lambda_n^1(a_1, a_1 \ldots, a_n) = 0$.

Since the definable closure of a set is P-special, we conclude the following result by Lemma 3.2.

{C:coh_pair}

Corollary 6.2. *Given two subfields A and B of K containing an $\mathcal{L}_p$-elementary substructure M of K such that $A \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_M^{\mathsf{ACFP}} B$, then the fields $E \cdot A$ and $E \cdot B$ are linearly disjoint over $E \cdot M$.*

Our candidates for the equations in the theory ACFP will be called *tame* formulae.

{D:tame_formel}

Definition 6.3. Let x be a tuple of variables. A formula $\varphi(x)$ in the language $\mathcal{L}_P$ is *tame* if there are polynomials $q_1, \ldots, q_m$ in $\mathbb{Z}[X, Z]$, homogeneous in the variables Z, such that

$$\varphi(x) \quad = \quad \exists\, \zeta \in P^r \Big(\neg \zeta = 0 \;\wedge\; \bigwedge_{j \le m} q_j(x, \zeta) = 0 \Big).$$

Let X, Y and Z be tuples of variables.

{L:segre}

Lemma 6.4. *Let $q_1, \ldots, q_m$ be polynomials in $\mathbb{Z}[X, Y, Z]$ homogeneous in Y and homogeneous in Z. The $\mathcal{L}_P$-formula*

$$\exists\, \upsilon \in P^r\; \exists \zeta \in P^s \Big(\neg \upsilon = 0 \;\wedge\; \neg \zeta = 0 \;\wedge\; \bigwedge_{k \le m} q_k(x, \upsilon, \zeta) = 0 \Big)$$

is equivalent in ACFP *to a tame formula.*

Proof. With the notation $\xi_{*,j} = \xi_{1,j}, \ldots, \xi_{r,j}$ and $\xi_{i,*} = \xi_{i,1}, \ldots, \xi_{i,s}$, the previous formula is equivalent in ACFP to the tame formula

$$\exists (\xi_{1,1}, \ldots, \xi_{r,s}) \in P^{rs} \setminus 0 \bigwedge_{i,j,k=1}^{r,s,m} q_k(x, \xi_{*,j}, \xi_{i,*}) = 0.$$

Indeed, given $\upsilon = (\upsilon_i)$ and $\zeta = (\zeta_j)$, set $\xi_{i,j} = \upsilon_i \cdot \zeta_j$, and use that each q_k is homogeneous both in Y and in Z. For the converse, given $(\xi_{i,j})$, choose indices i_0 and j_0 with $\xi_{i,j} \neq 0$, and set $\upsilon = (\xi_{i,j_0})$ and $\zeta = (\xi_{i_0,j})$.

□

Observe that a polynomial $q(X, Y)$ homogeneous in Y can be seen as a polynomial in X, Y and Z, which is both homogeneous in Y and in Z. If $q_1(X, Y)$ is homogeneous in Y and $q_2(X, Z)$ is homogeneous in Z, then $q_1(X, Y) \cdot q_2(X, Z)$ is homogeneous both in Y and in Z. Therefore, we deduce the following result:

{C:conj_tame}

Corollary 6.5. *The collection of tame formulae is closed under conjunctions and disjunctions.*

Proof. Given

$$\varphi(x) = \exists\, \upsilon \in P^r \Big(\neg \upsilon = 0 \ \wedge \ \bigwedge_{j \leq m} q_j(x, \upsilon) = 0\Big)$$

and

$$\psi(x) = \exists\, \zeta \in P^s \Big(\neg \zeta = 0 \ \wedge \ \bigwedge_{k \leq n} r_k(x, \zeta) = 0\Big),$$

then the conjunction $(\varphi \wedge \psi)(x)$ is equivalent to

$$\exists\, \upsilon \in P^r \,\exists\, \zeta \in P^s \Big(\neg \upsilon = 0 \ \wedge \ \neg \zeta = 0 \ \wedge \ \bigwedge_{j \leq m} q_j(x, \upsilon) \wedge \bigwedge_{k \leq n} r_k(x, \zeta) = 0\Big)$$

and the disjunction $(\varphi \vee \psi)(x)$ is equivalent to

$$\exists\, \upsilon \in P^r \,\exists\, \zeta \in P^s \Big(\neg \upsilon = 0 \ \wedge \ \neg \zeta = 0 \ \wedge \ \bigwedge_{\substack{j \leq m \\ k \leq n}} q_j(x, \upsilon) \cdot r_k(x, \zeta) = 0\Big).$$

