# Pseudo-Finite Homogeneity and Saturation

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**Abstract**

When analyzing database query languages a property of theories, the pseudo-finite homogeneity property has been introduced and applied (cf. [3]). We show that a stable theory has the pseudo-finite homogeneity property just in case its expressive power for finite states is bounded. Moreover, we introduce the corresponding pseudo-finite saturation property and show that a theory fails to have the finite cover property iff it has the pseudo-finite saturation property.

## 1 Pseudo-finite homogeneity

Throughout let $T$ be a *complete* first-order theory in a countable language $L$ with infinite models. Suppose that $\rho$ is a finite non-empty set of relation symbols not contained in $L$. Set $L(\rho) := L \cup \rho$. If $M$ is a model of $T$, and $(M, \bar{P})$ is an $L(\rho)$-structure with, say, $\bar{P} = P_1 \dots P_r$, then $\bar{P}$ is a $(\rho)$-*state* in $M$. $\mathrm{fld}(\bar{P})$, the *field* or *active domain* of the state $\bar{P}$, is the set $\mathrm{fld}(P_1) \cup \dots \cup \mathrm{fld}(P_r)$, where $\mathrm{fld}(P_j)$ is the field of the relation $P_j$. $\bar{P}$ is a *finite* state, if every $\mathrm{fld}(P_j)$ is finite and non-empty. In the following we will denote finite states by $\bar{s}, \bar{s}' \dots$.

A state $\bar{P}$ in $M$ is *pseudo-finite*, if $(M, \bar{P})$ is a model of $F(T, \rho)$, the theory of all finite states, i.e.,

$$F(T, \rho) \quad := \quad \mathrm{Th}(\{(N, \bar{s}) \mid N \models T,\ (N, \bar{s}) \text{ an } L(\rho)\text{-structure, } \bar{s} \text{ a finite state}\}).$$

In general, we use $\bar{r}, \bar{r}', \bar{t} \dots$ to denote pseudo-finite states.

**Example 1.1** For $L := \emptyset$, $T$ the $L$-theory of infinite sets, and $\rho := \{P\}$ with unary $P$, a subset $r$ of a model $M$ of $T$ is pseudo-finite iff $M \setminus r$ is infinite. In fact, for every $k$, the complement of every finite subset contains at least $k$ elements, hence the same holds for a pseudo-finite subset. If $M \setminus r$ and $r$ are infinite and $l \geq 1$, then $(M, r)$ satisfies the same sentences of quantifier rank $\leq l$ as $(M, s)$, where $|s| = l$. Therefore, $(M, r) \models F(T, \rho)$.

We collect some properties of pseudo-finite states:

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**Proposition 1.2** (cf. [3])

a) *For every $L(\rho)$-sentence $\varphi$ there is a sequence $(\varphi_n)_{n\geq 1}$ of $L$-sentences such that*

$$\varphi \in F(T,\rho) \quad \textit{iff} \quad \textit{for all } n \geq 1,\ T \models \varphi_n.$$

b) *Let $r$ be a pseudo-finite subset of $M$ (i.e., $\rho = \{P\}$ with unary $P$). If $s \subseteq M$ is finite, then $r \cup s$ is pseudo-finite. If $s$ is definable (in particular, finite) then $r \setminus s$ is pseudo-finite.*

c) *If $\bar{r}$ is pseudo-finite in $M$, then* $\mathrm{fld}(\bar{r})$ *is pseudo-finite in $M$.*

d) *If $\bar{r}$ is a pseudo-finite $\rho$-state in $M$ and $\rho' \subseteq \rho$, then $\bar{r}|\rho'$ is pseudo-finite.*

*Proof.* a) For simplicty, let $\rho := \{P\}$ with unary $P$. Fix an $L(\rho)$-sentence $\varphi$. Now, let the $L$-sentence $\varphi_n$ express that $\varphi$ holds, if $P$ has at most $n$ elements, e.g., $\varphi_n := \forall x_1 \ldots \forall x_n \varphi^*$, where $\varphi^*$ is obtained from $\varphi$ by replacing each subformula of the form $Pu$, $u$ a term, by $(u = x_1 \vee \ldots \vee u = x_n)$.

The proofs of b) – d) make use of the corresponding facts for finite $r$ and $\bar{r}$, respectively (see [3] for details). □

The *pseudo-finite homogeneity property* $\mathrm{H}(T)$ was introduced in [3]: For an infinite cardinal $\lambda$ denote by $\mathrm{H}(T,\lambda)$ the property

> if $r$ and $t$ are pseudo-finite subsets of a model $M$ of $T$, $h : r \to t$ is bijective and $L$-elementary, and $(M, r, t, h)$ is $\lambda$-saturated, then for every $a \in M$ there is $b \in M$ such that $h \cup \{(a, b)\}$ is $L$-elementary.

$\mathrm{H}(T)$ means that $\mathrm{H}(T,\lambda)$ holds for some $\lambda$. $T$ has the *pseudo-finite homogeneity property*, if $\mathrm{H}(T)$ holds.

**Proposition 1.3**

- *If $\mu \leq \lambda$ then* $\mathrm{H}(T,\mu)$ *implies* $\mathrm{H}(T,\lambda)$.
- $\mathrm{H}(T,\lambda)$ *implies* $\mathrm{H}(T,\omega)$. *Hence,* $\mathrm{H}(T)$ *is equivalent to* $\mathrm{H}(T,\omega)$.

