Annals of Pure and Applied Logic 30 (1986) 63–82
North-Holland

QUASI FINITELY AXIOMATIZABLE TOTALLY CATEGORICAL THEORIES

Gisela AHLBRANDT
Dept. of Mathematics and Computer Science, Eastern Michigan University, Ypsilanti, MI 48197, USA

Martin ZIEGLER
Mathematisches Institut, Beringstrasse 4, 5300 Bonn 1, FRG

Communicated by A. Prestel
Received 12 January 1985

0. Introduction

As was shown in [2], totally categorical structures (i.e. which are categorical in all powers) are not finitely axiomatizable. On the other hand, the most simple totally categorical structures: infinite sets, infinite projective or affine geometries over a finite field, are quasi finitely axiomatizable (i.e. axiomatized by a finite number of axioms and the schema of infinity, we will use the abbreviation 'qfa'. Since all totally categorical structures are 'built up' from these simple structures, it was conjectured in [2] that all totally categorical structures are quasi finitely axiomatizable (which from now on means: being interdefinable with a qfa structure).

We prove in this paper

Theorem. *All totally categorical almost strongly minimal theories are quasi finitely axiomatizable.*

This includes the case of totally categorical structures of modular type (i.e. which do not have affine geometries attached), for example, structures having a disintegrated set attached.

In Sections 1 and 4 we deal with transfer theorems, which allow us to infer qfa of one structure from the qfa from a related structure.

The results of Section 1 are essentially known: qfa is invariant under bi-interpretability of structures and bi-interpretability of structures can easily be checked by looking at the automorphism groups: Two countable $\aleph_0$-categorical structures are bi-interpretable iff their automorphism groups are isomorphic as topological groups. In Section 4 we look at countable $\aleph_0$-categorical structures $\mathfrak{A} = (A, W)$ with an n-to-one surjection $\pi: A \to W$. If W (with the structure induced by $\mathfrak{A}$) has a 'nice' enumeration, we can lift qfa from W to $\mathfrak{A}$. A modular Grassmannian W is the structure of N-element (dimensional) subsets (subspaces) of a countable set (projective geometry over a finite field). By the methods of Section 1, W is again qfa and in Section 3 we show that W has a nice enumeration.

The proof of the theorem is now completed in Section 2 where it is proved that every countable totally categorical strongly minimal structure is bi-interpretable with $\mathfrak{A}$ as considered in Section 4, where W is a modular Grassmannian. For arbitrary totally categorical structures a similar theorem is not known. Thus the question whether all totally categorical structures are qfa remains open.

It is also not known if – up to interdefinability – there are only countably many totally categorical theories. Our theorem implies that there are only countably many totally categorical almost strongly minimal theories. But we were not able to find such theories explicitly.

Remark. Our proof also shows that every totally categorical strongly minimal structure is interdefinable with a structure which is model complete and has a finite language.

Special cases of our theorem were proved earlier: the disintegrated case by the second author (1983), the case where W is a projective geometry over the field with 2 elements by the first author [1].

1. Interpretations

All structures which are considered in this section are countable, $\aleph_0$-categorical and have a countable language.

Definition. Let the L-structure $\mathfrak{A}$ and the L'-structure $\mathfrak{A}'$ have the same universe A. We say that $\mathfrak{A}'$ is *definable* in $\mathfrak{A}$, if every relation (function, constant) $R^{\mathfrak{A}'}$ ($R \in L'$) is $\mathfrak{A}$-definable.

Theorem 1.1. *$\mathfrak{A}'$ is definable in $\mathfrak{A}$ iff* $\operatorname{Aut}\mathfrak{A} \subset \operatorname{Aut}\mathfrak{A}'$

Proof. The relations definable in $\mathfrak{A}$ are just the relations which are invariant under all automorphisms of $\mathfrak{A}$.

Definition. If $\mathfrak{A}'$ is definable in $\mathfrak{A}$, and $\mathfrak{A}$ is definable in $\mathfrak{A}'$, then we call $\mathfrak{A}'$ and $\mathfrak{A}$ *interdefinable.*

A lot of properties are invariant under interdefinability e.g. ω-stability, $\aleph_1$-categoricity (and $\aleph_0$-categoricity).

Definition. A structure $\mathfrak{A}$ is *quasi finitely axiomatizable,* if there is a structure $\mathfrak{A}'$ interdefinable with $\mathfrak{A}$ such that $\mathrm{Th}(\mathfrak{A}')$ is axiomatized by finitely many axioms and the scheme of infinity

$$\exists x_1, \ldots, x_n \bigwedge_{i \neq j} x_i \neq x_j \qquad (n = 2, 3, \ldots).$$

(Note, that the language of $\mathfrak{A}'$ must be finite. If the language of $\mathfrak{A}$ is finite, we can take $\mathfrak{A} = \mathfrak{A}'$.)

Interpretability is a generalization of definability. In many cases one checks interpretability most easily by looking at the automorphism-groups. This is the reason for the following study, where we turn the class $\mathcal{K}$ of all structures (i.e. countable, $\aleph_0$-categorical, countable languages) into a category and Aut into a functor from $\mathcal{K}$ to the category of topological groups.

The morphisms of $\mathcal{K}$ are interpretations $f: \mathfrak{A} \rightsquigarrow \mathfrak{B}$. This is a surjection $f: U \to B$, where B is the universe of $\mathfrak{B}$, U is an $\mathfrak{A}$-definable subset of A^n and the following relations are definable in $\mathfrak{A}$:

$$=_f = \{(a_1, \ldots, a_n, a'_1, \ldots, a'_n) \in U^2 \mid f(a_1, \ldots, a_n) = f(a'_1, \ldots, a'_n)\},$$
$$Rf = \{(a^1_1, a^1_2, \ldots, a^1_n, a^2_1, \ldots, a^2_n, \ldots, a^m_1, \ldots, a^m_n) \in U^m \mid R^{\mathfrak{B}}(f(a^1_1, \ldots, a^1_n), \ldots, f(a^m_1, \ldots, a^m_n))\}$$

for all R in the language of $\mathfrak{B}$. (For simplicity we consider here only relational languages.)

The identical interpretation $1_{\mathfrak{A}}$ is given by $\mathrm{id}_A: A \to A$.

The composition of two interpretations $f: \mathfrak{A} \rightsquigarrow \mathfrak{B}$ and $g: \mathfrak{B} \rightsquigarrow \mathfrak{C}$ is defined as follows: If $f: U \to B$, $U \subset A^n$ and $g: V \to C$, $V \subset B^m$, we define

$$g \circ f: \mathfrak{A} \rightsquigarrow \mathfrak{C}$$

by $g * f: W \to C$, where

$$W = \{(a^1_1, \ldots, a^1_n, \ldots, a^m_1, \ldots, a^m_n) \in U^m \mid (f(a^1_1, \ldots, a^1_n), \ldots, f(a^m_1, \ldots, a^m_n)) \in V\}$$

and

$$g \circ f(a^1_1, \ldots) = g(f(a^1_1, \ldots), \ldots, f(a^m_1, \ldots)).$$

It is easy to check that this makes $\mathcal{K}$ a category.

Now we turn Aut into a functor. First note that Aut $\mathfrak{A}$ is a topological group whose basis of open neighbourhoods of 1 are the subgroups $\mathrm{Aut}(\mathfrak{A}, a_1, \ldots, a_k)$.

If f is an interpretation $f: \mathfrak{A} \rightsquigarrow \mathfrak{B}$ and $\sigma \in \mathrm{Aut}\, \mathfrak{A}$, then there is a unique permutation ρ of B which makes the following diagram commutative (σ operates in a natural way on U):

$$\begin{array}{ccc} U & \overset{f}{\rightsquigarrow} & B \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\rho} \\ U & \underset{f}{\rightsquigarrow} & B. \end{array}$$

ρ is always an automorphism of $\mathfrak{B}$. And, if we define $\operatorname{Aut} f(\sigma)=\rho$, $\operatorname{Aut} f$ is a continuous homorphism from $\operatorname{Aut}\mathfrak{A}$ to $\operatorname{Aut}\mathfrak{B}$. As one sees easily Aut is now a functor from $\mathscr{K}$ to the category of topological groups.

Remark. Let B be a trivial structure (i.e. having the empty language or — more generally — such that $\operatorname{Aut} B = \operatorname{Sym} B$ (=full permutation group)). If $f:\mathfrak{A}\rightsquigarrow B$ is an interpretation, the closure of the image of $\operatorname{Aut} f$ is already the automorphism group of an $\aleph_0$-categorical structure with universe B: the induced structure on B. This structure is uniquely defined up to interdefinability.