□

Before proving that tame formulae determine types in ACFP, we first need some basic notions from linear algebra (cf. [4, Résultats d'Algèbre]) in order to describe by tame formulae the E-annihilator of a (possibly infinite) tuple.

Let V be a vector subspace of E^n with basis $\{v_1, \ldots, v_k\}$. Observe that

$$V = \big\{ v \in E^n \ \big|\ v \wedge (v_1 \wedge \cdots \wedge v_k) = 0 \text{ in } \bigwedge^{k+1} E^n \big\}.$$

The vector $v_1 \wedge \cdots \wedge v_k$ depends only on V, up to scalar multiplication, and determines V completely. *The Plücker coordinates* $\mathrm{Pk}(V)$ of V are the homogeneous coordinates of $v_1 \wedge \cdots \wedge v_k$ with respect to the canonical basis of $\bigwedge^k E^n$. The k^{th}*-Grassmannian* $\mathrm{Gr}_k(E^n)$ of E^n is the collection of Plücker coordinates of all k-dimensional subspaces of E^n. Clearly $\mathrm{Gr}_k(E^n)$ is contained in $\mathbb{P}^{r-1}(E)$, for $r = \binom{n}{k}$.

The k^{th}-Grassmannian is Zariski-closed. Indeed, given an element ζ of $\bigwedge^k E^n$, there is a smallest vector subspace V_ζ of E^n such that ζ belongs to $\bigwedge^k V_\zeta$. The vector space V_ζ is the collection of vectors $e \lrcorner \zeta$, for e in $\bigwedge^{k-1}(E^n)^*$, where the interior product

$$\lrcorner : \bigwedge^{k-1}(E^n)^* \times \bigwedge^k (E^n) \to E$$

is a bilinear map uniquely determined by the relation

$$\langle \mu, e \lrcorner \zeta \rangle_1 = \langle \mu \wedge e, \zeta \rangle_k$$

for every μ in E^*, with $\langle \cdot, \cdot \rangle_i$ the dual pairing between $\bigwedge^i(E^n)$ and $\bigwedge^i(E^n)^*$.

A non-trivial element ζ of $\bigwedge^k E^n$ determines a k-dimensional subspace of E^n if and only if

$$\zeta \wedge (e \lrcorner \zeta) = 0,$$

for every e in $\bigwedge^{k-1}(E^n)^*$. Letting e run over a fixed basis of $\bigwedge^{k-1}(E^n)^*$, we see that the k^{th}-Grassmannian is the zero-set of a finite collection of homogeneous polynomials.

Fix some enumeration $(M_i(x_1,\ldots,x_s))_{i=1,2,\ldots}$ of all monomials in s variables. Given a tuple a of length s, denote

$$\mathrm{Ann}_n(a)=\left\{(\lambda_1,\ldots,\lambda_n)\in E^n \;\middle|\; \sum_{i=1}^n \lambda_i\cdot M_i(a)=0\right\}.$$

Notation. If we denote the scalar multiplication of two tuples x and y of length n by

$$x\cdot y=\sum_{i=1}^n x_i\cdot y_i,$$

then

$$\mathrm{Ann}_n(a)=\{\lambda\in E^n\mid \lambda\cdot(M_1(a),\ldots,M_n(a))=0\}.$$