*Proof.* The first assertion is clear. For the last one, suppose that $(M, r, t, h)$ is $\omega$-saturated, $r$ and $t$ are pseudo-finite, and $h : r \to t$ is onto and elementary. Let $a \in M$. Choose a $\lambda$-saturated elementary extension $(M', r', t', h')$ of $(M, r, t, h)$. By $\mathrm{H}(T,\lambda)$, there is $b' \in M'$ such that $h' \cup \{(a, b')\}$ is elementary. Using $\omega$-saturation, choose $b \in M$ such that $(M, r, t, h, a, b) \equiv (M', r', t', h', a, b')$. Then, $h \cup \{(a, b)\}$ is elementary. □

In [3] it is shown that $\mathrm{H}(T)$ holds for *o*-minimal $T$ (even for quasi-*o*-minimal $T$). There are theories $T$ without $\mathrm{H}(T)$ (variants of the following example will play a role in the next section):

**Example 1.4** Let $T$ be the theory of an equivalence relation which, for every $n \geq 1$, has exactly one equivalence class of cardinality $n$. $\mathrm{H}(T)$ does not hold: Let $\lambda$ be an infinite cardinal and $M$ a $\lambda$-saturated model of cardinality $\lambda$. Let $A_1$ and $A_2$ be two infinite equivalence classes and let $a \in A_1$. Set $r := A_1 \setminus \{a\}$

and $t := A_2$. Choose a bijection $h : r \to t$. Then, $r$ and $t$ are pseudo-finite in $M$, $h$ is elementary and $(M, r, t, h)$ is $\lambda$-saturated. But, there is no $b \in M$ such that $h \cup \{(a, b)\}$ is elementary.

# 2 Some collapsing results

Let $\bar{s}$ be a finite state in a model $M$ of $T$. In the theory of constraint databases the question has been addressed whether every $L(\rho)$-formula ("query") is equivalent in $M$ for finite states to a formula whose quantifiers are restricted to $\mathrm{fld}(\bar{s})$. Moreover, in case an order relation $<$ is in $L$, one wants to know whether every $L(\rho)$-formula preserved under partial order-isomorphisms is expressible in terms of $<$ and the symbols in $\rho$. We present some positive results (collapsing results) and some negative ones. More or less explicitly, these results are contained in [3], [4], [5], [6]. Moreover, for stable theories, we show that pseudo-finite homogeneity is equivalent to one of the collapsing properties.

$L(\rho)$-formulas $\varphi(\bar{x})$ and $\psi(\bar{x})$ are *equivalent in $T$ for finite states* if for all models $M$ of $T$, all finite states $\bar{s}$ and all $\bar{a} \in \mathrm{fld}(\bar{s})$, $(M, \bar{s}) \models \varphi(\bar{a}) \leftrightarrow \psi(\bar{a})$. Of course, by completeness of $T$ and finiteness of the states, it suffices to require the condition just for one model $M$ of $T$. By the same reasons, the following fact, tacitly used in the next proofs, holds:

> If, for $i = 1, 2$, $N_i \models T$, $N_i \models \varphi_i(\bar{s}_i)$, where $\bar{s}_i$ is a finite $\rho_i$-state and $\rho_1 \cap \rho_2 = \emptyset$, then in every model $M$ of $T$ there are finite states $\bar{s}'_1$ and $\bar{s}'_2$ such that $M \models \varphi_1(\bar{s}'_1) \wedge \varphi_2(\bar{s}'_2)$.

An $L(\rho)$-formula is *$\rho$-bounded*, if it has the form

$$Q_1\bar{x}_1 \in P_1 \dots Q_m\bar{x}_m \in P_m\psi$$

where $Q_1, \dots, Q_m \in \{\forall, \exists\}$, $P_1, \dots, P_m \in \rho$, and $\psi$ is an $L$-formula (and $\bar{x}_1, \dots, \bar{x}_m$ have the appropriate length). By our assumption that the relations in a finite state are non-empty, every Boolean combination of $\rho$-bounded formulas is equivalent for finite states to a $\rho$-bounded formula.

The first collapsing theorem reads as follows (the idea of the proof was used in [5] to show Corollary 2.2 below for o-minimal theories):

**Theorem 2.1** *If* $\mathrm{H}(T)$ *holds then, for every $\rho$, every $L(\rho)$-formula is equivalent in $T$ for finite states to a $\rho$-bounded sentence.*

*Proof.* For notational simplicity, let $\varphi$ be an $L(\rho)$-sentence. If $\varphi$ is not equivalent to a $\rho$-bounded sentence, then by a standard compactness argument, one obtains pseudo-finite $\bar{r}$ and $\bar{r}'$ in a model $M$ of $T$ such that $M \models \varphi(\bar{r}) \wedge \neg\varphi(\bar{r}')$ and such that $(M, \bar{r})$ and $(M, \bar{r}')$ satisfy the same $\rho$-bounded sentences. An appropriate back and forth argument (together with further applications of the compactness theorem) shows that, in addition, we can assume that there is a partial $L(\rho)$-isomorphism $h$ with $h : \mathrm{fld}(\bar{r}) \to \mathrm{fld}(\bar{r}')$, which is onto and $L$-elementary, and that $(M, \mathrm{fld}(\bar{r}), \mathrm{fld}(\bar{r}'), h)$ is $\omega$-saturated. Now, for every $l \geq 1$, the pseudo-finite homogeneity shows that $h$ can be extended $l$ times back and forth to an $L$-elementary map, which by $\mathrm{fld}(\bar{r}) \subseteq \mathrm{do}(h)$ and $\mathrm{fld}(\bar{r}') \subseteq \mathrm{rg}(h)$ is a partial $L(\rho)$-isomorphism. Therefore, $h$ is $L(\rho)$-elementary, which contradicts $M \models \varphi(\bar{r}) \wedge \neg\varphi(\bar{r}')$. □