Theorem 1.2. *A continuous homorphism* $\varphi:\operatorname{Aut}\mathfrak{A}\to\operatorname{Aut}\mathfrak{B}$ *is of the form* $\operatorname{Aut} f$ *for an interpretation* $f:\mathfrak{A}\rightsquigarrow\mathfrak{B}$ *iff the image of* φ *has only finitely many orbits.*

Proof. The necessity follows from the above remark: in fact $\operatorname{im}\varphi$ is again the automorphism group of an $\aleph_0$-categorical structure.

For sufficiency choose representatives $b_1, \ldots, b_k$ for the orbits of $\operatorname{im}\varphi$. Choose $a_1, \ldots, a_m$ such that $\operatorname{aut}(\mathfrak{A}, a_1, \ldots, a_m)$ is mapped into $\operatorname{Aut}(\mathfrak{B}, b_1, \ldots, b_k)$ by φ. We can assume that $k \leq m$ and all a_i are different. Let $U \subset A^{m+1}$ consist of all the conjugates of $(a_1, a_1, \ldots, a_m)$, $(a_2, a_1, \ldots, a_m), \ldots, (a_k, a_1, \ldots, a_m)$. Define $f:U\to B$ by

$$f(\sigma(a_i), \sigma(a_1), \ldots, \sigma(a_m)) = \varphi(\sigma)b_i, \quad i=1,\ldots,k, \quad \sigma\in\operatorname{Aut}\mathfrak{A}.$$

Remark. We do not have to assume that $\mathfrak{B}$ is $\aleph_0$-categorical. This will follow.

Definition. Two interpretations $f:\mathfrak{A}\rightsquigarrow\mathfrak{B}$ and $g:\mathfrak{A}\rightsquigarrow\mathfrak{B}$ are homotopic ($f\sim g$), if the relation

$$(f=g)=\{(a_1,\ldots,a_m,b_1,\ldots,b_n) \in U\times V \mid f(a_1,\ldots,a_m)=g(b_1,\ldots,b_n)\}$$

is $\mathfrak{A}$-definable.

Theorem 1.3. *f and g are homotopic iff* $\operatorname{Aut} f=\operatorname{Aut} g$.

Proof. This is a simple matter to check, using the fact that definable = $\operatorname{Aut}\mathfrak{A}$-invariant.

Definition. $\mathfrak{A}$ is called a *retraction* of $\mathfrak{B}$ if there are interpretations $f:\mathfrak{A} \rightsquigarrow \mathfrak{B}$ and $g:\mathfrak{B} \rightsquigarrow \mathfrak{A}$ such that $g \circ f \sim 1_{\mathfrak{A}}$. If moreover $f \circ g \sim 1_{\mathfrak{B}}$, then $\mathfrak{A}$ and $\mathfrak{B}$ are called *bi-interpretable*.

Corollary 1.4 [4]. (i) *$\mathfrak{A}$ is a retraction of $\mathfrak{B}$ iff there are continuous homomorphisms*

$$\text{Aut}\,\mathfrak{A} \xrightarrow{\varphi} Aut\,\mathfrak{B} \xrightarrow{\psi} \text{Aut}\,\mathfrak{A} \quad \textit{such that} \quad \psi \circ \varphi = 1.$$

(ii) *$\mathfrak{A}$ and $\mathfrak{B}$ are bi-interpretable iff* Aut $\mathfrak{A}$ *and* Aut $\mathfrak{B}$ *are isomorphic as topological groups.*

Theorem 1.5 [4]. *If $\mathfrak{A}$ is a retraction of $\mathfrak{B}$ and $\mathfrak{B}$ is quasi finitely axiomatizable, then also $\mathfrak{A}$ is quasi finitely axiomatizable.*

Proof. We can assume that $\mathfrak{B}$ is of finite language. Now, besides saying that $\mathfrak{A}$ is infinite, the axioms of $\mathfrak{A}$ will tell that (via f) there is a structure ($\equiv\mathfrak{B}$) interpreted in $\mathfrak{A}$, which satifies a certain finite number of axioms (=the axioms of $\mathfrak{B}$) and which again has in it a structure interpreted (via g) which is definably isomorphic to $\mathfrak{A}$ itself. The latter we need only to express for those relation symbols of the language of $\mathfrak{A}$ which occur in the previous axioms. In fact $\mathfrak{A}$ is interdefinable with its restriction to this finite language.

Theorem 1.6. *For any open subgroup G of* Aut $\mathfrak{A}$ *there is a structure $\mathfrak{B}$, an isomorphism $\Phi:\text{Aut}\,\mathfrak{A} \to \text{Aut}\,\mathfrak{B}$ and an element $b \in B$ such that $\varphi(G) =$* $\text{Aut}(\mathfrak{B}, b)$.

Proof. We find $\text{Aut}(\mathfrak{A}, a_1, \ldots, a_n) \subset G$. If $C \subset A^n$ is the G-orbit of $(a_1, \ldots, a_n)$, we have $G = \{\sigma \in \text{Aut}\,\mathfrak{A} \mid \sigma(C) = C\}$. Note that C is definable with parameters $a_1, \ldots, a_n$. Let D be the set of all conjugates of C. Set $\mathfrak{B} = (\mathfrak{A}, D, \epsilon)$ – a 2-sorted structure, where $\epsilon \subset A \times D$. There is an obvious isomorphism $\varphi:\text{Aut}\,\mathfrak{A} \to \text{Aut}\,\mathfrak{B}$. ($\varphi$ is continuous, since all elements of D are definable with parameters in $\mathfrak{A}$.) One sees also that Aut $\mathfrak{B}$ has only finitely many orbits. By the remark following 1.2 we conclude that $\mathfrak{B}$ is $\aleph_0$-categorical. Clearly $\varphi(G) = \text{Aut}(\mathfrak{B}, C)$.

Corollary 1.7 (Essentially in [5]). *If $\mathfrak{A}'$ is definable in $\mathfrak{A}$ and* Aut $\mathfrak{A}$ is open in Aut $\mathfrak{A}'$, *then $\mathfrak{A}$ is quasi finitely axiomatizable iff $\mathfrak{A}'$ is.*

Proof. We find a $\mathfrak{B}$ and $b \in B$ such that $\text{Aut}\,\mathfrak{A}' \cong \text{Aut}\,\mathfrak{B}$ and $\text{Aut}\,\mathfrak{A} \cong \text{Aut}(\mathfrak{B}, b)$. Having the same automorphism group preserves qfa by 1.4 and 1.5. So we have to know that $\mathfrak{B}$ is qfa iff $(\mathfrak{B}, b)$ is. But this is trivially true (for $\aleph_0$-categorical structures of course).

Remark. We can derive the $\aleph_0$-categoricity of $\mathfrak{A}$ from the $\aleph_0$-categoricity of $\mathfrak{A}'$.

Example 1. Let H be an $\aleph_0$-categorical minimal set (i.e. a structure whose universe is a minimal set.) Let W be the set of all algebraically closed subsets of H of dimension N. There is an obvious map $\varphi: \mathrm{Aut}\, H \to \mathrm{Sym}\, W$. The image of φ operates transitively on W. By 1.2 (and the remark) we obtain an interpretation of W in H, which induces an $\aleph_0$-categorical structure on W having $\mathrm{im}\, \varphi$ as automorphism group. We call this structure W the N-Grassmannian of H.

Example 2. Let V be an $\aleph_0$-dimensional vector space over a finite field F and H the set of all one-dimensional subspaces. $\mathrm{GL}(V)$ induces a group Γ_0 of permutations of H. Also – if we fix a basis of V – $\mathrm{Aut}\, F$ induces a group of permutations of H, which we also call $\mathrm{Aut}\, F$. Let Γ be the permutation group generated by Γ_0 and $\mathrm{Aut}\, F$ (in a semi-direct way). The fundamental theorem of projective geometry tells us that Γ is the automorphism group of the projective space (H, Coll), where $\mathrm{Coll} = \{(a, b, c) \mid \dim(\{a, b, c\}) \leq 2\}$. Also, by well known axioms, (H, Coll) is qfa and totally categorical. Since Γ_0 is of finite index in Γ $(=|\mathrm{Aut}\, F|)$, Γ_0 is open in Γ. Whence, by 1.6 and 1.7 every group G between Γ_0 and Γ is the automorphism group of an $\aleph_0$-categorical structure on H and all these structures are qfa. We call these structures projective geometries.

Remark. All these structures have the same notion of algebraic closure: $\mathrm{acl}(S) = \langle S \rangle$ = the subspace spanned by S.

To define affine geometries, we start with $H = V$ and the permutation groups Γ_0, generated by $\mathrm{GL}(V)$ and the transvections $x + a$ $(a \in V)$, and Γ, generated by Γ_0 and $\mathrm{Aut}\, F$. Again every G between Γ_0 and Γ is the automorphism group of a qfa, totally categorical structure on H: the affine geometries.

Example 3. If H is either a disintegrated set or a projective geometry over a finite field, then H is minimal and the map from $\mathrm{Aut}\, H$ to the automorphism group of the Grassmannian W is an isomorphism. Therefore W and H are bi-interpretable and we can conclude that W is quasi finitely axiomatizable.