{L:tp_ann}

Lemma 6.6. *Two tuples a and b of K have the same* ACFP*-type if and only if*

$$\mathrm{ldim}_E\,\mathrm{Ann}_n(a)=\mathrm{ldim}_E\,\mathrm{Ann}_n(b)$$

and the type $\mathrm{tp}(\mathrm{Pk}(\mathrm{Ann}_n(a)))$ *equals* $\mathrm{tp}(\mathrm{Pk}(\mathrm{Ann}_n(a)))$ *(in the pure field language), for every n in $\mathbb{N}$.*

Proof. We need only prove the right-to-left implication. Since $\mathrm{Pk}(\mathrm{Ann}_i(a))$ is determined by $\mathrm{Pk}(\mathrm{Ann}_n(a))$, for $i\le n$, we obtain an automorphism of E mapping $\mathrm{Pk}(\mathrm{Ann}_n(a))$ to $\mathrm{Pk}(\mathrm{Ann}_n(b))$ for all n. This automorphism maps $\mathrm{Ann}_n(a)$ to $\mathrm{Ann}_n(b)$ for all n and hence extends to an isomorphism of the rings $E[a]$ and $E[b]$. It clearly extends to a field isomorphism of the tame subfields $E(a)$ and $E(b)$ of K, which in turn can be extended to an automorphism of (K,E). So a and b have the same ACFP-type, as required. □

{P:Tame_Type}

Proposition 6.7. *Two tuples a and b of K have the same* ACFP*-type if and only if they satisfy the same tame formulae.*

Proof. Let $q_1(Z),\ldots,q_m(Z)$ be homogeneous polynomials over $\mathbb{Z}$. By Lemma 6.6, it suffices to show that

« $\mathrm{Ann}_n(x)$ has a k-dimensional subspace V such that $\bigwedge_{j\le m} q_j(\mathrm{Pk}(V))=0$ »

is expressible by a tame formula. Indeed, if the q_j's are all the 0 polynomial, this expression encodes the linear dimension of $\mathrm{Ann}_n(x)$. Furthermore, for $k=\mathrm{ldim}_E\,\mathrm{Ann}_n(x)$, the above encodes a finite fragment of the type $\mathrm{tp}(\mathrm{Pk}(\mathrm{Ann}_n(x)))$.

To see that the above expression is equivalent to a tame formula, it suffices to guarantee that there is an element ζ in $\mathrm{Gr}_k(E^n)$ such that

$$(e\lrcorner\zeta)\cdot(M_1(x),\ldots,M_n(x))=0$$

for all e from a a fixed basis of $\bigwedge^{k-1}(E^n)^*$, where $M_i(X)$ is the above enumeration of all monomials in the tuple x, and

$$\bigwedge_{j\le m} q_j(\zeta)=0.$$

In particular, the tuple ζ is not trivial, so we conclude that the above is a tame formula. □

By compactness, we conclude the following:

{C:Tame_BK}

Corollary 6.8. *In the theory* ACFP *of proper pairs of algebraically closed fields, every formula is a Boolean combination of tame formulae.*

{S:EqPairs}

7. Equationality of belles paires of algebraically closed fields

Though the theory of algebraically closed fields has elimination of quantifiers, the projection of a Zariski-closed set need not be again closed. For example, the closed set

$$V = \{(x, z) \in E \times E \mid x \cdot z = 1\}$$

projects onto the open set $\{x \in E \mid x \neq 0\}$.

{R:proj_complete}

Remark 7.1. An algebraic variety Z is *complete* if, for all varieties X, the projection $X \times Z \to X$ is a Zariski-closed map. Projective varieties are complete.

In order to show that the theory ACFP of proper pairs of algebraically closed fields is equational, we need only to show that tame formulae are equations with respect to any partition of the variables, by Corollary 6.8. As before, work inside a sufficiently saturated model (K, E) of ACFP in the language $\mathcal{L}_P = \mathcal{L}_{rings} \cup \{P\}$, where $E = P(K)$ is the proper subfield.

Consider the following special case as an auxiliary result.

{L:tame_vble_in_P}

Lemma 7.2. *Let $\varphi(x; y)$ be a tame formula. The formula*

$$\varphi(x; y) \wedge x \in P$$

is an equation.