An $L(\rho)$-sentence is *$\rho$-restricted*, if all quantifiers are relativized to $\mathrm{fld}(\rho)$. Clearly, every $\rho$-restricted formula is equivalent to a $\rho$-bounded one, and if $T$ admits quantifier elimination, the converse holds, i.e., every $\rho$-bounded formula is equivalent to a $\rho$-restricted one. Therefore:

**Corollary 2.2** (cf. [5]) *If $T$ admits quantifier elimination and $\mathrm{H}(T)$ holds then, for every $\rho$, every $L(\rho)$-formula is equivalent in $T$ for finite states to a $\rho$-restricted formula.*

For stable theories the converse of the preceding theorem holds (we do not know whether the assumption of stability can be omitted):

**Theorem 2.3** *Assume $T$ is stable. If, for every $\rho$, every $L(\rho)$-formula is equivalent in $T$ for finite states to a $\rho$-bounded formula, then $\mathrm{H}(T)$ holds.*

*Proof.* We show $\mathrm{H}(T, \omega)$. So, let $r$ and $t$ be pseudo-finite subsets of a model $M$ of $T$, $h : r \to t$ onto and elementary, and $(M, r, t, h)$ $\omega$-saturated. Given $a \in M$ let $p := \mathrm{tp}(a/r)$ ($\mathrm{tp}(a/r)$ denotes the type of $a$ in $(M, (e)_{e \in r})$). For an $L$-formula $\varphi(x, \bar{y})$ set

$$p_\varphi \quad := \quad \{\varphi(x, \bar{a}) \mid \bar{a} \in r,\ \varphi(x, \bar{a}) \in p\} \cup \{\neg\varphi(x, \bar{a}) \mid \bar{a} \in r,\ \neg\varphi(x, \bar{a}) \in p\}.$$

By stability of $T$, the type $p_\varphi$ is definable over $r$, i.e., there is an $L$-formula $\delta_\varphi(\bar{y}, \bar{z})$ and $\bar{c}_\varphi \in r$ such that for all $\bar{a} \in r$,

$$M \models \delta_\varphi(\bar{a}, \bar{c}_\varphi) \quad \text{iff} \quad \varphi(x, \bar{a}) \in p.$$

For $\rho = \{P\}$ with unary $P$ and $L$-formulas $\varphi_1(x, \bar{y}), \ldots, \varphi_m(x, \bar{y})$, the formula

$$\alpha(P, \bar{c}_{\varphi_1}, \ldots, \bar{c}_{\varphi_m}) \quad := \quad \exists x \bigwedge_{1 \le i \le m} \forall \bar{y} \in P(\delta_{\varphi_i}(\bar{y}, \bar{c}_{\varphi_i}) \leftrightarrow \varphi_i(x, \bar{y}))$$

expresses in $(M, r)$ that $p_{\varphi_1} \cup \ldots \cup p_{\varphi_m}$ is realized. By assumption, for finite, and hence, for pseudo-finite $P$, $\alpha(P, \bar{c}_{\varphi_1}, \ldots, \bar{c}_{\varphi_m})$ is equivalent to a $\rho$-bounded formula. But, $M \models \alpha(r, \bar{c}_{\varphi_1}, \ldots, \bar{c}_{\varphi_m})$ and $h$ preserves $\rho$-bounded formulas, therefore, $M \models \alpha(t, h(\bar{c}_{\varphi_1}), \ldots, h(\bar{c}_{\varphi_m}))$. Thus, there is $b_0 \in M$ such that for $i = 1, \ldots, m$ and $\bar{a} \in r$,

$$M \models \varphi_i(a, \bar{a}) \leftrightarrow \varphi_i(b_0, h(\bar{a})).$$

Hence,

$$q(x) \quad := \quad \{\forall \bar{y} \in P(\varphi(a, \bar{y}) \leftrightarrow \varphi(x, h(\bar{y}))) \mid \varphi(x, \bar{y}) \text{ an } L\text{-formula}\}$$

is finitely satisfiable in $(M, r, h, a)$. Therefore, by $\omega$-saturation, there is $b \in M$ satisfying $q(x)$. Then, $h \cup \{(a, b)\}$ is elementary. □

**Example 2.4** (cf. [5]) Let $T$ be the theory of the ordered field of real numbers. $T$ admits elimination of quantifiers and is $o$-minimal and therefore, H(T) holds. By 2.2, for $\rho := \{P\}$ with binary $P$ the sentence

$$\varphi \quad := \quad \exists u \exists v \forall x \forall y (Pxy \to y = u \cdot x + v)$$

($\varphi$ expresses that the elements of $P$ lie on some line not parallel to the $y$-axis) must be equivalent in $T$ for finite states to a $\rho$-restricted sentence $\psi$. In fact, as $\psi$ we can take

$$\begin{aligned} \psi \quad := \quad & \exists x_1 y_1 \in P \exists x_2 y_2 \in P \forall xy \in P((x = x_1 \wedge y = y_1) \vee \\ & (x_1 \neq x_2 \wedge x \neq x_1 \wedge \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1})). \end{aligned}$$

