Jet spaces of varieties over differential and difference fields

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Abstract

We give elementary proofs, using suitable jet spaces, of some old and new structural results concerning finite-dimensional differential algebraic varieties (characteristic zero). We prove analogous results for difference algebraic varieties in characteristic zero. We also mention partial results and problems in the positive characteristic case.

1 Introduction and preliminaries

It is by now well-known that certain structure theorems for finite-dimensional differential algebraic and difference algebraic varieties have implications for diophantine geometry (Mordell-Lang over function fields and Manin-Mumford over number fields). These structure theorems give conditions under which certain finite-dimensional differential (difference) varieties "come from" the constants (fixed field). In the differential algebraic context, the existing

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proofs of these various structural results make use of analytic arguments in the case of Buium (characteristic zero)[3], [5], and of "Zariski geometries" in the case of Hrushovski (all characteristics) ([14], [13], [11]). The characteristic 0 difference algebraic case is dealt with in [7] by somewhat involved arguments, and in the positive characteristic "Zariski geometries" again intervene [8]. Also in many cases, the arguments use properties of groups and fields definable in the relevant structures.

In this paper we obtain these structure theorems (and somewhat more) fairly directly in the characteristic zero case, using differential and difference analogues of jet spaces. The main result, stated in model-theoretic language, is:
(*) in the relevant structure (a model of DCF_0 or $ACFA_0$), and working inside a finite-dimensional set X, for any a and b the type of $Cb(tp(a/b))$ over a is internal (almost internal) to the field of constants (fixed field).

It should be said that (*) is in a sense strictly stronger than the Zilber dichotomy (for the relevant structures) which says:
(**) any type of U-rank 1 (SU-rank 1) is either modular or internal (almost internal) to the constants (fixed field).

(*) implies (**) immediately. But using (**) one can only formally derive that $tp(Cb(tp(a/b))/a)$ is *analyzable* (*almost analyzable*) in the constants (fixed field). Analyzability in the constants (fixed field) does not in general imply internality. In fact the statement (*) directly yields that if G is a finite-dimensional group and X a differential (difference) subvariety of G with trivial (or even finite) stabilizer, then X is internal to the constants (fixed field). So the "socle argument" from [11] is also subsumed by (*).

The jet spaces we use for the proof are *not* the prolongations of Buium, but rather analogues of the standard jet spaces or jets from algebraic geometry: dual spaces to $\mathcal{M}/\mathcal{M}^n$ where $\mathcal{M}$ is the maximal ideal of the local ring of a variety at a given point, and $n \geq 2$. Analogous results on cycle spaces in compact complex spaces were proved by Campana [6] and Fujiki [9] independently, using complex analytic jet bundles. (See [19] for the model-theoretic interpretation.) After reading Campana's paper, the first author saw the possibility of adapting the ideas to the differential and difference algebraic contexts. The details (that is, the definition and properties of the *differential* jet spaces) were worked out in a very enjoyable collaboration with the second author.

In the remainder of this section we recall the standard jet spaces and their properties.

Let K be an algebraically closed field of any characteristic. Let $X \subseteq K^n$ be an irreducible affine variety, with ideal $I_X \subset K[x_1,..,x_n]$ and coordinate ring $K[X] = K[x_1,..,x_n]/I_X$. Let $a \in X$. $\mathcal{M}_{X,a}$ is by definition $\{f \in K[X] : f(a) = 0\}$. For each $m \geq 2$, $\mathcal{M}_{X,a}/\mathcal{M}_{X,a}^m$ is a finite-dimensional K-vector space, and we define $J^{m-1}(X)_a$, the $(m-1)st$ jet space of X at a, to be its dual space. For $m = 2$, we get the tangent space to X at a. (In the literature the kth jet space to X at a is often called the k-jet of X at a. We hope there is no confusion with our terminology.)

The following basic fact is crucial to us.

Fact 1.1 *(With above notation.) Let $a \in X$. Then $\bigcap_m \mathcal{M}_{X,a}^m = (0)$.*

Proof. Corollary 10.18 of [2] for example says that if R is a Noetherian domain and I a proper ideal of R then $\bigcap_n I^n = (0)$. (So Fact 1.1 also holds for the maximal ideal of the local ring of X at a, in place of $\mathcal{M}_{X,a}$.)

In the special case where $X = K^n$ (affine n-space), let $\mathcal{M}_a$ denote $\mathcal{M}_{X,a}$, and J_a^m the corresponding jet space. So for arbitrary X we have canonical linear embeddings of $J^m(X)_a$ in J_a^m for all m. We will identify $J^m(X)_a$ with its image.

Fact 1.2 *Suppose X, Y are irreducible subvarieties of K^n, and $a \in X \cap Y$. Suppose $J^m(X)_a = J^m(Y)_a$ for all m. Then $X = Y$.*

Proof. If $f \in I_X$ then $f/\mathcal{M}_a^m$ is annihilated by $J^m(X)_a$ for all m and thus by $J^m(Y)_a$ for all m. It follows that for all m, $f/I_Y \in \mathcal{M}_{Y,a}^m$ for all m. By Fact 1.1 $\cap_m \mathcal{M}_{Y,a}^m = (0)$. Thus $f \in I_Y$.