Proof. Let b be a tuple in K of length $|y|$, and suppose that the formula $\varphi(x, b)$ has the form

$$\varphi(x, b) \quad = \quad \exists \zeta \in P^r \Big(\neg\zeta = 0 \ \wedge \ \bigwedge_{j \leq k} q_j(x, b, \zeta) = 0 \Big).$$

for some polynomials $q_1, \ldots, q_k$ with integer coefficients and homogeneous in ζ. Express each of the monomials in b appearing in the above equation as a linear combination of a basis of K over E. We see that there are polynomials $r_1, \ldots, r_s$ with coefficients in E, homogeneous in ζ, such that the formula $\varphi(x, b) \wedge x \in P$ is equivalent to

$$\exists \zeta \in P^r \Big(\neg\zeta = 0 \ \wedge \ \bigwedge_{j \leq s} r_j(x, \zeta) = 0 \Big) \ \wedge \ x \in P.$$

Working inside the algebraically closed subfield E, the expression inside the brackets is a projective variety, which is hence complete. By Remark 7.1, its projection is again Zariski-closed, as desired. $\square$

{P:tame_ic}

Proposition 7.3. *Let $\varphi(x; y)$ be a tame formula. The formula $\varphi(x; y)$ is an equation.*

Proof. We need only show that every instance $\varphi(a, y)$ of a tame formula is indiscernibly closed. By Lemma 2.4, it suffices to consider a Morley sequence $(b_i)_{i \leq \omega}$ over an elementary substructure M of (K, E) with

$$a \underset{M}{\overset{\mathsf{ACFP}}{\mathop{\smile\hspace{-0.9em}\mid}}} b_i \text{ with } \models \varphi(a, b_i) \text{ for } i < \omega.$$

Suppose that the formula $\varphi(a,y)$ has the form

$$\varphi(a,y) \quad = \quad \exists\,\zeta \in P^r\Big(\neg\zeta = 0 \ \wedge \ \bigwedge_{j\leq k} q_j(a,y,\zeta) = 0\Big),$$

for polynomials $q_1,\ldots,q_k$ with integer coefficients and homogeneous in ζ.

Let (c_ν) be a basis of $E\cdot M(a)$ over $E\cdot M$ By appropriately writing each monomial in a in terms of the basis, and after multiplication with a common denominator, we have that $\varphi(a,y)$ is equivalent to

$$\exists\,\zeta \in P^r\Big(\neg\zeta = 0 \ \wedge \ \bigwedge_{j}\sum_{\nu} r_{j,\nu}(e,m,y,\zeta)\cdot c_\nu = 0\Big),$$

where the polynomials $r_{j,\nu}(X,Y',Y,Z)$ are homogeneous in Z, the tuple e is from E and m is a tuple from M. By Corollary 6.2, the fields $E\cdot M(a)$ and $E\cdot M(b_i)$ are linearly disjoint over $E(M)$ for every $i<\omega$. Hence,

$$K \models \ \exists\,\zeta \in P^r\Big(\neg\zeta = 0 \ \wedge \ \bigwedge_{j,\nu} r_{j,\nu}(e,m,b_i,\zeta) = 0\Big) \text{ for } i<\omega.$$

By Lemma 7.2, the formula

$$\varphi'(e,y',y) = \exists\,\zeta \in P^r\Big(\neg\zeta = 0 \ \wedge \ \bigwedge_{j,\nu} r_{j,\nu}(e,y',y,\zeta) = 0\Big)$$

is indiscernibly closed. Since the sequence $(m,b_i)_{i\leq\omega}$ is indiscernible, we have $K\models \varphi'(e,m,b_\omega)$, so $K\models\varphi(a,b_\omega)$, as desired.

□

Corollary 6.8 and Proposition 7.3 yield now the equationality of ACFP.

{T:eq}

Theorem 7.4. *The theory of proper pairs of algebraically closed fields is equational.*

8. Linear Formulae

{S:LinFor}

A stronger relative quantifier elimination for ACFP_0 was provided by Günaydın [5, Theorem 1.1], which yields a nicer description of the equations to consider in ACFP_0. We will provide an alternative approach to his result, valid in arbitrary characteristic. We work inside a sufficiently saturated model (K,E) of ACFP.