**Example 2.5** Let $T$ be the theory of $(\mathbb{Z}, <, +)$ and $\rho := \{P\}$ with unary $P$. H($T$) holds by quasi-$o$-minimality of $T$ (cf. [3]). By 2.1, the $L(\rho)$-sentence

$$\varphi \quad := \quad \text{“}P \text{ contains even and odd numbers”}$$

is equivalent for finite states to a $\rho$-bounded sentence $\psi$, e.g., to

$$\psi \quad := \quad \exists x \in P \exists y \in P \exists u \exists v (x = u + u \wedge y = v + v + 1).$$

But, $\varphi$ is not equivalent to a $\rho$-restricted sentence. Therefore, the assumption of quantifier elimination cannot be omitted in the preceding corollary.

**Example 2.6** Let $T$ be the theory of Example 1.4, i.e., the theory of an equivalence relation $E$ that, for every $n \geq 1$, has exactly one equivalence class of cardinality $n$. $T$ is stable and H($T$) fails. Hence, by 2.3, there is a formula that is not equivalent in $T$ for finite states to a bounded one. In fact, the sentence

$$\varphi \quad := \quad \text{“}P \text{ is an equivalence class”}$$

is not equivalent to a $\{P\}$-bounded one for finite states. To prove this, e.g., for each $n \geq 1$, introduce a new unary relation $R_{\geq n}$ with

$$R_{\geq n} \quad := \quad \{a \in M \mid \text{the equivalence class of } a \text{ has} \geq n \text{ elements}\}.$$

Then, $T_1 := \mathrm{Th}(M, E, (R_{\geq n})_{n \geq 1})$ allows elimination of quantifiers. But $\varphi$ is not equivalent in $T_1$ for finite states to a restricted formula.

A theory $T$ has the finite cover property (we denote this by fcp($T$)) if there is a formula $\varphi(x, \bar{y})$ such that for all $k \geq 1$ there is a sequence $(\bar{a}_i)_{i \in I}$ in some model of $T$ such that the set $\{\varphi(x, \bar{a}_i) \mid i \in I\}$ is not finitely satisfiable but every subset of at most $k$ formulas is.

The theory $T$ of the preceding example is a standard example of a stable theory with the finite cover property. We have proved that this theory does not have the pseudo-finite homogeneity property. This proof does not generalize to arbitrary theories with the fcp. The theory constructed in the following example is stable, has the fcp and the pseudo-finite homogeneity property. In the next section we shall see that H($T$) holds for every theory $T$ which does not have fcp.

**Example 2.7** Let $L = \{E, F\}$ with binary relation symbols $E$ and $F$. Take an $L$-structure $M_0$ such that

> $E$ is an equivalence relation, every equivalence class is finite, and for every $n \geq 1$ there is exactly one class of cardinality $n$; $F$ is a symmetric and antireflexive relation (a graph relation), $F \subseteq E$, and $F$ is a cycle of length $n$ in the equivalence class of cardinality $n$.

Let $T$ be the theory of $M_0$. Clearly, $T$ is stable and has the fcp. We show that H($T$) holds. So, let $(M, r, t, h)$ be $\omega$-saturated, $M \models T$, $r$ and $t$ pseudo-finite in $M$, and $h : r \to t$ onto and elementary. Furthermore, let $a \in M$ be arbitrary. We have to find $b \in M$ such that $h \cup \{(a, b)\}$ is elementary. Note that every infinite equivalence class, with respect to $F$ consists of infinitely many "$\mathbb{Z}$-components" (connected components). An analysis of the different possibilities for $\mathrm{tp}(a/r)$ shows that we find such an element $b$ in all cases but one: namely, if $a$ is in the equivalence class of an element $a_0 \in r$, but is an element of a new $\mathbb{Z}$-component (that is, no element of this component lies in $r$), and moreover, in the equivalence class of $h(a_0)$ every $\mathbb{Z}$-component contains an element of $t$. We show that this cannot happen. First note that for every finite subset $s$ in the model $M_0$, every $c_0 \in s$ and $k \geq 1$, the following statements $(1)_k$ and $(2)_k$ are equivalent:

$(1)_k$ The union of the $k$-balls whose center are elements of $s$ equivalent to $c_0$ is the whole equivalence class of $c_0$, i.e.,

$$\begin{aligned} M \quad \models \quad & \forall x (Exc_0 \to (sx \vee \exists y \in s \bigvee_{1 \leq l < k} \exists y_1 \ldots \exists y_l \\ & (Fyy_1 \wedge \ldots \wedge Fy_{l-1}y_l \wedge y_l = x))). \end{aligned}$$

$(2)_k$ For every $d \in s$ in the equivalence class of $c_0$, there are "at both sides of $d$" elements in $s$ at a distance $< 2 \cdot k$, i.e.,

$$\begin{aligned} M \quad \models \quad & \forall x \in s \exists y \in s \exists z \in s \\ & (Exc_0 \to \bigvee_{1 \leq l < 2k} \bigvee_{1 \leq m < 2k} \exists y_1 \ldots \exists y_l \exists z_1 \ldots \exists z_m (\neg y_1 = z_1 \wedge \\ & (Fxy_1 \wedge \ldots \wedge Fy_{l-1}y_l \wedge y_l = y \wedge \bigwedge_{1 \leq i \leq l} \neg y_i = x) \wedge \\ & (Fxz_1 \wedge \ldots \wedge Fz_{m-1}z_m \wedge z_m = z \wedge \bigwedge_{1 \leq i \leq m} \neg z_i = x))). \end{aligned}$$