Example 4. Let $f: \mathfrak{A} \rightsquigarrow B$ be an interpretation. f induces in a natural way an interpretation of the pair (A, B) in $\mathfrak{A}$. We denote the induced structure by $(\mathfrak{A}, B)$. Clearly this structure is bi-interpretable with $\mathfrak{A}$. We say that B (with the induced structure) is attached to $\mathfrak{A}$.

2. Almost strongly minimal structures

In [2] it is shown that every totally categorical structure has a strictly minimal set H attached to it, i.e. H is strongly minimal and $\{a\} = \mathrm{acl}(a) \cap H$ for all $a \in H$.

The classification theorem of Cherlin and Zil'ber says that every (countable)

$\aleph_0$-categorical strictly minimal set is (up to interdefinability) disintegrated, a projective geometry or an affine geometry over a finite field. [2]

Disintegrated and projective strictly minimal sets are called modular.

Definition. Let $\mathfrak{A}$ be a countable, totally categorical structure.

(i) $\mathfrak{A}$ is of *modular type*, if for every finite sequence $a_1, \ldots, a_n \in A$ every strictly minimal set attached to $(\mathfrak{A}, a_1, \ldots, a_n)$ is modular.

(ii) $\mathfrak{A}$ is *almost strongly minimal*, if there is a finite sequence $a_1, \ldots, a_n \in A$ and a strongly minimal set H definable with parameters $a_1, \ldots, a_n$ such that A is algebraic over $H \cup \{a_1, \ldots, a_n\}$.

The following theorem is shown in [1].

Theorem 2.1. *If the totally categorical structure $\mathfrak{A}$ is of modular type, then it is almost strongly minimal.*

If a disintegrated set is attached to some $(\mathfrak{A}, a_1, \ldots, a_n)$, then $\mathfrak{A}$ is of modular type (and whence almost strongly minimal.)

It is well known that there are totally categorical structures which are not almost strongly minimal, e.g. $(\mathbb{Z}/4\mathbb{Z})^{\aleph_0}$. By [1] these are exactly those structures for which there exists no finite sequence $a_1, \ldots, a_n$ for which $(\mathfrak{A}, a_1, \ldots, a_n)$ is of modular type.

Theorem 2.2. *For every (countable) $\aleph_0$-categorical, almost strongly minimal structure $\mathfrak{B}$ there exists a two-sorted structure $\mathfrak{A} = (A, W)$ such that*

(i) *$\mathfrak{A}$ is bi-interpretable with $(\mathfrak{B}, b_1, \ldots, b_n)$ for some finite sequence $b_1, \ldots, b_n$ in B.*

(ii) *There is a 0-definable surjection $\pi: A \to W$ with fibers of a fixed finite cardinality.*

(iii) *W (with the induced structure) is a modular Grassmannian.*

Proof. Let H be strongly minimal, definable with parameters $b_2, \ldots, b_n$, such that B is algebraic over $H \cup \{b_2, \ldots, b_n\}$. Choose b_1 to be a non-algebraic element of H. Now we argue in the structure $\mathfrak{B}' = (\mathfrak{B}, b_1, \ldots, b_n)$.

Call two elements of H equivalent, if they have the same algebraic closure. The set H' of equivalence classes of non-algebraic elements of H is a strictly minimal set attached to $\mathfrak{B}'$. It is modular, by the choice of b_1. Since the equivalence classes are finite, B is algebraic over H'. Choose N minimal such that every element of B is algebraic over an N-dimensional, algebraically closed subset of H'. Let W be the N-Grassmannian of H' and $A = \{(b, w) \mid b \in \operatorname{acl} w\}$. The obvious map $\operatorname{Aut} \mathfrak{B}' \to \operatorname{Sym}(A, W)$ is easily seen to be an isomorphism onto a subgroup of $\operatorname{Sym}(A, W)$, which is the automorphism group of an $\aleph_0$-categorical structure $\mathfrak{A}$ with universe (A, W). Clearly the projection π to the second component is invariant under all automorphisms, whence definable, and all fibers have the same cardinality, since W is transitive.

Remark. The minimal possible N is the Morley rank of $\mathfrak{B}'$.

3. Nice enumerations

Definition. A set P together with a transitive and reflexive relation $\leq$ is called a *partial well ordering,* if every subset A of P is generated by finitely many elements a_i $(i = 1, \ldots, n)$, i.e., $\bar{a}_i \in A$ and for all $b \in A$ there is an a_i below: $a_i \leq b$. This is equivalent to saying that P is well founded and has no infinite antichains.

Let w_i^* $(i \in \omega)$ be an enumeration of the structure W. We introduce the following notations:
$w_{<n}^* = \{w_0^*, w_1^*, \ldots, w_{n-1}^*\}$.
Σ is the set of all pairs (w, S) which are conjugate to a pair $(w_n^*, w_{<n}^*)$.
$(w', S') \leq (w, S)$, if for some subset S'' of S, (w', S') is conjugate to (w, S'').

Definition. An enumeration w_i^* $(i \in \omega)$ of a structure W is called *nice,* if the following three properties hold.

(i) $(\Sigma, \leq)$ is a partial well ordering.

(ii) There is a finite bound k_0 such that for $(w, S) \in \Sigma$ with $|S| \geq k_0$, $\mathrm{tp}(w/S)$ is either algebraic or minimal.

Notation: A set S conjugate to some $w_{<n}^*$ is called 'nice'.

(iii) For all k there is a k' such that between any pair of sets $T \subset S$, where $|T| < k$ and S is nice, we can find a nice set S' with $|S'| < k'$.

Theorem 3.1. *Let H be either a countable disintegrated set (i.e. with trivial structure) or an $\aleph_0$-dimensional projective geometry over a finite field. Then any Grassmannian W of H has a nice enumeration.*

We will make use of a trivial and a non-trivial lemma. The trivial one is

Lemma 3.2. *If $f: P_1 \to P_2$ satisfies $f(x) \leq_2 f(y) \to x \leq_1 y$, then $(P_1, \leq_1)$ is a partial well ordering, if $(P_2, \leq_2)$ is.*

Lemma 3.3 [3]. *Let Ω be a finite alphabet. Then Ω^* – the set of all finite words in Ω – becomes a partial well ordering, if we define: $(a_0, \ldots, a_{m-1}) \leq (b_0, \ldots, b_{n-1})$ iff there is a monotone map $\varepsilon: m \to n$ such that $a_i = b_{\varepsilon(i)}$ $(i < m)$.*

We split the proof of 3.1 into three cases: the case when H is disintegrated, the case when H is a projective geometry and $W = H$, and the general case of a Grassmannian W of a projective geometry.

Lemma 3.4. *W has a nice enumeration, if W is the structure of all N-element subsets of a disintegrated set H.*

Proof. We identify H with the set ω of natural numbers. We then write every $w \in W$ as $\{w_1, \ldots, w_N\}$, where $w_1 < w_2 < \cdots < w_N$. This gives rise to a lexicographical ordering of W: $v < w$ iff there is an i such that $v_i < w_i$ and $v_j = w_j$ for all $j > i$. This orders W of type ω and defines our enumeration. Notations:

$$w_< = \{v \in W \mid v < w\}, \qquad \Sigma' = \{(w, w_<) \mid w \in W\}$$

We have to show that (i), (ii) and (iii) of the definition are true.

(i) It is enough to show that $(\Sigma', \leqslant)$ is a partial well ordering. We will apply Lemma 3.2 and Lemma 3.3.

The alphabet we use is $\Omega = \{0, 1\}$. We attach a word w^* to every $w \in W$ as follows:

$$w^* : w_N + 1 \to \Omega, \qquad w^*(i) = \begin{cases} 0, & \text{if } i \notin w, \\ 1, & \text{if } i \in w. \end{cases}$$

It is enough to show that

$$v^* \leqslant w^* \quad \Rightarrow \quad (v, v_<) \leqslant (w, w_<)$$

Proof. If $v^* \leqslant w^*$ via $\varepsilon : v_N + 1 \to w_N + 1$, we choose a permutation σ of H which extends ε. σ yields an automorphism of W, which is also denoted by σ. It is easy to see that

$$u < v \quad \Rightarrow \quad \sigma(u) < \sigma(v) = w.$$

This shows that $(v, v_<) \leqslant (w, w_<)$ via σ (i.e. $\sigma(v) = w$, $\sigma(v_<) \subset w_<$).