A tame formula $\varphi(x)$ (cf. Definition 6.3) is *linear* if the corresponding polynomials in φ are linear in Z, that is, if there is a matrix $(q_{i,j}(X))$ of polynomials with integer coefficients such that

$$\varphi(x) \quad = \quad \exists\zeta\in P^s\left(\neg\zeta = 0 \ \wedge \ \bigwedge_{j=1}^{k}\zeta_1 q_{1,j}(x)+\cdots+\zeta_s q_{s,j}(x) = 0\right).$$

A linear formula is *simple* if $k=1$, that is, if it has the form

$$\mathrm{Dep}_s(q_1(x),\ldots,q_s(x)),$$

for polynomials q_i in $Z[X_1,\ldots,X_n]$.

We will show that every tame formula is equivalent in ACFP to a conjunction of simple formulae. For this, we first need a definition: Every ideal I of $K[X_1,\ldots,X_n]$

admits an *E-hull*, which is the smallest ideal I^E containing I and generated by elements of $E[X_1, \ldots, X_n]$. Note that, if I is *homogeneous*, i.e. it is the sum of all

$$I_d = \{f \in I \mid h \text{ homogeneous of degree } d\},$$

then so is I^E, with $(I^E)_d = (I_d)^E$.

{L:linear} **Lemma 8.1.** *Every tame formula is equivalent in* ACFP *to a linear formula.*

Proof. Consider a tame formula

$$\varphi(x) \quad = \quad \exists\, \zeta \in P^r \Big(\neg \zeta = 0 \ \wedge \ \bigwedge_{j \le m} q_j(x, \zeta) = 0 \Big).$$

Denote by Z the tuple of variables $(Z_1, \ldots, Z_{\mathrm{length}(\zeta)})$. For a tuple a in K of length $|x|$, denote by $I(a, Z)$ the ideal in $K[Z]$ generated by $q_1(a, Z), \ldots q_m(a, Z)$. Since $I(a, Z) \subset I(a, Z)^E$, a zero of $I(a, Z)^E$ is a zero of $I(a, Z)$. A relative converse holds: If the tuple ζ in E^r is a zero of the ideal $I(a, Z)$, then $I(a, Z)$ is contained in the ideal generated by all $Z_i - \zeta_i$'s, which is E-defined, so ζ is a zero of $I(a, Z)^E$. So $(K, E) \models \varphi(a)$ if an only if $I(a, Z)^E$ has a non-trivial zero in E^r. Since E is an elementary substructure of K, this is equivalent to $I(a, Z)^E$ having a non-trivial zero in K^r.

The ideal $I(a, Z)^E$ is generated by polynomials from $q_j(a, Z)^E$. In particular, there is a degree d, independent from a, such that $I(a, Z)^E$ has a non-trivial zero if and only if the E-hull $(I(a, Z)^E)_d$ of $I(a, Z)_d$ is not all of $K[Z]_d$, the homogeneous polynomials of degree d. As a vector space, the ideal $I(a, Z)_d$ is generated by all products $M \cdot q_j(a, Z)$, with M a monomial in Z such that $\deg(M) + \deg_Z(q_j(X, Z)) = d$. Given an enumeration $M_1, \ldots, M_s$ of all monomials in Z of degree d, the vector space $I(a, Z)_d$ is generated by a sequence of polynomials $f_1, \ldots, f_k$ of the form

$$f_j = M_1 r_{1,j}(a) + \cdots + M_s r_{s,j}(a),$$

for polynomials $r_{i,j}(X) \in \mathbb{Z}[X]$ which do not depend of a. Thus, the tuple a realises $\varphi(x)$ if and only if $(I(a, Z)^E)_d \neq K[Z]_d$, that is, if and only if there is a tuple $\xi \in E^s \setminus 0$ such that $\xi_1 r_{1,j}(a) + \cdots + \xi_s r_{s,j}(a) = 0$ for all $j = 1, \ldots, k$. The latter is expressible by a linear formula. □