We come back to $(M, r, t, h)$. By assumption, in the equivalence class of $h(a_0)$ every $\mathbb{Z}$-component contains an element of $t$. Hence, by $\omega$-saturation of $(M, r, t, h)$, there is $k \geq 1$ such that $(1)_k$ holds for $s := t$ and $c_0 := h(a_0)$. Therefore, since $t$ is pseudo-finite, $(2)_k$ is true for $t$ and $h(a_0)$. But the formula in $(2)_k$ is bounded and hence is preserved by $L$-elementary mappings, thus, $(2)_k$ and therefore, $(1)_k$ are true for $s := r$ and $c_0 := a_0$, a contradiction.

We turn to the second collapsing result. In the next definition and theorem we fix a binary relation symbol $<$. If $<$ is in $L$, then we asssume that $T$ contains the axioms of orderings for $<$ and we set $L_0 = \{<\}$. Otherwise, let $L_0 = \emptyset$.

An $L(\rho)$-sentence $\varphi$ is *locally $L_0$-generic* in $T$, if for some (or, equivalently, all) models $M$ of $T$ and all finite states $\bar{s}$, all partial $L_0$-isomorphisms $h$ (i.e., $h$ is injective and $<$-preserving, in case $L_0 = \{<\}$, and injective, in case $L_0 = \emptyset$) with $\mathrm{fld}(\bar{s}) \subseteq \mathrm{do}(h)$, we have

$$M \models \varphi(\bar{s}) \quad \text{iff} \quad M \models \varphi(h(\bar{s})).$$

Part a) of the next theorem is shown in [3] (and in [6] for $o$-minimal theories). We do not know, whether in part b) the assumption $\mathrm{H}(T)$ can be omitted.[1]

**Theorem 2.8** *a) Assume $< \in L$ and $T$ contains the theory of orderings. Set $L_0 = \{<\}$. If $\mathrm{H}(T)$ holds, then every $L(\rho)$-sentence $\varphi$, which is locally $L_0$-generic, is equivalent in $T$ for finite states to an $L_0(\rho)$-sentence.*

*b) Set $L_0 = \emptyset$. If $T$ is stable and $\mathrm{H}(T)$ holds, then every $L(\rho)$-sentence $\varphi$, which is locally $L_0$-generic, is equivalent in $T$ for finite states to an $L_0(\rho)$-sentence.*

*Proof.* If not, a compactness argument gives a pseudo-finite $(\bar{r}, \bar{r}')$, a partial $L_0$-isomorphism $h$ with $\mathrm{do}(h) = \mathrm{fld}(\bar{r})$, $h(\bar{r}) = \bar{r}'$ such that

$$M \models \varphi(\bar{r}) \wedge \neg\varphi(\bar{r}') \quad \text{and} \quad (M, \bar{r})|L_0(\rho) \equiv (M, \bar{r}')|L_0(\rho).$$

It suffices to show that one can further assume that $\mathrm{fld}(\bar{r})$ and $\mathrm{fld}(\bar{r}')$ are subsets of one set of $L_0$-indiscernibles for $L$-formulas (i.e., $<$-indiscernibles in a) and total indiscernibles in b)). Then, $h$ is $L$-elementary, and therefore, by $\mathrm{H}(T)$, we see, arguing as in the preceeding theorem, that $h$ is $L(\rho)$-elementary, a contradiction.

To get the pseudo-finite states into indiscernibles, we take a disjoint copy $\rho'$ of $\rho$ and let $T_1$ consist of the $L \cup \rho \cup \rho' \cup \{I, f, f'\}$-sentences in (1)–(7):

(1) $\mathrm{Th}(M, \bar{r}, \bar{r}')$

(2) $I$ is infinite and $L_0$-indiscernible for $L$-formulas

(3) $f$ and $f'$ are partial $L_0$-isomorphisms

(4) $\mathrm{fld}(\bar{r}) \subseteq \mathrm{do}(f)$, $\mathrm{fld}(\bar{r}') \subseteq \mathrm{do}(f')$

(5) $\mathrm{rg}(f)$, $\mathrm{rg}(f') \subseteq I$

(6) $\varphi(f(\bar{r}))$, $\neg\varphi(f(\bar{r}'))$

(7) $(f(\bar{r}), f(\bar{r}'))$ is pseudo-finite.

[1] Added in proof: Meanwhile Baldwin and Benedikt ([1]) have proved that the assumption $\mathrm{H}(T)$ can be omitted. Compare also [7].

We show that $T_1$ is satisfiable and hence, by Robinson's joint consistency lemma, $T_1 \cup \mathrm{Th}(M, \bar{r}, \bar{r}', h)$ is satisfiable. In a corresponding model, $g := f' \circ h \circ f^{-1}$ is a partial $L_0$-isomorphism with $g(f(\bar{r})) = f'(\bar{r}')$, and $\mathrm{fld}(f(\bar{r}))$ and $\mathrm{fld}(f'(\bar{r}'))$ are contained in a set of indiscernibles.