Remark 1.3 *(i) Let X be a subvariety of K^n. Fix $m \geq 1$. Let $\mathcal{D}$ be the set of differential operators of the form*

$$\frac{\partial^s}{\partial x_{i_1}^{s_1} \partial x_{i_2}^{s_2} ... \partial x_{i_r}^{s_r}}$$

where $0 < s \leq m$, $1 \leq i_1 < i_2 < .. < i_r \leq n$, $s_1 + ... + s_r = s$, and $0 < s_i$. Let $a \in X$. Let $d = |\mathcal{D}|$. Then $J^m(X)_a$ identifies with the subspace $\{(u_D)_{D \in \mathcal{D}}$:

$\sum_{D\in\mathcal{D}} DP(a)u_D = 0, P \in I_X\}$ of K^d. Moreover, if X is defined over $k < K$, then we can restrict the polynomials P to those in $I_X \cap k[X_1,..,X_n]$.
(ii) For X a (not necessarily affine) algebraic variety over K and $a \in X(K)$, we can again define $J^m(X)_a$ working either in an affine neighbourhood of a in X, or equivalently working with the local ring of X at a.
(iii) Suppose X, Y are varieties over K, $f : X \to Y$ is a morphism over K, and $a \in X(K)$, then f induces a canonical linear map $J(f)_a : J^m(X)_a \to J^m(Y)_{f(a)}$. $J(f)_a$ is an embedding if f is a closed immersion.
(iv) (Bearing in mind (ii) and (iii).) Let X, Y be subvarieties of Z, all over K. Let $a \in X \cap Y$ be nonsingular on X, Y, Z. Then $X = Y$ iff $J^m(X)_a = J^m(Y)_a$ (as subspaces of $J^m(Z)_a$) for all m.
(v) In fact the jet spaces of X at a as a varies fit together to give the jet bundle $J(X)$ of X. J is a functor.

Proof. Left to the reader.

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2 Differential fields of characteristic zero

In this section we establish the main results for finite-dimensional differential algebraic varieties in characteristic zero. We refer the reader to [4] for background on "differential algebraic geometry" and to [16] for background on the model-theoretic approach to this subject.

We will work inside a large (saturated) differentially closed field $(\mathcal{U}, +, \cdot, \partial)$ (of characteristic zero). $\mathcal{C}$ denotes the field of constants of $\mathcal{U}$. The field $\mathcal{U}$ is also a universal domain for algebraic geometry and we will treat it as such freely.

Let us start by giving an informal description of what we wish to consider as the jet spaces of *differential algebraic* varieties. A (affine) differential algebraic variety X is roughly speaking the zero set (in $\mathcal{U}^n$) of a finite system of differential polynomial equations, that is polynomial equations over $\mathcal{U}$ in $x_1,..,x_n, \partial x_1,..,\partial x_n, \partial^2 x_1,..,\partial^2 x_n,....$ Given a point $a \in X$, we obtain the local ring $\mathcal{O}$ of X at a, that is the ring of differential rational functions on X which are defined at a. $\mathcal{O}$ comes equipped naturally with an action of ∂ extending the action on $\mathcal{U}$, making $\mathcal{O}$ into a differential ring. The maximal ideal $\mathcal{M}$ of $\mathcal{O}$ consists of those $f \in \mathcal{O}$ which are zero at a. $\mathcal{M}$ is a differential ideal of $\mathcal{O}$, as is $\mathcal{M}^m$ for any m. Thus for any m, the $\mathcal{U}$-vector space $V_m = \mathcal{M}/\mathcal{M}^m$ is acted on by ∂ making it a possibly infinite-dimensional "∂-module" over $\mathcal{U}$. Then the *mth* jet space to X at a should consist of those elements of the dual space to V_{m+1} which commute with ∂. In the case $m = 1$ (differential tangent space) this notion was introduced and studied by Cassidy and Kolchin [15].

This general notion is worth studying. It makes sense also for fields equipped with several commuting derivations and may give insight into the structure of differential varieties in this more general context. However here we have more limited aims. We are interested in the case of one derivation and where X is "finite-dimensional", that is, its differential function field has finite transcendence degree (over $\mathcal{U}$). In this case X can be considered as an algebraic variety X_1 say, equipped with a "connection" s. The jet space of X at a (as described informally in the previous paragraph) is a subset of the standard jet space of the algebraic variety X at a, and we want to study its properties, in particular prove Zariski-denseness and finite-dimensionality over the field $\mathcal{C}$ of constants. This turns out to be simply an issue of linear differential equations. So we start with some remarks on "∂-modules".

Definition 2.1 *Let $(K, +, \cdot, \partial)$ be a differential field.*
(i) By a ∂-module over K we mean a finite-dimensional K-vector space V together with an additive endomorphism $D_V : V \to V$ such that for any $c \in K$ and $v \in V$, $D_V(c \cdot v) = \partial(c) \cdot v + c \cdot D_V(v)$.
(ii) Let (V, D_V) be as in (i). By $(V^*)^\partial$ we mean the subset of the dual space V^* of V consisting of those f such that $f(D_V(v)) = \partial(f(v))$ for all $v \in V$.

Lemma 2.2 *Suppose K to be differentially closed with field k of constants. Let V be a ∂-module over K. Then $(V^*)^\partial$ is a finite-dimensional k-vector space and is Zariski-dense in V^*. In fact V^* can be given a k-structure in such a way that $(V^*)^\partial$ is precisely $V^*(k)$, the set of k-rational points of V^*.*

Proof. It is clear that $(V^*)^\partial$ is a k-vector subspace of V^*.

Fix $\lambda \in V^*$. It is easy to check that
$\lambda'(x) = \partial(\lambda(x)) - \lambda(D_V(x))$

defines an element λ' of V^*. Fix a basis $e_1, .., e_n$ of V. Then $\lambda \in (V^*)^\partial$ iff $\lambda' = 0$ which is equivalent to $\lambda'(e_i) = 0$ for $i = 1, .., n$. Let $e_1^*, .., e_n^*$ be the dual basis, and suppose $\lambda = \sum y_i e_i^*$. Then $\lambda'(e_i) = 0$ iff $\partial y_i - \lambda(D_V(e_i)) = 0$. So, writing $D_V(e_i)$ as $\sum d_{i,j} e_j$, we see that $\lambda' = 0$ iff $\partial y_i = \sum d_{i,j} y_j$ for $i = 1, .., n$.