(ii) Set $k_0 = N + 1$. Assume that $(w, S) \in \Sigma'$ and $|S| \geqslant N + 1$. Since the set of all N-element subsets of $\{0, \ldots, N\}$ forms the first $N + 1$ elements of our enumeration, all these sets belong to S and we have $N < w_N$. Choose two different elements a, b from $\{0, \ldots, N\} \setminus \{w_1, \ldots, w_{N-1}\}$. Then $\{a, w_1, \ldots, w_{N-1}\}$ and $\{b, w_1, \ldots, w_{N-1}\}$ belongs to S. Whence

$$\{w_1, \ldots, w_{N-1}\} = \{a, w_1, \ldots, w_{N-1}\} \cap \{b, w_1, \ldots, w_{N-1}\}$$

is definable in S (as an element of the $N - 1$)-Grassmannian of H, which is attached to W).

$w \notin \operatorname{acl} S$ means $w \not\subset \bigcup S$. In our case this implies $w_N \notin \bigcup S$. Thus $\operatorname{tp}(w_N/S)$ is minimal. (Here we think of H being attached to W.) By the above this implies that $\operatorname{tp}(w/S)$ is minimal.

(iii) Let k be given. $k' = \binom{kN}{N}$ will do.

So let T be a subset of $w_<$ and $|T| < k$. Then, if $J = \bigcup T \cup w$ we have $|J| = k'' \leqslant kN$. Choose a permutation σ of H which induces a monotone map from k'' onto J. Let $v \in W$ be the preimage of w (σ being extended to an automorphism of W). Then, if we set $S' = \sigma(v_<)$, we have $T \subset S' \subset w_<$ and $|S'| \leqslant \binom{k''}{N} \leqslant k'$.

Lemma 3.5. *W has a nice enumeration, if W is an $\aleph_0$-dimensional projective geometry over a finite field F.*

Proof. Let V be a vector space over F with basis $v_0, v_1, \ldots, v_i, \ldots$ $(i \in \omega)$. Let, for $a \in V\setminus\{0\}$, $\tilde{a}$ denote the 1-dimensional subspace generated by a. We can assume that W consists of all $\tilde{a}$ and that every linear automorphism of V induces an automorphism of W.

We fix a linear ordering of F with 0 as the least and 1 as the second least element.

Then, a lexicographical ordering of V is defined by

$$a = \sum \alpha_i v_i < b = \sum \beta_i v_i \quad \text{iff} \quad \text{for some } i,\ \alpha_i < \beta_i \text{ and } \alpha_j = \beta_j \text{ for all } j > i.$$

Our enumeration of W is given by an ordering of W, which is defined as follows:

$$v < w \quad \text{iff} \quad a < b, \text{ where } a, b \text{ are minimal elements of } V \text{ representing } v \text{ resp. } w.$$

Note that a is minimal representing a iff a has the form

$$a = \sum_{i<m} \alpha_i v_i + v_m.$$

(i) We show that $(\Sigma', \leqslant)$ is a partial wellordering, using 3.2 and 3.3. Set $\Omega = F \cup \{\infty\}$. Let $b = \sum_{i<n} \beta_i v_i + v_n$ represent $w \in W$. We define then w^* by

$$w^*: n+1 \to \Omega, \qquad w^*(i) = \begin{cases} \beta_i, & i < n, \\ \infty, & i = n. \end{cases}$$

Claim. $v^* \leqslant w^* \Rightarrow (v, v_<) \leqslant (w, w_<)$.

Proof. Let $a = \sum_{i<m} \alpha_i v_i + v_m$ represent v. If $v^* \leqslant w^*$ via $\varepsilon: m+1 \to n+1$, then $\alpha_i = \beta_{\varepsilon(i)}$ and $\varepsilon(m) = n$.

Choose an automorphism σ of V with

$$\sigma(v_i) = \begin{cases} v_{\varepsilon(i)}, & i < m, \\ v_n + \sum \{\beta_j v_j \mid j < n, j \notin \{\varepsilon(0), \ldots, \varepsilon(m)\}\}, & i = m. \end{cases}$$

Clearly $\sigma(a) = b$. If $u < v$ take c minimal representing u. Then $c < a$. By the next lemma we have $\sigma(c) < \sigma(a)$. Whence $\sigma(u) = \widetilde{\sigma(c)} < \tilde{b} = w$. This proves the claim.

Lemma 3.6. *Let $a = \sum_{i \leqslant m} \alpha_i v_i$, an increasing sequence of numbers $\varepsilon(0) < \varepsilon(1) < \cdots < \varepsilon(m)$ and an automorphism σ of V be given. Suppose that for all $i \leqslant m$, $\sigma(v_i) \in \langle v_0, v_1, \ldots, v_{\varepsilon(i)} \rangle$ and, if $\alpha_i \neq 0$, that*

$$\sigma(v_i) - v_{\varepsilon(i)} \in \langle v_j \mid j < \varepsilon(i), j \notin \{\varepsilon(0), \ldots, \varepsilon(i)\} \rangle.$$

Then $c < a$ implies $\sigma(c) < \sigma(a)$.

Proof. If $c = \sum \gamma_i v_i$, $\sigma(c) = \sum \delta_i v_i$, $\sigma(a) = \sum \beta_i v_i$, we have for some i, $\gamma_i < \alpha_i$ and $\gamma_j = \alpha_j$ for all $j > i$. Note that $\alpha_i \neq 0$. Now it is easy to see that $\delta_{\varepsilon(i)} = \gamma_i < \alpha_i = \beta_{\varepsilon(i)}$ and $\delta_j = \beta_j$ for all $j > \varepsilon(i)$. Whence $\sigma(c) < \sigma(a)$.

Now we continue the proof of Lemma 3.5

(ii) We can use $k_0 = 0$: Since W is a minimal set, we have $\mathrm{tp}(w/T)$ algebraic or minimal.

(iii) If k is given and $f = |F|$, set

$$k' = \frac{f f^k - 1}{f - 1}.$$

Now suppose $T \subset w_<$, $|T| < k$ and $w_< \subset \langle \bar{v}_0, \ldots, \bar{v}_n \rangle$. Define an equivalence relation E on $\{0, \ldots, n\}$ by

$$iEj \quad \text{iff} \quad \gamma_i = \gamma_j \quad \text{for all } c = \sum \gamma_i v_i, \quad \bar{c} \in T \cup \{w\}.$$

We enumerate the equivalence classes $E_0, E_1, \ldots, E_{k''-1}$ in such a way that $\max E_i < \max E_j$ whenever $i < j$. Choose an automorphism σ of V such that $\sigma(v_i) = \sum \{v_j \mid j \in E_i\}$ $(i < k'')$. σ restricted to $\langle v_0, \ldots, v_{k''-1} \rangle$, which is an initial segment of W, respects the ordering. Furthermore we have $\sigma(\{\bar{v}_0, \ldots, \bar{v}_{k''-1}\rangle) \supset T \cup \{w\}$. Whence, if v is the preimage of w, we have $T \subset (v_<) \subset w_<$. Finally note that $k'' \leqslant f^k$ and

$$|v_<| < |\langle \bar{v}_0, \ldots, \bar{v}_{k''-1} \rangle| = \frac{f^{k''} - 1}{f - 1} \leqslant k'.$$

Lemma 3.7. *W has a nice enumeration, if W is the N-Grassmannian of an $\aleph_0$-dimensional projective geometry over a finite field F.*

Proof. Let W be the N-Grassmannian of the geometry H, V as in the proof of Lemma 3.5 such that $H = \{\bar{a} \mid a \in V \setminus \{0\})$.

As in the proof of Lemma 3.4 the ordering of V gives rise to an ordering of all N-element linearly independent subsets

$$\bar{a} = \{a_1, \ldots, a_n\} \qquad (a_1 < a_2 < \cdots < a_N)$$

of V. We use the notation $\tilde{a} = \langle \bar{a}_1, \ldots, \bar{a}_N \rangle \in W$.

This gives us an ordering of W, which defines our nice enumeration: For v, w in W choose minimal $\bar{a}$, $\bar{b}$ such that $v = \tilde{a}$ and $w = \tilde{b}$. Then set $v < w$ iff $\bar{a} < \bar{b}$.

(i) Let Ω be the alphabet used in the proof of 3.5. We will use 3.2 and 3.3, now for the alphabet Ω^N.

For any $w \in W$ choose $\bar{b}$ minimal such that $w = \tilde{b}$. Write $b_l = \sum \beta_{l,i} v_i + v_{n_l}$, Define $w^*: n_N + 1 \to \Omega^N$ by

$$w^*(i) = (\beta'_{1,i}, \ldots, \beta'_{N,i}), \quad \text{where } \beta'_{l,i} = \begin{cases} \beta_{l,i} & i < n_1, \\ \infty, & i = n_1, \\ 0, & (i > n_l. \end{cases}$$

Remark. Since otherwise we could produce a smaller $\bar{b}$ representing w (using 'row operations'), we have

(a) $n_1 < n_2 < \cdots < n_N$,

(b) $\beta_{l,n_k} = 0$, if $l \neq k$.