$T_1$ is satisfiable: Let $\gamma \in \mathrm{Th}(M, \bar{r}, \bar{r}')$, $m \in \omega$, and $\psi_1(\bar{x}), \ldots, \psi_k(\bar{x})$ be $L$-formulas. It suffices to show that there exists a model of $\gamma$ and of the sentences in (3)–(7) such that $I$ contains at least $m$ elements and is a set of $L_0$-indiscernibles with respect to $\psi_1(\bar{x}), \ldots, \psi_k(\bar{x})$.

By pseudo-finiteness of $(\bar{r}, \bar{r}')$ there are finite $\bar{s}, \bar{s}'$ in some model $N$ of $T$ such that

$$N \models \gamma \wedge \varphi(\bar{s}) \wedge \neg\varphi(\bar{s}').$$

By Ramsey's theorem, there is a set $I$, $|I| \geq \max\{m, |\mathrm{fld}(s)|, |\mathrm{fld}(s')|\}$ of $L_0$-indiscernibles with respect to $\psi_1(\bar{x}), \ldots, \psi_k(\bar{x})$. Choose partial $L_0$-isomorphisms $f$ and $f'$ with domain $\mathrm{fld}(s)$ and $\mathrm{fld}(s')$, respectively, and range in $I$. Set $\bar{s}_1 := f(\bar{s}), \bar{s}'_1 := f(\bar{s}')$. By local genericity of $\varphi$, $N \models \varphi(\bar{s}_1) \wedge \neg\varphi(\bar{s}'_1)$. □

**Example 2.9** Let $L := \{\epsilon, <\}$ with binary $\epsilon$. Consider the structure $(V_\omega, \epsilon, <)$, where $V_\omega$ is the set of hereditarily finite sets, where $\epsilon$ is the $\in$-relation on $V_\omega$ and $<$ a total ordering of $V_\omega$, say, of order type $\omega$. Let $T$ be the theory of $(V_\omega, \epsilon, <)$. For $\rho = \{P\}$ with unary $P$, let $\varphi$ be the $L(\rho)$-sentence stating that there is a bijection between $P$ and an even natural number which, thus, expresses that the cardinality of $P$ is even. Clearly, $\varphi$ is locally $L_0$-generic in $T$, but not equivalent to an $L_0(\rho)$-sentence. Hence, $\mathrm{H}(T)$ fails.

# 3 Pseudo-finite saturation

We already remarked that there is a relationship between the pseudo-finite homogeneity property and the finite cover property. In fact, in this final section, we show that the saturation property that corresponds to the pseudo-finite homogeneity property even is equivalent to the finite cover property.

We introduce $\mathrm{S}(T, \lambda)$ by

> if $r$ is a pseudo-finite subset of a model $M$ of $T$ and $(M, r)$ is $\lambda$-saturated, then every type in $S_1(r)$ is realized in $(M, (a)_{a \in r})$

($S_1(r)$ denotes the set of complete types in a variable $x$ with parameters from $r$). $\mathrm{S}(T)$ means that $\mathrm{S}(T, \lambda)$ holds for some $\lambda$. $T$ has the *pseudo-finite saturation property*, if $\mathrm{S}(T)$ holds.

Clearly, if $\mu \leq \lambda$ then $\mathrm{S}(T, \mu)$ implies $\mathrm{S}(T, \lambda)$. Below we shall see that $\mathrm{S}(T)$ is equivalent to $\mathrm{S}(T, \omega_1)$.

**Proposition 3.1** $\mathrm{S}(T)$ *implies* $\mathrm{H}(T)$.

*Proof.* Assume $\mathrm{S}(T, \lambda)$ holds. Let $(M, r, t, h)$ be $\lambda$-saturated with pseudo-finite $r$ and $t$ and elementary and bijective $h$. For $a \in M$ and $p := \mathrm{tp}(a/r)$, $h(p) := \{\varphi(x, h(\bar{a})) \mid \varphi(x, \bar{a}) \in p\}$ is a type in $S_1(t)$. By $\mathrm{S}(T, \lambda)$, there is $b \in M$

realizing $h(p)$. Then, $h \cup \{(a,b)\}$ is elementary. $\square$

There are theories with $\mathrm{H}(T)$ but whithout $\mathrm{S}(T)$: By *o*-minimality, $\mathrm{H}(T)$ holds for the theory $T$ of the ordering of the rational numbers. But $\mathrm{S}(T)$ fails, as shown by:

**Lemma 3.2** *If* $\mathrm{S}(T)$ *holds then* $T$ *is stable.*[2]

*Proof.* By contradiction, suppose that $T$ is unstable. Let $\lambda$ be a cardinal. We show that $\mathrm{S}(T,\lambda^+)$ fails. By unstability, there is a model $M$ of $T$ and a subset $A$ of $M$ such that

$$(+) \qquad |M| = |A| = 2^\lambda \ \text{ and } |S_1(A)| > 2^\lambda.$$

We may assume that there is a pseudo-finite subset $r$ of $M$ with $A \subseteq r$ (since, for $\rho := \{P\}$ with a new unary predicate $P$,

$$\mathrm{Th}(M,(a)_{a\in A}) \cup F(T,\rho) \cup \{Pa \mid a \in A\}$$

is finitely satisfiable). Choose a $\lambda^+$-saturated elementary extension $(M',r')$ of $(M,r)$ of cardinality $2^\lambda$. By(+), there are types in $S_1(A)$, and hence in $S_1(r')$, that are not realized in $M'$. $\square$