In order to show that every tame formula is equivalent to a conjunction of simple formulae, we need the following result:

{P:simple} **Proposition 8.2.** *For all natural numbers m and n, there is a natural number N and an $n \times N$-matrix $(r_{j,k})$ of polynomials from $\mathbb{Z}[x_{1,1}, \ldots, x_{m,n}]$ such that the linear formula*

{linpaar}
$$\exists\, \zeta \in P^m \left(\neg \zeta = 0 \ \wedge \ \bigwedge_{j=1}^{n} \zeta_1 x_{1,j} + \cdots + \zeta_m x_{m,j} = 0 \right) \tag{1}$$

is equivalent in ACFP *to the conjunction of*

{minoren}
$$\bigwedge_{j_1 < \cdots < j_m} \det((x_{i,j_{i'}})) = 0 \tag{2}$$

and

{reduktion}
$$\bigwedge_{k=1}^{N} \mathrm{Dep}_m\big(\sum_{j=1}^{n} x_{1,j} r_{j,k}(\bar{x}), \ldots, \sum_{j=1}^{n} x_{m,j} r_{j,k}(\bar{x})\big). \tag{3}$$

Proof. The implication (1) $\Rightarrow$ ((2) $\wedge$ (3)) always holds, regardless of the choice of the polynomials $r_{j,k}$: Whenever a matrix $A = (a_{i,k})$ over K is such that there is a non-trivial vector ζ in E^m with

$$\bigwedge_{j=1}^{n} \sum_{i=1}^{m} \zeta_i a_{i,j} = 0,$$

then the rows of A are linearly dependent, so $\det((a_{i,j_{i'}})) = 0$ for all $j_1 < \cdots < j_m$. For all k, we have that

$$\sum_{i=1}^{m} \zeta_i \big(\sum_{j=1}^{n} a_{i,j} r_{j,k}(\bar{a})\big) = \sum_{j=1}^{n} \big(\sum_{i=1}^{m} \zeta_i a_{i,j}\big) r_{j,k}(\bar{a}) = 0.$$

For the converse, an easy compactness argument yields the existence of the polynomials $r_{j,k}$, once we show that (1) follows from (2) together with the infinite conjunction

$$\bigwedge_{r_1,\ldots r_n \in \mathbb{Z}[\bar{x}]} \mathrm{Dep}_m\big(\sum_{j=1}^{n} x_{1,j} r_j(\bar{x}), \ldots, \sum_{j=1}^{n} x_{m,j} r_j(\bar{x})\big). \tag{4}$$ {infreduktion}

Hence, let $A = (a_{i,k})$ be a matrix over K witnessing (2) and (4). The rows of A are K-linearly dependent, by (2). If the matrix were defined over E, its rows would then be E-linearly dependent, which yields (1). Thus, if we R is the subring of K generated by the entries of A, we may assume that the ring extension $E \subset E[R]$ is proper.

Claim 1. There is a non-zero element r in R which is not a unit in $E[R]$. {C:nonunit}

Proof of Claim 1. The field $E(R)$ has transcendence degree $\tau \geq 1$ over E. As in the proof of Noether's Normalisation Theorem [9, Theorem X 4.1], there is a transcendence basis $r_1, \ldots, r_\tau$ of R over E, such that $E[R]$ is an integral extension of $E[r_1, \ldots, r_\tau]$. If r_1 were a unit in $E[R]$, its inverse would u be a root of a polynomial with coefficients in $E[r_1, \ldots, r_\tau]$ and leading coefficient 1. Multiplying by a suitable power of r_1, we obtain a non-trivial polynomial relation among the r_j's, which is a contradiction. □ Claim 1

Claim 2. Given a sequence $V_1, \ldots V_n$ of finite dimensional E-subvector spaces of $E[R]$, there is a sequence $z_1, \ldots, z_n$ of non-zero elements of R such that the subspaces $V_1 z_1, \ldots, V_n z_n$ are independent. {C:seq_ind}

