subgroup of M. (pp-definable subgroups were introduced in [34] as 'endlich matrizielle Untergruppen'.)

Lemma 1.2. *Let $\varphi(x, \mathbf{y})$ be a pp-formula and $\boldsymbol{a} \in M$. Then $\varphi(M, \boldsymbol{a})$ is empty or a coset of $\varphi(M, \mathbf{0})$. ($\boldsymbol{a}$ stands for a finite sequence $a_1, \ldots, a_n$ of elements of M.)*

Proof. $M \models \varphi(x, \boldsymbol{a}) \to (\varphi(y, \mathbf{0}) \leftrightarrow \varphi(x+y, \boldsymbol{a}))$.

Corollary 1.3. *Let $\boldsymbol{a}, \boldsymbol{b} \in M$, $\varphi(x, \mathbf{y})$ a ppf. Then (in M) $\varphi(x, \boldsymbol{a})$ and $\varphi(x, \boldsymbol{b})$ are equivalent or contradictory.*

Note. The pp-definable subgroups are closed under $\cap$ and $+$. If $\varphi(x)$, $\psi(x)$ are ppf, we write

$$\varphi \cap \psi = \varphi \wedge \psi,$$

$$\varphi + \psi = \exists y, z\, \varphi(y) \wedge \psi(z) \wedge y + z \doteq x.$$

By $\varphi \subset \psi$ we mean that $\vdash \varphi(x) \to \psi(x)$.

For the proof of 1.1 we need two further lemmas:

Lemma 1.4 (B.H. Neumann). *Let H_i denote abelian groups. If $H_0 + a_0 \subset \bigcup_{i=1}^{n} H_i + a_i$ and $H_0/(H_0 \cap H_i)$ is infinite for $i > k$, then $H_0 + a_0 \subset \bigcup_{i=1}^{k} H_i + a_i$.*

Lemma A (for sets A_i). *If A_0 is finite, then $A_0 \subset \bigcup_{i=1}^{k} A_i$ iff*

$$\sum_{\Delta \subset \{1,\ldots,k\}} (-1)^{|\Delta|} \left| A_0 \cap \bigcap_{i \in \Delta} A_i \right| = 0. \qquad \text{(Easy)}$$

Proof of Theorem 1.1. Fix M. We have to show: If $\psi(x, \mathbf{y})$ is in M equivalent to a boolean combination of ppf, then also $\forall x\, \psi$ is. Since ppf are closed under conjunction, ψ is M-equivalent to a conjunction of formulas

$$\varphi_0(x, \mathbf{y}) \to \varphi_1(x, \mathbf{y}) \vee \cdots \vee \varphi_n(x, \mathbf{y}), \qquad \varphi_i \text{ ppf.}$$

We can assume that already ψ has this form.

Let $H_i = \varphi_i(M, \mathbf{0})$. By 1.2 the $\varphi_i(M, \mathbf{y})$ are empty or cosets of H_i. (Think of $\mathbf{y}$ as being fixed in M.) Let $H_0/(H_0 \cap H_i)$ be finite for $i = 1, \ldots, k$ and infinite for $i = k+1, \ldots, n$ $(k \geq 0)$. By 1.4

$$M \models \forall x\, \psi \leftrightarrow \forall x\, (\varphi_0(x, \mathbf{y}) \to \varphi_1(x, \mathbf{y}) \vee \cdots \vee \varphi_k(x, \mathbf{y})).$$

We apply Lemma A to the sets $A_i = \varphi_i(M, \mathbf{y})/(H_0 \cap \cdots \cap H_k)$: $\varphi_0(M, \mathbf{y}) \cap \bigcap_{i \in \Delta} \varphi_i(M, \mathbf{y})$ is empty or consists of N_Δ cosets of $H_0 \cap \cdots \cap H_k$, where

$$N_\Delta = \left| H_0 \cap \bigcap_{i \in \Delta} H_i / (H_0 \cap \cdots \cap H_k) \right|.$$

$\leftrightarrow \bigvee_{\mathcal{N} \in P(\{1,\ldots,k\})} \left(\bigwedge_{\Delta \in \mathcal{N}} \exists x (\varphi_0(x,y) \wedge \bigwedge_{i \in \Delta} \varphi_i(x,y)) \wedge \bigwedge_{\Delta \notin \mathcal{N}} \neg \exists x (\varphi_0(x,y) \wedge \bigwedge_{i \in \Delta} \varphi_i(x,y)) \right)$

Whence $\sum_{\Delta \in \mathcal{N}} (-1)^{|\Delta|} N_\Delta = 0$

$$M \models \forall x\, \psi \leftrightarrow \sum_{\Delta \in \mathcal{N}} (-1)^{|\Delta|} N_\Delta = 0,$$

where

$$\mathcal{N} = \left\{ \Delta \subset \{1, \ldots, k\} \,\middle|\, \exists x \left(\varphi_0(x, \mathbf{y}) \wedge \bigwedge_{i \in \Delta} \varphi_i(x, \mathbf{y}) \right) \right\}.$$

The resulting formula depends only on the indices N_Δ. Since pp-sentences are always true, the above proof shows:

Corollary 1.5 (Monk [14]). *M_1 and M_2 are elementarily equivalent iff*

$$\varphi/\psi(M_1) = \varphi/\psi(M_2) \quad \textit{for all ppf } \psi \subset \varphi.$$

(*Notation*: $\varphi/\psi(M) = (\varphi(M) : \psi(M)) \bmod \infty$. We assume $\varphi/\psi(M)$ to be a natural number or $= \infty$. *Convention*: $n \cdot \infty = \infty \cdot n = \infty$ $(n \geq 1)$ etc.)

Definition. M is a *pure* submodule of N, if $M \subset N$ and

$$N \models \varphi(\boldsymbol{a}) \Leftrightarrow M \models \varphi(\boldsymbol{a}) \quad \text{for all ppf } \varphi \text{ and } \boldsymbol{a} \in M.$$

Examples. $M \prec N$, M a direct factor of N.

Corollary 1.6 (Sabbagh [29]). *M is an elementary substructure of N iff M is pure in N and elementarily equivalent to N.*

Proof. Since $M \equiv N$, every L_R-formula is – in M and in N – equivalent to the same boolean combination of ppfs.

Corollary 1.7. *Suppose $L \subset M \subset N$. If $L \prec N$ and M pure in N, then $M \prec N$.*

Proof. $\varphi/\psi(L) \leq \varphi/\psi(M) \leq \varphi/\psi(N)$ by pureness, whence $M \equiv L \equiv N$.

Corollary 1.8. *Let κ be an infinite cardinal. Denote by $\prod_{i \in I}^{\kappa} M_i$ the product of the M_i restricted to sequences with $< \kappa$ members $\neq 0$. Then $\prod_{i \in I}^{\kappa} M_i \prec \prod_{i \in I} M_i$.*

Proof. $\prod_{i \in I}^{\kappa} M_i$ is the directed union of the modules $\prod_{i \in J} M_i$, $|J| < \kappa$, which are direct factors of $\prod_{i \in I} M_i$. Whence $\prod_{i \in I}^{\kappa} M_i$ is pure $\prod_{i \in I} M_i$. One computes easily

$$\varphi\left(\prod_{i \in I}^{\kappa} M_i\right) = \prod_{i \in I}^{\kappa} \varphi(M_i).$$

Whence

$$\varphi/\psi\left(\prod_{i \in I} M_i\right) = \prod_{i \in I} \varphi/\psi(M_i) = \varphi/\psi\left(\prod_{i \in I}^{\kappa} M_i\right).$$