**Theorem 3.3** *If* $T$ *does not have the finite cover property, then* $\mathrm{S}(T,\omega_1)$ *holds.*

*Proof.* Let $(M,r) \models T$, where $r$ is pseudo-finite and $(M,r)$ is $\omega_1$-saturated. Fix $p \in S_1(r)$. Since a theory without the fcp is stable, $p$ is definable over $r$, i.e., for every $L$-formula $\varphi(x,\bar{y})$ there is an $L$-formula $\delta_\varphi(\bar{y},\bar{z})$ and $\bar{c}_\varphi \in r$ such that for all $\bar{a} \in r$,

$$M \models \delta_\varphi(\bar{a},\bar{c}_\varphi) \quad \text{iff} \quad \varphi(x,\bar{a}) \in p.$$

Let $P$ be a new unary relation symbol. For an $L$-formula $\varphi(x,\bar{y})$ set

$$\chi_\varphi(x,\bar{c}_\varphi) \quad := \quad \forall \bar{y} \in P(\delta_\varphi(\bar{y},\bar{c}_\varphi) \rightarrow \varphi(x,\bar{y})).$$

By $\omega_1$-saturation of $(M,r)$, it suffices to show that

$$\{\chi_\varphi(x,\bar{c}_\varphi) \mid \varphi(x,\bar{y}) \text{ an } L\text{-formula}\}$$

is finitely satisfiable in $(M,r,(\bar{c}_\varphi)_{\varphi\in L})$. If not, there are $\varphi_1(x,\bar{y}),\ldots,\varphi_m(x,\bar{y})$ such that

$$\{\chi_{\varphi_1}(x,\bar{c}_{\varphi_1}),\ldots,\chi_{\varphi_m}(x,\bar{c}_{\varphi_m})\}$$

is not satisfiable. First we show that we can assume $m = 1$. Let

$$\begin{aligned} \varphi_0(x,\bar{y},\bar{u}) \quad := \quad & ((u_0 = u_1 \wedge \neg u_0 = u_2 \wedge \ldots \wedge \neg u_0 = u_m) \rightarrow \varphi_1(x,\bar{y})) \wedge \\ & ((\neg u_0 = u_1 \wedge u_0 = u_2 \wedge \ldots \wedge \neg u_0 = u_m) \rightarrow \varphi_2(x,\bar{y})) \wedge \\ & \vdots \\ & ((\neg u_0 = u_1 \wedge \neg u_0 = u_2 \wedge \ldots \wedge u_0 = u_m) \rightarrow \varphi_m(x,\bar{y})). \end{aligned}$$

[2] In [3] the pseudo-finite isolation property $\mathrm{I}(T)$ is considered and related to $\mathrm{H}(T)$. Both $\mathrm{S}(T)$ and $\mathrm{I}(T)$ imply $\mathrm{H}(T)$. But $\mathrm{S}(T)$ and $\mathrm{I}(T)$ contradict each other, since $\mathrm{I}(T)$ implies that $T$ is unstable.

Then, $\chi_{\varphi_0}(x, \bar{c}_{\varphi_0})$ is not satisfiable $(M, r, \bar{c}_{\varphi_0})$.

So let $\varphi(x, \bar{y})$ be an $L$-formula such that $\chi_\varphi(x, \bar{c}_\varphi)$ is not satisfiable in $(M, r, \bar{c}_\varphi)$. Since non-fcp($T$), there is a natural number $k$ such that

> for every finite sequence $(\bar{a}_i)_{i \in I}$ in a model of $T$, if every subset of $\{\varphi(x, \bar{a}_i) \mid i \in I\}$ of at most $k$ formulas is satisfiable, so is the whole set.

But then, for every $N \models T$ and any finite $s \subseteq N$, $(N, s)$ is a model of

$$\forall \bar{z}((\forall \bar{y}_1 \in P \ldots \forall \bar{y}_k \in P(\delta_\varphi(\bar{y}_1, \bar{z}) \wedge \ldots \wedge \delta_\varphi(\bar{y}_k, \bar{z}) \rightarrow \exists x \bigwedge_{1 \leq i \leq k} \varphi(x, \bar{y}_i))) \\ \rightarrow \exists x \forall \bar{y} \in P(\delta_\varphi(\bar{y}, \bar{z}) \rightarrow \varphi(x, \bar{y})),$$

and hence, $(M, r)$ is a model of this sentence. For $\bar{z} = \bar{c}_\varphi$, the hypothesis of this sentence is satisfied in $(M, r)$ (since $p$ is a type), and hence, $\chi_\varphi(x, \bar{c}_\varphi)$ is satisfiable in $(M, r, \bar{c}_\varphi)$, contrary to our assumption. □

By 3.1 and 3.3 we get

**Corollary 3.4** *If $T$ does not have the finite cover property, then $T$ has the pseudo-finite homogenity property.*

**Theorem 3.5** *If $T$ has the finite cover proerty, then* S($T$) *does not hold.*

*Proof.* If $T$ is unstable then S($T$) fails by 3.2. So assume that $T$ is stable and has the fcp($T$). Then, (cf. [8]) there is a formula $\varphi(x, y, \bar{z})$ such that

> for every model $M$ of $T$ and every $\bar{a} \in M$, $\varphi(\cdot, \cdot, \bar{a})$ is an equivalence relation and for every natural number $n$ there is $\bar{a}_n \in M$ such that $\varphi(\cdot, \cdot, \bar{a}_n)$ has $\geq n$ but only finitely many equivalence classes.