Proof of Claim 2. Assume that $z_1, \ldots, z_{k-1}$ have been already constructed. Let z be as in Claim 1. If we consider the sequence of ideals $z^k E[R]$, an easy case of Krull's Intersection Theorem ([9, Theorem VI 7.6]) applied to the noetherian integral domain $E[R]$ yields that

$$0 = \bigcap_{k \in \mathbb{N}} z^k E[R].$$

Choose some natural number N_k large enough such that

$$(V_1 z_1 + \cdots + V_{k-1} z_{k-1}) \cap z^{N_k} E[R] = 0,$$

and set $z_k = z^{N_k}$. □ Claim 2

Let us now prove that the matrix A satisfies (1). Let V_j be the E-vector space generated by $a_{1,j}, \ldots, a_{m,j}$, that is, by the j-th column of A. Choose $0 \neq z_j$ in R as in Claim 2, and write each $z_j = r_j(\bar{a})$, for some polynomial $r_j(\bar{x})$ with integer coefficients. Since A satisfies (4), there is a non-trivial tuple ζ in E^m such that

$$\sum_{i=1}^{m} \zeta_i \big(\sum_{j=1}^{n} a_{i,j} z_j\big) = \sum_{j=1}^{n} \big(\sum_{i=1}^{m} \zeta_i a_{i,j}\big) z_j = 0.$$

Observe that $\left(\sum_{i=1}^m \zeta_i a_{i,j}\right) z_j$ belongs to $V_j z_j$. The subspaces $V_i z_1, \ldots, V_n z_n$ are independent, so each $\left(\sum_{i=1}^m \zeta_i a_{i,j}\right) z_j$ must equal 0. Therefore so is

$$\sum_{i=1}^{m} \zeta_i a_{i,j} = 0,$$

as desired. □

Question. Can the integer N and the polynomials $r_{i,j}$ in Proposition 8.2 be explicitly computed?

{T:simple}

Theorem 8.3. *Every tame formula is equivalent in* ACFP *to a conjunction of simple formulae.*

Proof. By Lemma 8.1, it suffices to show that every linear formula is equivalent in ACFP to a conjunction of simple formulae. This follows immediately from Proposition 8.2, since the polynomial equation $q(x) = 0$ is equivalent in ACFP to the simple formula $\mathrm{Dep}_1(q(x))$. □

Lemma 8.1 and Corollary 6.5 yield that a finite conjunction of linear formulae is again linear. However, we do not think that the same holds for simple formulae.

Together with Corollary 6.8, we deduce another proof of [5, Theorem 1.1], valid in all characteristics:

{C:simple}

Corollary 8.4. *In the theory* ACFP *of proper pairs of algebraically closed field, every formula is equivalent in to a boolean combination of simple formulae.*

We can use the above theorem to give another proof of Proposition 7.3 in characteristic 0. Indeed, it suffices to show that every simple formula $\varphi(x; y)$ is an equation in every model (K, E) of ACFP. Consider a differential field (K, δ) with algebraically closed field of constants $E = \{x \in K \mid \delta(x) = 0\}$. As noted in the example 2.2, it suffices to show that every simple formula is equivalent to a differential equation. Now, the elements $a_1, \ldots, a_k$ of K are linearly dependent over E if and only if their *Wronskian*

$$\mathrm{W}(a_1, \ldots, a_k) = \det \begin{pmatrix} a_1 & a_2 & \ldots & a_k \\ \delta(a_1) & \delta(a_2) & \ldots & \delta(a_k) \\ \vdots & & \vdots & \\ \delta^{k-1}(a_1) & \delta^{k-1}(a_2) & \ldots & \delta^{k-1}(a_k) \end{pmatrix}$$

equals 0. Thus, the formula $\mathrm{Dep}_s(x_1, \ldots, x_s)$ is equivalent to the differential equation $\mathrm{W}(x_1, \ldots, x_s) = 0$. In [11, Proposition 9.9], we give a more explicit transformation of a tame formula into a system of differential equations to avoid using Lemma 8.1 and Proposition 8.2.