So with respect to the basis $(e_i^*)_i$, $(V^*)^\partial$ is precisely the set of solutions of the linear differential equation $\partial(y^t) = Ay^t$ where y^t is $(y_1, .., y_n)$ as a column vector and A is the matrix $(d_{i,j})_{i,j}$. As K is differentially closed this system has a fundamental matrix U of solutions in K. That is U is a nonsingular n-by-n matrix over K whose columns form a k-basis for $(V^*)^\partial$. So, choosing the elements of $(V^*)^\partial$ represented by the columns of U as a basis of V^*, gives V^* a k-structure such that $(V^*)^\partial = V^*(k)$. The lemma is proved.

Remark 2.3 *The above construction turns V^* into a ∂-module (V^*, D_{V^*}) by setting $D_{V^*}(\lambda) = \lambda'$. What we called $(V^*)^\partial$ above is precisely the solution set of the linear differential equation $D_{V^*} = 0$ on V^*.*

We can now give a differential version of Fact 1.2. It is convenient to introduce first "finite-dimensional affine differential algebraic varieties". Such an object is an irreducible variety $X \subset \mathcal{U}^n$, together with a polynomial map $s : X \to \mathcal{U}^n$ such that $(X, s)^\sharp =_{def} \{x \in X : \partial(x) = s(x)\}$ is Zariski dense in X. We think of $(X, s)^\sharp$ as the set of $\mathcal{U}$-points of (X, s). We write $X^\sharp$ when s is understood. $X^\sharp$ is a definable set in $\mathcal{U}$ of finite Morley rank, and moreover up to Boolean combination any such definable set in $\mathcal{U}$ of finite Morley rank has this form.

It is well-known that the only condition on $s = (s_1, .., s_n)$ required to guarantee Zariski-denseness of $(X, s)^\sharp$ in X is that the polynomials $\sum_i^n (\partial P/\partial x_i)(x) s_i(x) + P^\partial(x)$ are in I_X for every $P \in I_X$, or for just those P in a given set of generators of I_X. Here P^∂ denotes the result of applying ∂ to the coefficients of P.

By virtue of s ∂ extends to a derivation of the coordinate ring $\mathcal{U}[X]$ of X: $\partial f = \sum_i (\partial f/\partial x_i) s_i + f^\partial$.

Then for $a \in X^\sharp$, $\mathcal{M}_{X,a}$ is a differential ideal of $\mathcal{U}[X]$, as is $\mathcal{M}_{X,a}^m$ for each m. It follows that each $\mathcal{M}_{X,a}/\mathcal{M}_{X,a}^m$ for $m \geq 2$ is (via the action of ∂) a ∂-module over $\mathcal{U}$. Let $J^m(X^\sharp)_a$ be the elements of $J^m(X)_a$ (the dual space of $\mathcal{M}_{X,a}/\mathcal{M}_{X,a}^{m+1}$) which commute with ∂. Then Lemma 2.2 applies. In particular $J^m(X^\sharp)_a$ is a finite-dimensional $\mathcal{C}$-space, Zariski-dense in $J^m(X)_a$.

Now suppose that $(Y, s|Y)$ is a differential algebraic subvariety of (X, s), in the obvious sense. That is Y is an irreducible subvariety of X (over $\mathcal{U}$) and $(Y, s|Y)^\sharp$ is Zariski-dense in Y. Then if $a \in Y$, we identify $J^m(Y)_a$ with its image in $J^m(X)_a$. If moreover $a \in Y^\sharp$ (that is $\partial(a) = s(a)$) then the canonical $\mathcal{U}$-linear surjection from $\mathcal{M}_{X,a}/\mathcal{M}_{X,a}^{m+1}$ to $\mathcal{M}_{Y,a}/\mathcal{M}_{Y,a}^{m+1}$ is a map of ∂-modules. Thus $J^m(Y^\sharp)_a$ is a $\mathcal{C}$-subspace of $J^m(X^\sharp)_a$. In fact $J^m(Y^\sharp)_a$ is precisely $J^m(Y)_a \cap J^m(X^\sharp)_a$.

With the above notation:

Lemma 2.4 *Let $(Y, s|Y), (Z, s|Z)$ be differential algebraic subvarieties of (X, s). Let $a \in Y^\sharp \cap Z^\sharp$ be nonsingular on each of Y, Z. Then $Y = Z$ if and only if $J^m(Y^\sharp)_a = J^m(Z^\sharp)_a$ for all m.*

Proof. If the right hand side holds, then by taking Zariski closures in $J^m(X)_a$ and using Lemma 2.2, we see that $J^m(Y)_a = J^m(Z)_a$ (as $\mathcal{U}$-subspaces of $J^m(X)_a$) for each m, and thus $Y = Z$ by 1.3(iv).

We can now state and prove the main result in the case of differentially closed fields of characteristic zero. The result, roughly speaking says the following: work inside a definable set of finite Morley rank in $\mathcal{U}$. Fix a generic point in this set. Then any definable family $\mathcal{F}$ of definable sets all of which pass through a, is (generically) internal to the constants.

It is convenient to use the language of types, canonical bases, and internality. We work in the structure $(\mathcal{U}, +, \cdot, \partial)$. K denotes a small differential subfield. If $a \in \mathcal{U}^n$, we say that $tp(a/K)$ is finite-dimensional if the transcendence degree of the differential field generated by K and a (namely $K(a, \partial a, \partial^2(a), ..))$ over K is finite. Equivalently $tp(a/K)$ has finite Morley rank, or finite U-rank.