Claim. $v^* \leq w^* \Rightarrow (v, v_<) \leq (w, w_<)$.

Proof. Let $\bar{a}$ be minimal representing v, $a_l = \sum \alpha_{l,i} v_i$. If $v^* \leq w^*$ via $\varepsilon : m_N + 1 \to n_N + 1$, then $\alpha_{l,i} = \beta_{l,\varepsilon(i)}$ and $\varepsilon(m_l) = n_l$. Choose an automorphism σ of V such that for all $i = 0, \ldots, m_N$,

$$\sigma(v_i) = \begin{cases} v_{\varepsilon(i)}, & i \notin \{m_1, \ldots, m_N\}, \\ v_{n_1} + \sum \{\beta_{l,j} v_j \mid j \notin \{\varepsilon(0), \varepsilon(1), \ldots, \varepsilon(m_N)\}\}, & (i = m_l). \end{cases}$$

By the above remark (applied to v) it is clear that $\sigma(a_l) = b_l$ and thus $\sigma(v) = w$.

Now suppose $u < v$. Choose $\bar{c}$ minimal such that $u = \bar{c}$. We then have $\bar{c} < \bar{a}$. (a) of the above remark applied to u shows that $\sigma(c_1) < \cdots < \sigma(c_l)$. There is an l such that $c_l < a_l$ and $c_l = a_k$ for all $k > l$. By Lemma 3.6 and the remark applied to v we can conclude that $\sigma(c_l) < \sigma(a_l)$. Whence $\bar{d} = \sigma(\bar{c}) < \bar{b}$. This implies $\sigma(u) = \bar{d} < \bar{b} = \sigma(w)$. Thus we have shown that $(v, v_<) \leq (w, w_<)$ via σ.

(ii) Set

$$k_0 = \frac{f^{N+1} - 1}{f - 1}.$$

Assume that $(w, S) \in \Sigma'$ and $|S| > k_0$. Since the set of all N-dimensional subspaces of $v_0, \ldots, v_N$ forms the first k_0 elements of our enumeration of W, all these sets belong to S. Since $a \leq v_N$ implies $a \in \langle v_0, \ldots, v_N \rangle$, we have $v_0 < v_1 < \cdots < v_N < b_N$. Pick two elements a, b from $\langle v_0, \ldots, v_N \rangle$ which are linearly independent over $\langle b_1, \ldots, b_{N-1} \rangle$. Then $\langle \tilde{a}, \tilde{b}_1, \ldots, \tilde{b}_{N-1} \rangle$ and $\langle \tilde{b}, \tilde{b}_1, \ldots, \tilde{b}_{N-1} \rangle$ belong to S. Whence

$$\langle \tilde{b}_1, \ldots, \tilde{b}_{N-1} \rangle = \langle \tilde{a}, \tilde{b}_1, \ldots, \tilde{b}_{N-1} \rangle \cap \langle \tilde{b}, \tilde{b}_1, \ldots, \tilde{b}_{N-1} \rangle$$

is definable in S (as an element of the $(N-1)$-Grassmannian of H, which is attached to W).

$w \notin \operatorname{acl} S$ implies therefore that $b_N \notin \operatorname{acl} S$. But $\operatorname{tp}(b_N/S)$ is minimal (in H being attached to W.) By the above this implies that $\operatorname{tp}(w/S)$ is minimal.

(iii) If k is given, take for k' the number of N-dimensional subspaces of an f^{Nk}-dimensional projective space over F, $(f = |F|)$.

Assume $T \subset w_<$ and $|T| < k$. By the proof of Lemma 3.5 (iii), there is a number

$s < f^{Nk}$ and an automorphism σ which preserves the ordering on $\langle v_0, \ldots, v_s \rangle$ and such that $\bigcup T \cup w$ is included in $\sigma(\langle \tilde{v}_0, \ldots, \tilde{v}_s \rangle)$. The preimage v of w is a subspace of $\langle \tilde{v}_0, \ldots, \tilde{v}_s \rangle$. Whence we have $T \subset \sigma(v_<) \subset w_<$. (Note that δ also preserves the ordering of the N-dimensional subspaces of $\langle \tilde{v}_0, \ldots, \tilde{v}_s \rangle$.)

4. Proof of the theorem

Let $\mathfrak{A}$ be an $\aleph_0$-categorical two-sorted structure (A, W) with the following properties:

(a) There is a 0-definable surjection $\pi : A \to W$ all fibers of which $A_w = \pi^{-1}(w)$ are finite, of cardinality κ.

(b) W has a nice enumeration (with the properties (i), (ii), (iii) of 3.1).

We look at W carrying the structure induced by $\mathfrak{A}$. This is determined up to inderdefinability.

We are going to show

Theorem 4.1. *$\mathfrak{A}$ is quasi finitely axiomatiable, if W is.*

Corollary. *$\aleph_0$-categorical almost strongly minimal theories are quasi finitely axiomatizable.*

Notations. If T is a subset of W, we denote by A_T the union of the A_w, $w \in T$.

$\bar{a}$ stands for a sequence $a^1 a^2, \ldots, a^\kappa$.

$\bar{a} = A_w$ means $\{a^1, a^2, \ldots, a^\kappa\} = A_w$.

Variables $x, y, \ldots$ range over A, $\bar{x}$ etc. is a sequence of variables $x^1, x^2, \ldots, x^\kappa$. The variables $\alpha, \beta, \ldots$ range over W.

$\operatorname{tp}(a/B)$ is the type of a over B in $\mathfrak{A}$. Note that, if a and B belong to W, we can read $\operatorname{tp}(a/B)$ also as the type of a over B in the structure W. Since W has the induced structure, this latter type is equivalent to the first one.

For types $p(x)$, $q(x)$ with finitely many parameters, $p \vdash q$ means that $\mathfrak{A} \models p(a) \Rightarrow \mathfrak{A} \models q(a)$ for all $a \in A \cup M$.

Σ is the set of nice pairs as defined in 3.1.

Lemma 4.2. $\operatorname{tp}(w/S) \vdash \operatorname{tp}(w/A_S)$ *for all* $(w, S) \in \Sigma$ *such that* $w \notin \operatorname{acl} S$ *and* $|S| \geq k_0$.

Proof. Since $A \cup W$ is atomic over W (in fact algebraic), every subset of W which is definable from parameters in $A \cup W$ is also definable with parameters from W, whence definable in the structure W. This shows that a type $\operatorname{tp}(w/S)$ which is minimal as a type in W is also minimal as a type in $\mathfrak{A}$. But $\operatorname{tp}(w/S)$ is minimal by 3.1(ii) and has therefore a unique extension to the algebraic closure of S in $\mathfrak{A}$, which includes A_S.

Lemma 4.3. *There is a finite constant* λ *such that for all* $(w, S) \in \Sigma$ *there is a subset* S' *of* S *with* $|S'| < \lambda$, $(w, S') \in \Sigma$ *and for* $\bar{a} = A_w$

(i) $\operatorname{tp}(\bar{a}/A_{S'} \cup \{w\}) \vdash \operatorname{tp}(\bar{a}/A_S \cup \{w\})$,

(ii) *if* $w \in \operatorname{acl} S$,

(a) $\operatorname{tp}(w/A_{S'}) \vdash \operatorname{tp}(w/A_S)$,

(b) $\operatorname{tp}(w/S') \vdash \operatorname{tp}(w/S)$.

(*Note*: (i) & (ii,a) *imply* $\operatorname{tp}(\bar{a}/A_{S'}) \vdash \operatorname{tp}(\bar{a}/A_S)$.)

Moreover, if $|S| \geq k_0$ *we can find* $|S'| \geq k_0$.

Remark. An example of Cherlin shows that given a projective geometry W we cannot bound λ by a function of κ.

Proof. First we prove that there is a bound k_1 for the degrees of all types $\operatorname{tp}(w/S)$ where $(w, S) \in \Sigma$, $w \in \operatorname{acl} S$: By 3.1(i), the set of these types is generated by a finite number of them. Let k_1 be the maximal degree of these generating types. Our claim now follows from the fact that, if $w \in \operatorname{acl} S'$ and $S' \subset S$, then $\deg \operatorname{tp}(w/S') \geq \operatorname{tp}(w/S)$.

Now look at the following sets:

$\Sigma' = \{(w, S) \in \Sigma \mid w \in \operatorname{acl} S\}$,

$\Sigma'_s = \{(w, S) \in \Sigma' \mid w$ has (exactly) s conjugates over $A_S\}$ $(1 \leq s \leq k_1)$,

$\Sigma''_+ = \{(w, S) \in \Sigma' \mid w$ has r conjugates over $S\}$ $(1 \leq s \leq k_1)$,

$\Sigma_t = \{(w, S) \in \Sigma \mid$ if $\bar{a} = A_w : \bar{a}$ has t conjugates over $A_S \cup \{w\}\}$ $(1 \leq t \leq \kappa)$.