A key point in the proof of [5, Theorem 1.1] is the fact that each $\mathcal{L}_D$-function λ_n^i defines, on its domain, a continuous function with respect to the topology generated by instances of simple formulae [5, Proposition 2.6]. We will conclude with an easy proof that all functions $\lambda_n^i \times \mathrm{id} \times \cdots \times \mathrm{id}$ are continuous with respect to this topology. For this, we need an auxiliary definition (cf. Definition 4.4):

{D:lambda_formula}

Definition 8.5. The collection of λ_P-formulae is the smallest collection of formulae in the language $\mathcal{L}_D$, closed under conjunctions and containing all polynomial equations, such that, for any natural number n and polynomials $q_0, \ldots, q_n$ in $\mathbb{Z}[x]$, given a λ_P-formula $\psi(x, z_1, \ldots, z_n)$, the formula

$$\varphi(x) = \mathrm{Dep}_n(q_1(x), \ldots, q_n(x)) \;\vee\; \big(\overline{\lambda}_n(q_0(x), \ldots, q_n(x))\!\downarrow \;\wedge\; \psi(x, \overline{\lambda}_n(q_0(x), \ldots, q_n(x)))\big)$$

is λ_P-tame, where $\overline{\lambda}_n(y_0, \ldots, y_n)\!\downarrow$ is an abbreviation for

$$\neg\,\mathrm{Dep}_n(y_1, \ldots, y_n) \wedge \mathrm{Dep}_{n+1}(y_0, \ldots, y_n).$$

{P:tame_lambda}

Proposition 8.6. *Up to equivalence in* ACFP, *tame formulae and λ_P-formulae coincide.*

Proof. Notice that every simple formula is λ_P-tame, since

$$\mathrm{Dep}_n(y_1, \ldots, y_n) \quad\Leftrightarrow\quad \mathrm{Dep}_n(y_1, \ldots, y_n) \vee \big(\overline{\lambda}_n(0, y_1, \ldots, y_n)\!\downarrow \wedge (1 = 0)\big).$$

By Theorem 8.3, we conclude that all tame formulae are λ_P-tame.

We prove the other inclusion by induction on the degree of the λ_P-formula $\varphi(x)$. Polynomial equations are clearly tame. By Corollary 6.5, the conjunction of tame formulae is again tame. Thus, we need only show that $\varphi(x)$ is tame, whenever

$$\varphi(x) = \mathrm{Dep}_n(q_1, \ldots, q_n) \;\vee\; \big(\overline{\lambda}_n(q_0, \ldots, q_n)\!\downarrow \;\wedge\; \psi(x, \overline{\lambda}_n(q_0, \ldots, q_n))\big),$$

for some tame formula $\psi(x, z_1, \ldots, z_n)$. Write

$$\psi(x, z) \;=\; \exists \zeta \in P^s \Big(\neg \zeta = 0 \;\wedge\; \bigwedge_{k \le m} p_k(x, z, \zeta) = 0\Big),$$

for some polynomials $p_1(x, z, u), \ldots, p_m(x, z, u)$ with integer coefficients and homogeneous in u.

Homogenising with respect to the variables $z_0, z_1, \ldots, z_n$, there is some natural number N such that, for each $k \le m$,

$$p_k(x, z_0^{-1} z, u) z_0^N = r_k(x, z_0, z, u),$$

where r_k is both homogeneous in (z_0, z) and in u. Thus,

$$\mathsf{ACFP} \models \Big(\varphi(x) \longleftrightarrow \Big(\exists (\zeta_0, \zeta) \in P^{n+1}\, \exists \upsilon \in P^s \Big(\neg(\zeta_0, \zeta) = 0 \wedge \neg \upsilon = 0 \wedge \zeta_0 q_0(x) + \cdots + \zeta_n q_n(x) = 0 \wedge \bigwedge_{k \le m} r_k(x, \zeta_0, \zeta, \upsilon) = 0\Big)\Big)\Big).$$

The right-hand expression is a tame formula, by Lemma 6.4, and so is φ, as desired. □

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