Theorem 2.5 *Suppose $tp(a/K)$ is finite-dimensional and stationary. Let b be a tuple such that $tp(a/K, b)$ is stationary. Let $c = Cb(tp(a/K, b))$. Then $tp(c/K, a)$ is internal to $\mathcal{C}$. That is, for some d independent from c over K, a, c is definable over K, a, d together with some tuple from $\mathcal{C}$.*

Proof. We may replace a by anything interdefinable with it. So replacing a by a suitable sequence $(a, \partial a, ...\partial^r(a))$, we may assume, using finite-dimensionality of $tp(a/K)$ and the properies of a derivation, that $\partial(a) \in K(a)$. Again adjoining to a a finite part of $K(a)$, we may assume that

$\partial(a) = s(a)$ where $s(x)$ is a polynomial function defined over K. Assume $a \in \mathcal{U}^n$. Let $X \subseteq \mathcal{U}^n$ be the locus of a over K, that is the irreducible variety over K with a as a K-generic point. Then (X, s) is a differential algebraic variety defined over K, and $a \in X^\sharp$. Let Y be the irreducible subvariety of X defined over $K < b >$ with a as a $K < b >$-generic point. So again $(Y, s|Y)$ is differential algebraic and $a \in Y^\sharp$. Note that if c generates the (algebraic-geometric) field of definition of Y then c is interdefinable over K with $Cb(tp(a/K, b))$. We must show that $tp(c/K, a)$ is internal to $\mathcal{C}$. As in Remark 1.3, $J^m(X)_a$ is a $\mathcal{U}$-vector space defined over K (all m). By the above remarks, $J^m(X^\sharp)_a$ is a finite-dimensional vector space over $\mathcal{C}$. Let d_m be a $\mathcal{C}$-basis for $J^m(X^\sharp)_a$, chosen such that $d = (d_1, d_2, ...)$ is independent from c over K, a. For each m, we have by virtue of the basis d_m an isomorphism of $J^m(X^\sharp)_a$ with $\mathcal{C}^{r_m}$ for some r_m. Let $e_m \subset \mathcal{C}$ be a finite tuple such that the image of $J^m(Y^\sharp)_a$ (a subspace of $J^m(X^\sharp)_a$) in C^{r_m} is defined over e_m. Let $e = (e_1, e_2, ...)$.
Claim $c \in dcl(K, a, d, e)$.
Proof. Let f be an automorphism of $\mathcal{U}$ fixing K, a, d, e pointwise. $(f(Y), s)$ is also a differential algebraic subvariety of (X, s) and $a \in f(Y)^\sharp = f(Y^\sharp)$. Clearly $f(J^m(Y^\sharp)_a) = J^m(f(Y)^\sharp)_a$ (any m). As f fixes d_m and e_m, f fixes $J_m(Y^\sharp)$ (setwise). Hence $J^m(Y^\sharp)_a = J^m(f(Y)^\sharp)_a$ for all m. By Lemma 2.4, $f(Y) = Y$. Thus $f(c) = c$.

The claim is proved and also the theorem.

Corollary 2.6 *(i) Let G be a connected finite-dimensional K-definable differential algebraic group (that is a connected group of finite Morley rank definable over K in $\mathcal{U}$). Let $a \in G$ with $p = tp(a/K)$ stationary. Let $H < G$ be the left-stabilizer of p. Then $tp(Ha/K)$ is internal to $\mathcal{C}$.*
(ii) Let $p(x) = tp(a/K)$ be a stationary type of U-rank 1 (in $\mathcal{U}$). Then p is either modular or it is nonorthogonal to $\mathcal{C}$.

Proof. This follows quickly from the theorem as in [19].

As is well-known, the particular case of Corollary 2.6(i) when G is a differential algebraic subgroup of a semiabelian variety gives (modulo the finite-dimensionality of the Manin kernel) the Mordell-Lang conjecture for function fields (characteristic zero) in the form stated in [11]: Suppose $k < K$ are algebraically closed fields of characteristic zero, A is a semi-abelian variety over

K, X is a subvariety of X defined over K, Γ is an abstract subgroup of $A(K)$ of finite rational rank with $A(K) \cap \Gamma$ Zariski-dense in X, and X has finite stabilizer. Then X is isotrivial, that is isomorphic to a variety defined over k. Moreover, up to translation this isomorphism is induced by an isomorphism (of algebraic groups) between some semiabelian subvariety of A and some semiabelian variety defined over k.

In fact having defined the differential set-up and a suitable finite-dimensional differential algebraic subgroup of A containing Γ one just plugs into the end of Hrushovski's proof in [11].

We complete this section by stating an isotriviality result whose proof requires very little from definability in differentially closed fields.

Proposition 2.7 *Let X be an affine variety defined over $K < \mathcal{U}$. Let s be a polynomial map on X over K such that $X^\sharp = \{x \in X : \partial x = s(x)\}$ is Zariski-dense in X. Let $a \in X^\sharp$. Let T be an irreducible subvariety of the Hilbert scheme of X such that that Y_t (the subvariety of X corresponding to t) contains a for all $t \in T$ and such that for generic $t \in T$, Y_t is irreducible and a is a generic point of Y_t over K, t. Then T is birationally isomorphic to a variety defined over $\mathcal{C}$.*

Sketch of proof. Firstly, for each m, let us choose, using 2.2, a basis d_m for $J^m(X)_a$ (over $\mathcal{U}$) such that, with the corresponding identification of $J^m(X)_a$ with some $\mathcal{U}^{r_m}$, $J^m(X^\sharp)_a$ is precisely $\mathcal{C}^{r_m}$. By Fact 1.2, for some sufficiently large m, and for suitable k, we obtain a birational isomorphism f of T with a subvariety Z of $Gr_k(\mathcal{U}^{r_m})$ (where Gr_k is the variety of k-dimensional linear subspaces of $\mathcal{U}^{r_m}$). Now for $t \in T$ generic over K, a, d_m (in the algebraic-geometric sense), a is a generic point of Y_t and thus $(Y_t, s|Y_t)^\sharp$ is Zariski-dense in Y_t. It follows that $J^m(Y_t^\sharp)_a$ is Zariski-dense in $J^m(Y_t)_a \subseteq J^m(X)_a = \mathcal{U}^{r_m}$. Thus $J^m(Y_t)_a$ is defined over $\mathcal{C}$ and so $f(t) \in Z(\mathcal{C})$. Thus Z has a Zariski-dense set of $\mathcal{C}$-rational points, so is defined over $\mathcal{C}$.