Applying 3.1(i), we see that there is a bound λ such that if K is any of the sets Σ_t or $\Sigma_t \cap \Sigma'_s \cap \Sigma''_s$, and if $(w, S) \in K$, then we find $S' \subset S$ such that $|S'| < \lambda$ and $(w, S') \in K$.

Now let $(w, S) \in \Sigma$ be given. Then one of the following two cases occur:

Case 1: $(w, S) \in \Sigma_t \backslash \Sigma'$.

Then we find $(w, S') \in \Sigma_t$ as above. Since $\bar{a}$ has over $A_{S'} \cup \{w\}$ the same number of conjugates as over $A_S \cup \{w\}$, every conjugate over $A_{S'} \cup \{w\}$ is also a conjugate over $A_S \cup \{w\}$. This proves (i).

Case 2: $(w. S) \in \Sigma_t \cap \Sigma'_s \cap \Sigma''_r$.

Here we find (w, S') in $\Sigma_t \cap \Sigma'_s \cap \Sigma''_r$. $(w, S) \in \Sigma_t$ again entails (i), $(w, S) \in \Sigma'_s$ in the same way implies (ii.a) and $(w, S) \in \Sigma''_r$ yields (ii.b).

For the supplement we take also $\Sigma'' = \{(w, S) \in \Sigma \mid |S| \geq k_0\}$ into consideration. Note that $k_0 < \lambda$.

Now in order to give a quasi finite axiom system for $\mathfrak{A}$ we define a new language for $\mathfrak{A}$.

For every type $p(\bar{x}_1, \ldots, \bar{x}_\lambda) = \operatorname{tp}(\bar{a}_1, \ldots, \bar{a}_\lambda)$ $(\bar{a}_i = A_{v_i})$, we introduce a new $\kappa \cdot \lambda$-place relation symbol R_p. We interpret it in the obvious way:

$$R_p(\bar{b}_1, \ldots, \bar{b}_\lambda) \quad \text{iff} \quad p = \operatorname{tp}(\bar{b}_1, \ldots, \bar{b}_\lambda).$$

In this way $\mathfrak{A}$ turns into an L^*-structure $\mathfrak{A}^*$, where

$$L^* = L_W \cup \{\underline{\pi}\} \cup \{\underline{R}_p \mid p \text{ a type as above}\}.$$

Here L_W denotes the language of the structure W and $\underline{\pi}$ is a function symbol for π.

Definition. Let $\mathfrak{A}^i$ $(i = 1, 2)$ be two L^*-structures both having the structure W as second sort and with surjections $\pi_i : A^i \to W$. A *$*$-map* σ between $\mathfrak{A}^1$ and $\mathfrak{A}^2$ is a partial isomorphism defined on a set of the form $A_S \cup S$ (S nice) such that $\sigma \restriction S$ is an elementary map between W and W.

Lemma 4.4. *A $*$-map σ between $\mathfrak{A}^*$ and $\mathfrak{A}^*$ is $\mathfrak{A}$-elementary (we will say simply 'elementary').*

Proof. (To get a better picture of the proof, the reader can imagine that σ is the identity on S. Note that $\sigma \restriction S$ can be extended to an automorphism of $\mathfrak{A}$.)

First we note that it suffices to show that $\sigma \restriction A_S$ is elementary.

Now we proceed by induction on the cardinality of S. By the choice of the R_p we already know the claim to be true, if $|S| \leq \lambda$.

Suppose now that the nice set S'' has more than λ elements, and that $\sigma : A_{S''} \cup S'' \to A \cup W$ is a $*$-map.

We decompose $S'' = S \cup \{w\}$, where $(w, S) \in \Sigma$. Then by induction $\sigma \restriction A_S \cup S$ is elementary. Choose S' as in Lemma 4.3, and $\bar{a} = A_w$.

Case 1: $w \notin \operatorname{acl} S$. Since $|S| \geq \lambda > k_0$, we have by Lemma 4.2,

$$\operatorname{tp}(w/S) \vdash \operatorname{tp}(w/A_S).$$

Whence, since $\sigma \restriction S \cup \{w\}$ is elementary and since $\sigma \restriction A_S \cup S$ is elementary, we can conclude that $\sigma \restriction A_S \cup \{w\}$ is elementary.

Case 2. $w \in \operatorname{acl} S$. Here we have, by 4.3(ii.a),

$$\operatorname{tp}(w/A_{S'}) \vdash \operatorname{tp}(w/A_S).$$

Whence, since $\sigma \restriction A_{S'} \cup \{w\}$ is elementary (for $|S' \cup \{w\}| \leq \lambda$), and since $\sigma \restriction A_S$ is elementary, we can conclude that $\sigma \restriction A_S \cup \{w\}$ is elementary.

Thus in both cases we have that $\sigma \restriction A_S \cup \{w\}$ is elementary. From this, the fact (4.3(i))

$$\operatorname{tp}(\bar{a}/A_{S'} \cup \{w\}) \vdash \operatorname{tp}(\bar{a}/A_S \cup \{w\}),$$

and the elementarity of $\sigma \restriction A_{S'} \cup \{w\} \cup A_w$ (for $|S' \cup \{w\}| \leq \lambda$), we can deduce that $\sigma \restriction A_S \cup \{w\} \cup A_w$ is elementary. This was to be shown.

Corollary. *$\mathfrak{A}$ and $\mathfrak{A}^*$ are interdefinable.*

Proof. $\mathfrak{A}^*$ is definable in $\mathfrak{A}$. For the converse let σ be an automorphism of $\mathfrak{A}^*$.

Since W is the union of an increasing chain of nice sets S, $A \cup W$ is the union of an increasing chain of sets of the form $A_S \cup S$ (S nice). Whence σ is an increasing union of the $*$-maps $\sigma \restriction A_S \cup S$, which are elementary by 4.4. Thus σ is elementary, and therefore an automorphism of $\mathfrak{A}$.

In order to state the axioms of $\mathfrak{A}^*$, we choose a number μ such that for all nice S and all $T \subset S$ with $|T| < (\kappa + 1)\lambda - 1$ there is a nice S' with $T \subset S' \subset S$ and $|S'| < \mu$. This is possible by the property 3.1(iii) of nice enumerations.

Also we make use of the following notion: An L^*-formula $\varphi(x_1, \ldots, x_m, \alpha_1, \ldots, \alpha_n)$ is called $*$-quantifier free ($*$-qf) if it is a boolean combination of quantifier free formulas and arbitrary L_W-formulas $\varphi(\alpha_1, \ldots, \alpha_n)$. Note that the $*$-maps are just the maps which preserve $*$-qf formulas (and are defined on the right sets.)

Now look at the following axioms.

(a) Infinite two-sorted structure: "π defines a surjection from the first sort onto the second sort, all fibres have cardinality κ."

The finitely many nontrivial axioms of W

(b) All sentences of the form

$$\exists \bar{x}_1, \ldots, \bar{x}_{k_0}, \alpha_1, \ldots, \alpha_{k_0} \quad \varphi(\bar{x}_1, \ldots, \bar{x}_{k_0}, \alpha_1, \ldots, \alpha_{k_0}) \qquad (\varphi \text{ *-qf}),$$

which are true in $\mathfrak{A}^*$.

(c) All sentences of the form

$$\exists \bar{x}_1, \ldots, \bar{x}_\mu, \alpha_1, \ldots, \alpha_\mu \quad \varphi(\bar{x}_1, \ldots, \bar{x}_\mu, \alpha_1, \ldots, \alpha_\mu) \qquad (\varphi \text{ *-qf}),$$

which are true in $\mathfrak{A}^*$.

(d) All sentences of the form

$$\exists \bar{x}_1, \ldots, \bar{x}_{\lambda-1}, \alpha_1, \ldots, \alpha_{\lambda-1} \, \exists \bar{y}, \beta \quad \varphi(\bar{x}_1, \ldots, \bar{x}_{\lambda-1}, \alpha_1, \ldots, \alpha_{\lambda-1}) \qquad (\varphi \text{ *-qf}),$$

which are true in $\mathfrak{A}^*$.

Note that on the basis of the axioms in (a) we have only finitely many $*$-qf formulas with a fixed set of free variables (up to equivalence). Therefore we have essentially only finitely many axioms of type (b), (c), (d).

We want to show that the complete theory of $\mathfrak{A}^*$ is axiomatized by these axioms. To this effect we show that the axioms yield an $\aleph_0$-categorical theory. Thus let $\mathfrak{B}^*$ be a countable model of the axioms. Since W is $\aleph_0$-categorical, and the axioms (a) hold in $\mathfrak{B}^*$, we can assume that $\mathfrak{B}^* = (B, W)$. Set $B_{w^*} = \pi^{-1}(w)$. We will show that the subset of $*$-maps between $\mathfrak{A}^*$ and $\mathfrak{B}^*$ with domain $A_S \cup S$ ($|S| \geq k_0$) is non-empty and has the back and forth property. This then implies that $\mathfrak{A}^*$ and $\mathfrak{B}^*$ are isomorphic.