Fix a model $N$ of T and $\bar{a}_n \in N$ according to the fcp. Let $s_n$ be a complete set of representatives of $\varphi(\cdot, \cdot, \bar{a}_n)$. Let $\lambda$ be an infinite cardinal. We show that S($T, \lambda$) fails. By compactness, there is a $\lambda$-saturated model $(M, r)$ and $\bar{a} \in M$ such that $r$ is infinite, pseudo-finite, and a complete set of representatives of $\varphi(\cdot, \cdot, \bar{a})$. So $t := r \cup \{\bar{a}\}$ is pseudo-finite, but the type

$$\{\neg\varphi(x, b, \bar{a}) \mid b \in r\}$$

is not realized in $(M, (e)_{e \in t})$. □

By 3.3 and 3.5:

**Corollary 3.6** *$T$ has the pseudo-finite saturation property iff $T$ fails to have the finite cover property.*

**Corollary 3.7** S($T$) *is equivalent to $S(T, \omega_1)$.*

We show that there are theories $T$ with $S(T, \omega_1)$ but without $S(T, \omega)$.

**Example 3.8** Let $T$ be the $L := \{E_n \mid n \geq 0\}$-theory stating that every $E_n$ is an equivalence relation, $E_0 \supseteq E_1 \supseteq \ldots$, $E_0$ has exactly one equivalence class, and, for every $n \geq 1$, every $E_n$-class splits into two $E_{n+1}$-classes.

Clearly, $T$ is complete and does not have the finite cover property, hence, $\mathrm{S}(T)$ holds. We show that $S(T, \omega)$ fails. For this purpose, let $\rho := \{P\}$ with unary $P$ and let $\varphi_n$ state that every $E_n$-class contains elements of $P$ and elements of the complement of $P$. Then, $T_1 := T \cup \{\varphi_n \mid n \in \omega\}$ is complete in $L(\rho)$ and every finite subset of $T_1$ has a model $(N, s)$, where $N$ is a model of $T$ and $s$ is finite. Take a countable $\omega$-saturated model $(M, r)$ of $T_1$. Then, $r$ is pseudo-finite in $M$, and it is not hard to see that there are $2^{\aleph_0}$-many types in $S_1(r)$; so, some of them are not realized in $(M, (a)_{a \in r})$.

We find that the pseudo-finite saturation property is a nice, conceptually clear reformulation of the non fcp. It also is a manageable one: We demonstrate this by reproving the result of [2] showing that a theory allows the elimination of all the Ramsey quantifiers in the $\omega$-interpretation in case it does not have the fcp.

**Proposition 3.9** *If* $\mathrm{S}(T)$ *holds then the Ramsey quantifiers are eliminable.*

*Proof.* Denote by $Q^n$ the $n$-th Ramsey quantifier, i.e.,

$$M \models Q^n x_1 \ldots x_n \varphi(\bar{x}, \bar{b}) \quad \text{iff} \quad \text{there is an infinite set } A \text{ homogeneous for } \varphi(\bar{x}, \bar{b})$$

($A$ is *homogeneous* for $\varphi(\bar{x}, \bar{b})$, if for all $\bar{a} = a_1 \ldots a_n \in A$, $M \models \varphi(\bar{a}, \bar{b})$).

Let $H^n_k$ be the corresponding $k$-th first-order definable approximation of $Q^n$, i.e.,

$$M \models H^n_k x_1 \ldots x_n \varphi(\bar{x}, \bar{b}) \quad \text{iff} \quad \text{there is a set } A \text{ of cardinality at least } k \text{ homogeneous for } \varphi(\bar{x}, \bar{b}).$$

By a standard compactness argument (cf. [2]), one can prove that the Ramsey quantifiers are eliminable in $T$ just in case

for every first-order formula $\varphi(\bar{x}, \bar{y})$ there is a natural number $k$ such that

$$T \models \forall \bar{y} (H^n_k \bar{x} \varphi(\bar{x}, \bar{y}) \rightarrow Q^n \bar{x} \varphi(\bar{x}, \bar{y})).$$

Now assume that $T$ does not allow the elimination of Ramsey quantifiers: By completeness of $T$, for some $\varphi(\bar{x}, \bar{y})$ there is a model $N$ of $T$ such that for every $k$ there are $\bar{c}_k \in N$ and a finite set $s_k$ homogeneous for $\varphi(\bar{x}, \bar{c}_k)$ with at least $k$ elements such that for all $b \in N \setminus s_k$, the set $s_k \cup \{b\}$ is not homogeneous for $\varphi(\bar{x}, \bar{c}_k)$. By compactness, there is an $\omega_1$-saturated $(M, r)$ and $\bar{c} \in M$ such that $r$ is infinite, pseudo-finite in $M$, and a maximal set homogeneous for $\varphi(\bar{x}, \bar{c})$. Therefore, $t := r \cup \{\bar{c}\}$ is pseudo-finite, $(M, t)$ is $\omega_1$-saturated, and the type $p \in S_1(t)$,

$$p \quad := \quad \{\neg x = a \mid a \in r\} \cup \{\varphi(\bar{y}, \bar{c}) \mid \bar{y} \in \{x\} \cup \{a \mid a \in r\}\}$$

is not realized in $(M, (a)_{a \in t})$. Thus, $\mathrm{S}(T)$ does not hold, a contradiction. □

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