3 Difference fields of characteristic 0

We work here in the context of existentially closed difference fields of characteristic zero, that is in a big model of $ACFA_0$. This theory was studied in detail in [7], and the Zilber dichotomy (types of SU-rank 1 are modular or nonorthogonal to the fixed field) was proved there, using various arguments. Here we work essentially at the quantifier-free level (that is with just difference equations). The required difference jet space theory turns out to be essentially trivial. $(\mathcal{U}, +, ., \sigma)$ denotes a saturated model of $ACFA_0$. We will use freely elementary facts about $ACFA_0$.

We need one more fact about jet spaces of varieties.

Fact 3.1 *Suppose V_1, V_2, and $W \subseteq V_1 \times V_2$ are irreducible varieties defined over a field K (characteristic zero). Suppose that the projections $\pi_i : W \to V_i$ are dominant and generically finite-to-one for $i = 1, 2$. Let (a, b) be a generic point of W over K. Then $J^m(W)_{(a,b)}$ induces an isomorphism between $J^m(V_1)_a$ and $J^m(V_2)_b$.*

We now work in the saturated model of $ACFA_0$. K denotes an (algebraically closed) small difference subfield of $\mathcal{U}$. For V a variety defined over K, V^σ denotes the image of V under σ (also defined over K). Rather than define difference jet spaces in general we will restrict ourselves to a special situation. Let V, W be irreducible varieties over K with the property that $W \subseteq V \times V^\sigma$, and the projections π_1, π_2 from W to V, V^σ are dominant and generically finite-to-one. Let a be a generic point of V over K such that $(a, \sigma(a)) \in W$ (so $(a, \sigma(a))$ is a generic point of W in the sense of algebraic geometry). By Fact 3.1, $J^m(W)_{(a,\sigma(a))}$ "is" (that is induces) an isomorphism f $(= f_{V,W,a,b})$ (in the algebraic-geometric sense) between $J^m(V)_a$ and $J^m(V^\sigma)_{\sigma(a)}$ as $\mathcal{U}$-vector spaces.

With this notation we have:

Lemma 3.2 $\{u \in J^m(V)_a : \sigma(u) = f(u)\}$ *is a finite-dimensional vector space over $Fix(\sigma)$ and is Zariski-dense in $J^m(V)_a$.*

Proof. As f is a linear map between the two finite-dimensional $\mathcal{U}$-vector spaces we get the first part. For the rest note that $J^m(V^\sigma)_{\sigma(a)}$ is the image of $J^m(V)_a$. As f is an isomorphism between these two spaces, the axioms for $ACFA$ yield a generic point $x \in J^m(V)_a$ such that $f(x) = \sigma(x)$.

We now discuss canonical bases of types in $ACFA$. On the one hand $ACFA$ is simple and there is a specific notion of the canonical base for Lascar strong types (amalgamation bases) in simple theories (see [10]). On the other hand, $ACFA$ is quantifier-free stable (in fact quantifier-free ω-stable). For K an algebraically closed difference field, and a a tuple, by $Cb(qftp(a/K))$ we mean the unique smallest difference subfield K_0 of K such that $qftp(a/K)$ does not fork over K_0 and $qftp(a/K_0)$ is stationary. It can also be described as the difference field generated by the field of definition of the proalgebraic variety V over K whose K-generic point is $(\sigma^i(a))_{i\in\mathbf{Z}}$.

In any case, working in $ACFA$ we have that $Cb(qftp(a/K)) \subseteq dcl(Cb(tp(a/K))$ and $Cb(tp(a/K)) \subseteq acl(Cb(qftp(a/K))$.

$acl_f(-)$ denotes field-theoretic algebraic closure.

Lemma 3.3 *Suppose that $\sigma(a) \in acl_f(K, a)$. Let $K_1 \supset K$ be algebraically closed (in $ACFA$). Let V_1 be the irreducible variety over K_1 with a as a K_1-generic point. Then the field of definition c of V_1 is contained in $Cb(qftp(a/K_1))$ and the latter is contained in the algebraic closure of K and c.*

Proof. It is clear that the field of definition of V_1 is contained in $Cb(qftp(a/K_1))$. For the second part, note that a is independent from K_1 over K, c in the sense of ACF. Our hypotheses imply that $(\sigma^i(a) : i \in \mathbf{Z})$ is contained in $acl_f(K, a)$, whereby $(\sigma^i(a) : i \in \mathbf{Z})$ is independent from K_1 over $K(c)$ in the sense of ACF.

Lemma 3.4 *Let K, V, V^σ, W, a, f be as in the discussion before 3.2. Let $K_1 \supseteq K$ be an algebraically closed difference field, and let V_1 be the algebraic variety over K_1 with generic point a. Then $f|J(V_1)_a$ is an isomorphism with $J(V_1^\sigma)_{\sigma a}$.*

Proof. Let W_1 be the irreducible variety over K_1 with generic point $(a, \sigma a)$. Then $W_1 \subseteq W$. By 3.2, $J(W_1)_{(a,\sigma a)}$ induces an isomorphism f_1 between $J(V_1)_a$ and $J(V_1^\sigma)_{\sigma a}$. But $J(W_1)_{(a,\sigma a)} \subseteq J(W)_{(a,\sigma a)}$ and the latter induces f. So $f_1 = f|J(V_1)_a$.

A type $p(x)$ over algebraically closed K is said to be almost internal to a $\emptyset$-definable set X (such as $Fix(\sigma)$), if for some $A \supset K$ and a realizing p such that a is independent from A over K, $a \in acl(A, X)$.