Lemma 4.5. *Let S be a nice set of cardinality k_0. Then there is a $*$-map σ between $\mathfrak{A}^*$ and $\mathfrak{B}^*$ defined on $A_S \cup S$.*

Proof. Let S be $\{v_1, \ldots, v_{k_0}\}$, $A_{v_i} = \bar{a}_i$. If the $*$-qf formula $\varphi(\bar{x}_1, \ldots, \alpha_{k_0})$ describes the $*$-qf type of $\bar{a}_1, \ldots, v_{k_0}$, the sentence

$$\exists \bar{x}_1, \ldots, \alpha_{k_0} \varphi$$

is an axiom of type (b). Choose for $\sigma(\bar{a}_1), \ldots, \sigma(v_{k_0})$ a realization of φ in $\mathfrak{B}^*$.

Lemma 4.6. *Let the $*$-map σ between $\mathfrak{A}^*$ and $\mathfrak{B}^*$ be defined on $A_S \cup S$ and $(w, S) \in \Sigma$, $w \in \operatorname{acl} S$. Then σ can be extended to a $*$-map defined on $A_{S\cup\{w\}} \cup S \cup \{w\}$.*

Proof. Choose $S' \subset S$ as in Lemma 4.3. Write $S' = \{v_1, \ldots, v_k\}$, $A_{v_i} = \bar{a}_i$ and $A_w = \bar{b}$, $(k \subset \lambda)$.

Claim 1. $\sigma \restriction A_{S'} \cup S'$ has a prolongation τ to $A_{S'\cup\{w\}} \cup S' \cup \{w\}$.

Proof. Let the $*$-qf formula $\varphi_1(\bar{x}_1, \ldots, \bar{x}_k, \alpha_1, \ldots, \alpha_k)$ describe the $*$-qf type of $\bar{a}_1, \ldots, \bar{a}_k, v_1, \ldots, v_k$ and let $\varphi_2(\bar{x}_1, \ldots, \bar{x}_k, \bar{y}, \alpha_1, \ldots, \alpha_k, \beta)$ describe the $*$-qf type of $\bar{a}_1, \ldots, \bar{a}_k, \bar{b}, v_1, \ldots, v_k, w$.

Subclaim. $\forall \bar{x}_1, \ldots, \bar{x}_k, \alpha_1, \ldots, \alpha_k \exists \bar{y}, \beta\ (\varphi_1 \rightarrow \varphi_2)$ is an axiom of type (d).

We have to show that this sentence is true in $\mathfrak{A}^*$. So let $\mathfrak{A}^* \models \varphi_1(\bar{a}'_1, \ldots, \bar{a}'_k, v'_1, \ldots, v'_k)$. Then the map ρ defined by $\rho(\bar{a}_i) = \bar{a}'_i$ and $\rho(v_i) = v'_i$ is a $*$-map. By Lemma 4.4, ρ is elementary. Therefore $\mathfrak{A}^* \models \exists \bar{y}, \beta\, \varphi_2(\bar{a}_1, \ldots, \bar{y}, v_1, \ldots, \beta)$ implies $\mathfrak{A}^* \models \exists \bar{y}, \beta\, \varphi_2(\bar{a}'_1, \ldots, \bar{y}, v'_1, \ldots, \beta)$. This proves the subclaim.

Now we know that $\mathfrak{B}^* \models \exists \bar{y}, \beta\, \varphi_2(\sigma(\bar{a}_1), \ldots, y, \sigma(v_1), \ldots, \beta)$, since $\mathfrak{B}^* \models \varphi_1(\sigma(\bar{a}_1), \ldots, \sigma(v_1), \ldots)$. Choose for $\tau(\bar{b})$, $\tau(w)$ a realization of $\varphi_2(\sigma(\bar{a}_1), \ldots, \bar{y}, \sigma(v_1), \ldots, \beta)$ in $\mathfrak{B}^*$. This proves Claim 1.

Claim 2. $\sigma \cup \tau$ is a $*$-map defined on $A_{S\cup\{w\}} \cup S \cup \{w\}$.

Proof. Clearly $\sigma \cup \tau$ is compatible with π and yields a bijection of corresponding fibers.

Since by 4.2(ii.b), $\operatorname{tp}(w/S') \vdash \operatorname{tp}(w/S)$ (in the structure W) the elementarity of $\tau \restriction S' \cup \{w\}$ and of $\sigma \restriction S$ implies that $(\sigma \cup \tau) \restriction S \cup \{w\}$ is elementary.

It remains to show that $\sigma \cup \tau$ is compatible with the R_p. Since the arity of the R_p is $\kappa\lambda$, it is enough to show that $(\sigma \cup \tau) \restriction A' \cup A_w$ is compatible with the R_p for every subset A' of A_S of cardinality smaller than $\kappa\lambda$. A' given, choose a nice S'' such that $S' \subset S'' \subset S$, $A' \subset A_{S''}$ and $|S''| \subset \mu$. (Note that $|S' \cup \pi(A')| < \kappa\lambda + \lambda - 1$.) We show that $\sigma \cup \tau$ restricted to $A_{S''\cup\{w\}} \cup S'' \cup \{w\}$ preserves $*$-qf formulas. Write $S'' = \{v_1, \ldots, v_k, \ldots, v_l\}$, $A_{v_i} = \bar{a}_i$ $(l < \mu)$.

Let the $*$-qf formula $\varphi_3(\bar{x}_1, \ldots, \bar{x}_l, \alpha_1, \ldots, \alpha_l)$ describe the $*$-qf type of $\bar{a}_1, \ldots, \bar{a}_l, v_1, \ldots, v_l$ and let $\varphi_4(\bar{x}_1, \ldots, \bar{y}, \alpha_1, \ldots, \beta)$ describe the $*$-qf type of $\bar{a}_1, \ldots, \bar{b}, v_1, \ldots, w$.

Subclaim. $\forall \bar{x}_1, \ldots, \bar{x}_l, \bar{y}, \alpha_1, \ldots, \alpha_l, \beta\ (\varphi_2 \wedge \varphi_3 \rightarrow \varphi_4)$ is an axiom of type (c).

To see that this sentence is true in $\mathfrak{A}^*$ we consider a realization $\bar{a}'_1, \ldots, \bar{a}'_l, \bar{b}', v'_1, \ldots, v'_l, w'$ of $\varphi_2 \wedge \varphi_3$ in $\mathfrak{A}^*$. By Lemma 4.4, the map ρ defined by $\rho(\bar{a}_1) = \bar{a}'_1, \ldots, \rho(\bar{b}) = \bar{b}'$, $\rho(v_1) = v'_1, \ldots, \rho(w) = w'$ is elementary restricted to $A_{S' \cup \{w\}} \cup S' \cup \{w\}$ and restricted to $A_{S''} \cup S''$. This together with (4.3)

$$\operatorname{tp}(\bar{b}/S_{S'}) \vdash \operatorname{tp}(\bar{b}/A_{S''}) \subset \operatorname{tp}(\bar{b}/A_S)$$

implies that ρ is elementary. Whence $\mathfrak{A}^* = \varphi_4(\bar{a}'_1, \ldots, w')$. This proves the subclaim.

Since τ is a $*$-map, we have

$$\mathfrak{B}^* \vDash \varphi_2(\sigma(\bar{a}_1), \ldots, \sigma(\bar{a}_k), \tau(\bar{b}), \sigma(v_1), \ldots, \sigma(v_k), \tau(w)).$$

Since σ is a $*$-map, we have

$$\mathfrak{B}^* \vDash \varphi_3(\sigma(\bar{a}_1), \ldots, \sigma(v_l)).$$

Whence, since $\mathfrak{B}^*$ is a model of our axioms,

$$\mathfrak{B}^* \vDash \varphi_4(\sigma(\bar{a}_1), \ldots, \sigma(\bar{a}_l), \tau(\bar{b}), \sigma(v_1), \ldots, \sigma(v_l), \tau(w)),$$

i.e. $\sigma \cup \tau$ restricted to $A_{S'' \cup \{w\}} \cup S'' \cup \{w\}$ preserves $*$-qf formulas.

Lemma 4.7. *Assume $(w, S) \in \Sigma$, $w \notin \operatorname{acl} S$ and $|S| \geqslant k_0$. Let σ be a $*$-map between $\mathfrak{A}^*$ and $\mathfrak{B}^*$ defined on $A_S \cup S$. Then*

(i) *$\sigma \restriction S$ can be extended to a W-elementary map σ', defined on $S \cup \{w\}$.*

(ii) *For every such σ', $\sigma \cup \sigma'$ can be extended to a $*$-map defined on $A_{S \cup \{w\}} \cup S \cup \{w\}$.*

Proof. (i) $\sigma \restriction S$ is W-elementary and W is $\aleph_0$-saturated.