Theorem 3.5 *Let $tp(a/K)$ be finite-dimensional, and b be such that $b = Cb(qftp(a/K, b))$, Then $tp(b/acl(K, a))$ is almost internal to $Fix(\sigma)$.*

Proof. Replacing a by some $(a, \sigma(a), .., \sigma^r(a))$ we may assume that $acl_f(K, a) = acl_f(K, \sigma(a))$. Let V be the variety of a over K, and W the variety of $(a, \sigma(a))$ over K. So V^σ is the variety of $\sigma(a)$ over K. (In fact by choosing r sufficiently large, the variety W will determine $qftp(a/K)$.) For $m \geq 1$ let $L^m = J^m(V)_a$, and $L^m_\sigma = \{u \in J^m(V)_a : \sigma(u) = f(u)\}$. By 3.2, L^m_σ is a finite-dimensional vector space over $Fix(\sigma)$, Zariski-dense in L^m. Let V_1 be the variety of a over $acl_\sigma(K, b)$, and b_1 the field of definition of V_1. So $J^m(V_1)_a$ is a K-subspace of L_m. By 3.3, $b \in acl(K, b_1)$. By 3.4 and the axioms for $ACFA$, $\{u \in J^m(V_1)_a : \sigma(u) = f(u)\}$ is a Zariski-dense subset of $J^m(V_1)_a$ as well as being a $Fix(\sigma)$ subspace of L^m_σ. The rest of the proof is a copy of that of Theorem 2.5.

Again as in [19] one deduces the Zilber dichotomy for types of SU-rank 1 in $ACFA_0$ (modular or nonorthogonal to the fixed field) as well as results for definable groups of finite SU-rank (if G is quantifier-free definable and X is a difference algebraic subvariety of G then $X/Stab(X)$ is almost internal to $Fix(\sigma)$). We do not see any improvement over the proof of the modularity of suitable difference algebraic subgroups of semi-abelian varieties relevant to Manin-Mumford in [12]. But, modulo this, our results give direct proofs of the results for arbitrary commutative algebraic groups (Corollary 4.4.2 of [12]).

4 Further remarks on jet bundles and prolongations

For the benefit of the interested reader we give another explicit description of $J^m(X^\sharp)_a$ (which was actually part of our original approach before we noticed the rather simpler approach using ∂-modules).

We work again in the saturated differentially closed field $(\mathcal{U}, +, ., \partial)$. X is an irreducible subvariety of $\mathcal{U}^n$ defined over K. The first (Buium) prolongation $\tau(X)$ (implicit in our treatement above) is the subvariety of $\mathcal{U}^{2n}$ defined by the equations $P(x_1, .., x_n) = 0$ for $P \in I_X$ together with the equations $\sum_i((\partial P/\partial x_i)(x))y_i + P^\partial(x) = 0$ for $P \in I_X$. $\tau(X)$ projects canonically onto X. Let $s : X \to \tau(X)$ be a section, defined over K, and $X^\sharp = \{a \in X : (a, \partial(a)) = s(a)\}$.

Let $J = J^m$ for some fixed $m \geq 1$. From 1.3(v) we have $J(s) : J(X) \to J(\tau(X))$. We will exhibit a reasonably canonical morphism $h : J(\tau(X)) \to \tau(J(X))$ such that for $a \in X^\sharp$, $J(X^\sharp)_a = \{u \in J(X)_a : \partial(u) = h \circ J(s)(u)\}$.

Let $\mathcal{D}$ be the set of differential operators of order at most m in variables $x_1, .., x_n$ as in 1.3(i), with the natural identifications. Let $\mathcal{D}_1$ be the same thing, but with variables $x_1, .., x_n, y_1, .., y_n$. Note that $\mathcal{D}$ is canonically a subset of $\mathcal{D}_1$. $J(X) = \{(a, u_D)_{D \in \mathcal{D}} : a \in X, \sum_D DP(a)u_D = 0, P \in I_X\}$ and $J(\tau(X)) = \{(a, b, u_D)_{D \in \mathcal{D}_1} : (a, b) \in \tau(X), \sum_{D \in \mathcal{D}_1} DQ(a, b)u_D = 0, Q \in I_{\tau(X)}\}$.

For $D \in \mathcal{D}$, let L_D be the set of those operators in $\mathcal{D}_1$ which can be obtained from D by replacing exactly one occurrence of some $\partial/\partial x_i$ by $\partial/\partial y_i$. So for example if D is $\partial^2/\partial x_1^2$ then $L_D = \{\partial^2/\partial x_1 \partial y_1\}$, and if $D = \partial^2/\partial x_1 \partial x_2$ then $L_D = \{\partial^2/\partial x_1 \partial y_2, \partial^2/\partial x_2 \partial y_1\}$.

Now for $(a, b, u_D)_{D \in \mathcal{D}_1} \in J(\tau(X))$, let $h(a, b, u_D)_{D \in \mathcal{D}_1} = (a, u_D, b, v_D)_{D \in \mathcal{D}}$ where $v_D = \sum_{D' \in L_D} u_{D'}$.

We leave the proof of the following to the interested reader:

Lemma 4.1 *(i) $h : J(\tau(X)) \to \tau(J(X))$. Moreover if π_i ($i = 1, 2$) are the natural projections from $J(\tau(X))$, $\tau(J(X))$ respectively to $\tau(X)$, then $\pi_2.h = \pi_1$.*
(ii) for $a \in X^\sharp$, $J(X^\sharp)_a = \{u \in J(X)_a : h(J(s)(a, u)) = (a, u, b, \partial u)\}$, where $s(a) = (a, b) \in \tau(X)$.

5 Remarks on the positive characteristic case

We mention here some problems and partial results concerning the generalization of the results above to the positive characteristic cases. The general problem here is that certain relevant finite morphisms need not be separable and so will not induce isomorphisms on jet spaces.

Let us first consider $ACFA_p$. As the π_i in Fact 3.1 need not be separable, we may not obtain a linear isomorphism between $J(V_1)_a$ and $J(V_2)_b$. In fact this cannot even be expected, as there are many different "fixed fields", $Fix(\sigma^n Fr^m)$ for $n, m \in \mathbf{Z}$. So the most one can hope for in 3.5 is almost internality to the union of all fixed fields. The obvious idea is as follows: suppose we are given finite-dimensional $tp(a/K)$ where a is a generic point of V over K and $(a, \sigma(a))$ is a generic point of W over K where the projections from W to both V and V^σ are finite-to-one. Replace σ by $\tau = \sigma^n Fr^m$ (still yielding a model of $ACFA_p$). $(a, \tau(a))$ is now a generic point of some W' over K which again projects finite-to-one to V and V^τ. Even though these projections need still not be separable, they may induce an isomorphism between some nonzero subspace of $J(V)_a$ and one of $J(V^\tau)_{\tau(a)}$. Try to show that the union of all these subspaces of $J(V)_a$ (as τ varies) generates (or is Zariski-dense) in $J(V)_a$. However it is not clear if this approach works even in some specific examples such as the following pointed out by Zoe Chatzidakis: Let V be 2-space and $W = \{(x_1, x_2, y_1, y_2) : y_1 = x_2, y_2^p + x_1^p + y_1 = 0\}$.

Let us now consider the characteristic p differential case. By this we mean separably closed fields of finite (nonzero) Ershov invariant e. It is convenient for our purposes to consider such fields equipped with several Hasse derivations (mainly because the ∂-module theory extends smoothly). A theory was developed by Messmer and Wood [18], and an alternative approach was recently developed by the second author [21]. In the case $e = 1$ these approaches essentially coincide. For convenience we restrict our attention to this case ($e = 1$) although everything we say generalizes (using the theory in [21]). So the relevant theory is the theory $SCH_{p,1}$ of separably closed fields K of characteristic p and Ershov invariant 1, equipped with a strict Hasse (or iterative) derivation. The Hasse derivation is by definition a sequence $\mathcal{D} = (D_0, D_1, ...)$ of additive maps from K to K such that $D_0 = id$, $D_n(xy) = \sum_{i+j=n} D_i(x)D_j(y)$ and
$D_i \circ D_j(x) = \begin{pmatrix} i+j \\ i \end{pmatrix} D_{i+j}(x)$ for all n, i, j.

Strictness means that K^p is the field of constants of D_1. The theory is complete with quantifier-elimination in the obvious language (the language of rings together with the D_i's), as well of course as being stable. If $\{t\}$ is a p-basis of K over K^p, then by considering a suitable Wronskian, we see that for any $x \in K$, the sequence $(D_i(x) : i < p^n - 1)$ is birational over $\{D_i(t^j) : i, j < p^n - 1\}$ with the sequence $(a_i : i < p^n - 1)$ of p^nth powers in K such that $x = \sum_i a_i t^i$.

Work in a saturated model $(\mathcal{U}, +, ., D_i)_i$ of $SCH_{p,1}$. K will denote a relatively algebraically closed substructure (or even a model). $\mathcal{C}$ is the field of absolute constants of $\mathcal{U}$, that is $\{a \in \mathcal{U} : D_i(a) = 0 \text{ for all } i\}$. $\mathcal{C}$ coincides with $\cap_n \mathcal{U}^{p^n}$. In fact more precisely $\mathcal{U}^{p^n}$ is the common zero set of $D_1, ..., D_{p^n - 1}$.

Definition 5.1 *(i) $tp(a/K)$ is thin if $trdeg(K(D_i(a) : i < \omega)/K)$ is finite.*
(ii) $tp(a/K)$ is very thin if $L = K(D_i(a) : i < \omega)$ is finitely separably generated over K, that is if there is a finite tuple b from K such that L is separably algebraic over $K(b)$.

Remark 5.2 *(i) This notion of thinness coincides with that in [11].*
(ii) Any extension of a very thin type is very thin.

Our partial result is:

Proposition 5.3 *Let $tp(a/K)$ be very thin. Then for any b, $tp(Cb(stp(a/K, b)/K, a)$ is internal to $\mathcal{C}$.*

We sketch how to adapt the previous arguments. The main point is that for any m the mth jet space at a of a suitable variety is (as a $\mathcal{U}$-vector space) equipped with a $\mathcal{D}$-module structure, whose solution set is finite-dimensional over $\mathcal{C}$ and Zariski-dense. The proof of Theorem 2.5 then goes through.

Let $L = K(D_i(a) : i < \omega)$. By the properties of the Hasse derivation $\mathcal{D}$ and as $tp(a/K)$ is assumed to be be very thin, we can find finite tuples $a_0 \subseteq a_1 \subseteq a_2 ...$ in L such that
(i) $a \subseteq a_0$,
(ii) $L = K(a_0, a_1, ...)$,
(iii) $K(a_i)$ is closed under $D_0, .., D_{p^i}$, and
(iv) L is separably algebraic over $K(a_0)$.

Let $\mathcal{D}_i = \{D_0, .., D_{p^i}\}$. Let X_i be the (absolutely irreducible) variety over K whose generic point is a_i. Let $f_i : X_i \to X_{i-1}$ be the surjective morphism induced by the inclusion $a_{i-1} \subseteq a_i$. Let $g_i : X_i \to X_0$ be likewise. Let $\mathcal{O}_i$ be the local ring of rational functions over $\mathcal{U}$ on X_i which are defined at a_i. As $K(a_i)$ is closed under the operators in $\mathcal{D}_i$, $\mathcal{O}_i$ is naturally equipped with a "truncated" Hasse ring structure, namely with an action of $D_0, .., D_{p^i}$ extending the action on $\mathcal{U}$ and satisfying the relevant properties. Clearly the maximal ideal $\mathcal{M}_i$ of $\mathcal{O}_i$ (that is, the functions which are 0 at a_i) is a $\mathcal{D}_i$ subring, as are all powers of $\mathcal{M}_i$. Let us fix $m \geq 1$. So the finite-dimensional $\mathcal{U}$-vector space $V_i = \mathcal{M}_i/\mathcal{M}_i^{m+1}$ is equipped with a $\mathcal{D}_i$-module structure. In particular, V_i is a $\mathcal{D}_j$-module for $j < i$. Now, as f_i is etale, it induces an isomorphism f_i^* between V_{i-1} and V_i, which is moreover an isomorphism of $\mathcal{D}_{i-1}$ modules. So, by virtue of the isomorphisms g_i^*, the $\mathcal{U}$-vector space is equipped with a $\mathcal{D}$-module structure. Let $J^m = J^m(X_0)_{a_0}$ be the dual space to V_0 (the mth jet space to X_0 at a_0). As in Remark 2.3, J^m is equipped with a $\mathcal{D}$-module structure $\mathcal{D}_{J^m}$ say. Either doing it oneself, or appealing to [17], the 0-set $J^{m,\sharp}$ of $\mathcal{D}_{J^m}$ is a finite-dimensional vector space over $\mathcal{C}$. In fact J^m can be given a $\mathcal{C}$ -structure such that $J^{m,\sharp} = J^m(\mathcal{C})$. That is, J^m has a $\mathcal{U}$-basis $v_1, .., v_r$ say, which is also a $\mathcal{C}$-basis for $J^{m,\sharp}$.