(ii) Choose S' and the notations as in the last lemma. We can find $|S'| \geqslant k_0$.

Claim 1. $(\sigma \cup \sigma') \restriction A_{s'} \cup S' \cup \{w\}$ can be extended to a $*$-map defined on $A_{S' \cup \{w\}} \cup S' \cup \{w\}$.

Proof. Let the $*$-qf formula $\varphi_5(\bar{x}_1, \ldots, \bar{x}_k, \alpha_1, \ldots, \alpha_k, \beta)$ describe the $*$-qf type of $\bar{a}_1, \ldots, \bar{a}_k, v_1, \ldots, v_k, w$. Take φ_2 from the proof of the last lemma.

Subclaim. $\forall \bar{x}_1, \ldots, \bar{x}_k, \alpha_1, \ldots, \alpha_k, \beta\ (\varphi_5 \rightarrow \exists \bar{y}\ (\varphi_2 \wedge \underline{\pi}(y^1) = \beta))$ is an axiom of type (d)

Indeed, if $\mathfrak{A}^* \vDash \varphi_5(\bar{a}'_1, \ldots, v'_1, \ldots, w')$, then the map ρ defined by $\rho(\bar{a}_1) = \bar{a}'_1, \ldots, \rho(v_1) = v'_1, \ldots, \rho(w) = w'$ preserves $*$-qf formulas. Whence, by 4.4, ρ is elementary restricted to $S \cup \{w\}$ and restricted to $A_S \cup S$. Together with (4.2)

$$\operatorname{tp}(w/S) \vdash \operatorname{tp}(w/A_S),$$

this shows that ρ is elementary. Whence

$$\mathfrak{A}^* \vDash \exists \bar{y}\ (\varphi_2(\bar{a}_1, \ldots, \bar{y}, v_1, \ldots, w) \wedge \underline{\pi}(y^1) = w)$$

implies

$$\mathfrak{A}^* \models \exists \bar{y}\, (\varphi_2(\bar{a}_1', \ldots, \bar{y}, v_1', \ldots, w') \wedge \underline{\pi}(y^1) = w').$$

This proves the subclaim.

Now $\sigma \cup \sigma'$ preserves $*$-qf formulas. Therefore

$$\mathfrak{B}^* \models \varphi_5(\sigma(\bar{a}_1), \ldots. \sigma(v_1), \ldots, \sigma'(w)).$$

$\mathfrak{B}^*$ being a model of the above axiom we find a realization $\tau(\bar{b}) = B_{\sigma'(w)}$ of $\varphi_2(\sigma(\bar{a}_1), \ldots, \bar{y}, \sigma(v_1), \ldots, \sigma'(w))$ in $\mathfrak{B}^*$. This proves Claim 1.

Claim 2. $\sigma \cup \tau$ is a $*$-map.

Proof. It remains to show that $\sigma \cup \tau$ is compatible with the R_p. We proceed as in the proof of the last lemma. We take S'' and the notations from there and show that $\sigma \cup \tau$ restricted to $A_{S'' \cup \{w\}} \cup S'' \cup \{w\}$ preserves $*$-qf formulas.

Let the $*$-qf formula $\varphi_6(\bar{x}_1, \ldots, \bar{x}_1, \alpha_1, \ldots, \alpha_l, \beta)$ describe the $*$-qf type of $\bar{a}_1, \ldots, \bar{a}_l, v_1, \ldots, v_l, w$. Take φ_4 from the proof of the last lemma.

Subclaim. $\forall \bar{x}_1, \ldots, \bar{x}_l, \bar{y}, \alpha_1, \ldots, \alpha_l, \beta\, (\varphi_2 \wedge \varphi_6 \rightarrow \varphi_4)$ is an axiom of type (c).

We want to show that this sentence is true in $\mathfrak{A}^*$. So take a realization $\bar{a}_1', \ldots, \bar{a}_l', \bar{b}', v_1', \ldots, v_l', w'$, of $\varphi_2 \wedge \varphi_6$ in $\mathfrak{A}^*$, and look at the map ρ defined by $\rho(\bar{a}_1) = \bar{a}_1', \ldots, \rho(\bar{b}) = \bar{b}', \rho(v_1) = v_1', \ldots, \rho(w) = w'$. Then, by Lemma 4.4, ρ is elementary restricted to the following sets:

(a) $A_{S' \cup \{w\}} \cup S' \cup \{w\}$, (b) $A_{S''} \cup S''$, (c) $S'' \cup \{w\}$.

Since both types $\mathrm{tp}(w/S') \subseteq \mathrm{tp}(w/S)$ are minimal (3.1(ii)), also $\mathrm{tp}(w/S'')$ is minimal. Whence, by the proof of Lemma 4.2,

$$\mathrm{tp}(w/S'') \vdash \mathrm{tp}(w/A_{S''}).$$

Together with (b) and (c) this implies that ρ is elementary restricted to $A_{S''} \cup S'' \cup \{w\}$. This and (a) allow us to deduce from

$$\mathrm{tp}(\bar{b}/A_{S'} \cup \{w\}) \vdash \mathrm{tp}(\bar{b}/A_{S''} \cup \{w\}) \subset \mathrm{tp}(\bar{b}/A_S \cup \{w\}) \qquad (4.3(\mathrm{i}))$$

that ρ is elementary. Whence $\mathfrak{A}^* \models \varphi_4(\bar{a}_1', \ldots, w')$. This proves the subclaim.

Since τ is a $*$-map, we have

$$\mathfrak{B}^* \models \varphi_2(\sigma(\bar{a}_1), \ldots, \sigma(\bar{a}_k), \tau(\bar{b}), \sigma(v_1), \ldots, \sigma(v_k), \tau(w)).$$

Since $\sigma \cup \sigma'$ preserves $*$-qf formulas,

$$\mathfrak{B}^* \models \varphi_6(\sigma(\bar{a}_1), \ldots, \sigma(\bar{a}_1), \sigma(v_1), \ldots, \sigma(v_l), \rho(w)).$$

Whence, since $\mathfrak{B}^*$ is a model of the above axiom,

$$\mathfrak{B}^* \models \varphi_4(\sigma(\bar{a}_1), \ldots, \sigma(\bar{a}_l), \tau(\bar{b}), \sigma(v_1), \ldots, \sigma(v_l), \tau(w)),$$

i.e., $\sigma \cup \tau$ preserves $*$-qf formulas.

Lemma 4.8. *The set of all $*$-maps between $\mathfrak{A}^*$ and $\mathfrak{B}^*$ defined on sets $A_S \cup S$ (S nice), where $|S| \geq k_0$, has the back and forth property.*

Proof. We call an enumeration $(v_i)_{i\in\omega}$ of W nice if it is conjugate to the given nice enumeration $(w_i^*)_{i\in\omega}$.

If σ is a $*$-map defined on $A_S \cup S$, there is a nice enumeration (v_i) such that $S = v_{<m}$. Using Lemma 4.6 in case that $v_i \in \operatorname{acl} v_{<i}$ and Lemma 4.7 in case that $v_i \notin \operatorname{acl} v_{<i}$, we can extend σ to $*$-maps defined on $A_{v_{<i}} \cup v_{<i}$ for arbitrarily large i. This shows that our set of $*$-maps has the forth-property.

If $w \notin \operatorname{acl} S$, then the back property is clear by Lemma 4.7. In the other case it follows from the following

Claim. If σ is a $*$-map between $\mathfrak{A}^*$ and $\mathfrak{B}^*$ defined on $A_S \cup S$, where $|S| \geq k_0$, and if $(v, \sigma(S)) \in \Sigma$, $v \in \operatorname{acl} \sigma(S)$ is given, then we can extend σ to a $*$-map τ defined on $A_{S\cup\{w\}} \cup S \cup \{w\}$ such that $\sigma(w) = v$.

Proof. Along a nice enumeration we can extend σ to a $*$-map σ' defined on $A_{s'} \cup S'$ such that $\operatorname{acl}_W S$ is included in S'. Now there must be a w in S' which is mapped to v. τ is the restriction of σ' to $A_{S\cup\{w\}} \cup S \cup \{w\}$.

References

[1] G. Ahlbrandt, Totally categorical theories of modular type, Ph.D. Thesis, Chicago (1984).

[2] G. Cherlin, L. Harrington and A.H. Lachlan, $\aleph_0$-categorical, $\aleph_0$-stable theories, Annals Pure Appl. Logic 28 (1985) 103–135.

[3] G. Higman, Ordering by divisibility in abstract algebra, Proc. London Math. Soc. (3) 2 (1952) 326–336.

[4] T. Coquand, Unpublished result

[5] G. Martin, Ph.D. Thesis, New Brunswick (forthcoming).