Now the proof of 2.5 goes through. Given a stationary extension $tp(a/K')$ of $tp(a/K)$, let Y_0 the variety over K' whose generic point is a_0. Let J'^m be the the mth jet space of Y_0 at a_0 as defined above. $J'^{m,\sharp}$ embeds in $J^{m,\sharp}$, and Y_0 is determined by the images of $J'^{m,\sharp}$ in $J^{m,\sharp}$ for all m. Thus $tp(Cb(tp(a/K'))/K, a)$ is internal to $\mathcal{C}$.

Remark 5.4 *Suppose that A is an ordinary semi-abelian variety over definably closed $K < \mathcal{U}$. Let $A^\sharp = \cap_n p^n(A(\mathcal{U}))$ (a type-definable connected subgroup of $A(\mathcal{U})$). Let a be a generic point of $A^\sharp$ over K. Then $tp(a/K)$ is very thin.*

Proof. We will make use of the "Verschiebung" as described in [1] to which the reader is referred for more background and references. First, what is the meaning of "ordinary"? Let us work for now geometrically, that is inside an algebraically closed field. The semi-abelian variety A is by definition an extension of an abelian variety B by an algebraic torus T. Let $b = dim(B)$. Then A is said to be ordinary if the group of p-torsion point of the abelian part B of A is precisely $(\mathbf{Z}/p\mathbf{Z})^b$ (which implies that the group of p^t-torsion points of B is $(\mathbf{Z}/p^t\mathbf{Z})^b$ for all $t > 0$).

Now for the Verschiebung. We can hit the coefficients defining A with the Frobenius Fr to obtain another semiabelian variety $A^{(p)}$. Moreover, acting on coordinates, Fr yields a bijective isogeny $Fr : A \to A^{(p)}$. The dual isogeny from $A^{(p)}$ to A is called the Verschiebung V, and we have that both $V \circ Fr : A \to A$ and $Fr \circ V : A^{(p)} \to A^{(p)}$ are just multiplication by p in the relevant groups (that is $x \to px$ in additive notation). Similarly the dual isogeny V_n to $Fr^n : A \to A^{(p^n)}$ has the feature that composition with Fr^n is multiplication by p^n.

The main fact we use is:
(*) if A is ordinary then each Verschiebung map $V_n : A^{(p^n)} \to A$ is *separable.*

Now let $A, K, A^\sharp, a$ be as in the hypotheses of the proposition. We will show that $D_i(a)$ is separably algebraic over $K(a)$ for all i. Fix n. Then by definition of $A^\sharp$ there is $c \in A(\mathcal{U})$ such that $a = p^n c$ (in the group A written additively). As multiplication by p^n in A is the same as $V_n \circ Fr^n$, it follows that $a = V_n(b)$ where $b = Fr^n(c) = c^{p^n}$, and so $a \in K(c^{p^n})$. It follows that $D_0(a), .., D_{p^{n-1}}(a) \in K(c^{p^n})$. But by (*) $b = c^{p^n}$ is separably algebraic over $K(a)$. The same is thus true of the $D_i(a)$. This completes the proof.

Proposition 5.3 and Remark 5.4 thus yield a relatively direct proof of the Mordell-Lang conjecture for function fields (with prime-to-p division points in place of all division points) in positive characteristic, assuming the semi-abelian variety A to be ordinary. This case however was already covered by part (3) of Theorem A in [1].

Finally let us give an example of a thin but not very thin type (in the context of $SCH_{p,1}$). Let K be an elementary submodel of $\mathcal{U}$. In particular K contains a p-basis $\{t\}$ of $\mathcal{U}$. We may assume that $D_{p^n}(t^{p^n}) = 1$ for all $n \geq 0$. Let $c \in \mathcal{C} \setminus K$. By saturation of $\mathcal{U}$ we can find an element $a \in \mathcal{U}$, transcendental over $K(c)$, such that $a - (ct + c^{p^{-1}}t^p + .. + c^{p^{-n}}t^{p^n})$ is a $p^{n+1}th$ power in $\mathcal{U}$ for all n. Then $D_{p^n}(x) = c^{p^{-n}}$ for all $n \geq 0$, and $K(D_0(a), D_1(a), ...)$ is not finitely separably generated over K (but is of course of transcendence degree 2 over K). On the other hand $tp(a/K)$ is 2-step analyzable in $\mathcal{C}$: first $c \in dcl(K, a)$. Let $K' = dcl(K, c) = K(c, c^{p^{-1}}, c^{p^{-2}}, ...)$. Then $a \notin K'$, but $D_i(a) \in K'$ for all $i > 0$, and so the difference of two realizations of $tp(a/K')$ is in $\mathcal{C}$.

Clearly a type of U-rank 1 which is nonorthogonal to $\mathcal{C}$ is very thin. We do not know of any type of U-rank 1 which is thin but not very thin.

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