On a question of Herzog and Rothmaler

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1 Introduction

Herzog and Rothmaler gave the following purely topological characterization of stable theories. (See the exercises 11.3.4 – 11.3.7 in [2]).

A complete theory T is stable iff for any model M and any extension $M \subset B$ the restriction map $\mathrm{S}(B) \to \mathrm{S}(M)$ *has a continuous section.*

In fact, if T is stable, taking the unique non-forking extension defines a continuous section of $\mathrm{S}(B) \to \mathrm{S}(A)$ for all subsets A of B, provided A is algebraically closed in T^{eq}. Herzog and Rothmaler asked, if, for stable T, there is a continuous section for *any* subset A of B. Or, equivalently, if for any A, $\mathrm{S}(\mathrm{acl}^{\mathrm{eq}}(A)) \to \mathrm{S}(A)$ has a continuous section .

This is an interesting problem, also for unstable T. Is it true that for any T and any set of parameters A the restriction map $\mathrm{S}(\mathrm{acl}(A)) \to \mathrm{S}(A)$ has a continuous section? We answer the question by the following two theorems.

Theorem 1 *Let A be a subset of a model of T. Assume that the Boolean algebra of* $\mathrm{acl}(A)$*–definable formulas is generated by*

- *some countable set of formulas,*
- *all A–definable formulas,*
- *all formulas which are atomic over* $\mathrm{acl}(A)$.

Then $\mathrm{S}(\mathrm{acl}(A)) \to \mathrm{S}(A)$ *has a continuous section.*

The conditions of the theorems are satisfied if, for example, L and A are countable, or, if there are only countably many non–isolated types over $\mathrm{acl}(A)$.

Theorem 2 *There is a theory of Morley rank* 2 *and Morley degree* 1 *such that* $\mathrm{S}(\mathrm{acl}(\emptyset)) \to \mathrm{S}(\emptyset)$ *has no continuous section.*

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2 Proof of Theorem 1

Theorem 1 follows immediately from the next lemma. (Note that the map $\mathrm{S}(\mathrm{acl}(A)) \to \mathrm{S}(A)$ is always open).

Lemma 3 *Let A be a subalgebra of the Boolean algebra B such that the projection of Stone spaces $\mathrm{S}(B) \to \mathrm{S}(A)$ is open. Assume that B can be generated by*

- *some countable subalgebra C of B,*
- *the elements of A,*
- *all atoms of B.*

Then the projection has a continuous section $\sigma : \mathrm{S}(A) \to \mathrm{S}(B)$

Without atoms as generators, the lemma is well–known. See for example Proposition 2.9 in S. Koppelberg's chapter on projective Boolean algebras in [1].

Proof: That the projection $\mathrm{S}(B) \to \mathrm{S}(A)$ is open means that for each b there is a smallest element $\pi(a)$ of A which contains b.[1]

We first define a homomorphism $h : C \to A$ which satisfies

$$h(c) \subset \pi(c) \tag{1}$$

for each $c \in C$. Since C is countable, it is enough to show that we can extend any h defined on a finite subalgebra D to any finite extension D'. Let $d_1, \ldots, d_n$ be the atoms of D. We can assume that D' just splits d_1 into two new atoms d' and d''. Since $h(d_1) \subset \pi(d_1) = \pi(d') \cup \pi(d'')$, we can extend h to D' by setting $h(d') = \pi(d') \cap h(d_1)$ and $h(d'') = h(d_1) \setminus h(d')$.

The condition (1) is equivalent to

$$c \subset a \Rightarrow h(c) \subset a \tag{2}$$

for all $c \in C$ and $a \in A$. Let C' be the subalgebra generated by C and A. (2) implies that we can extend h (uniquely) to a homomorphism $h : C' \to A$ which is the identity on A.

We have constructed so far a continuous section $\tau : \mathrm{S}(A) \to \mathrm{S}(C')$. We claim that any $p \in \mathrm{S}(C')$ which has two different extensions to $\mathrm{S}(B)$ is projected to an isolated point of $\mathrm{S}(A)$. Indeed, let q_1 and q_2 be two extension of p. Since B is generated over C' by atoms, q_1 and q_2 can be separated by an atom of B, so one, say q_1, is isolated. Since the projection from $\mathrm{S}(B) \to \mathrm{S}(A)$ is open, q_1 is projected to an isolated point of $\mathrm{S}(A)$.

Now we can apply the next lemma, which shows that any section σ' : $\mathrm{S}(C') \to \mathrm{S}(B)$ gives a continuous section $\sigma = \sigma' \circ \tau : \mathrm{S}(A) \to \mathrm{S}(B)$. QED.

[1] A is then called a *relatively complete* subalgebra of B.

Lemma 4 *Let $\tau : X \to Y$ and $\pi : Z \to Y$ be continuous maps, π surjective and closed. Assume that for every non-isolated element $x \in X$, $\tau(x)$ has exactly one preimage in Z. Then for every section of σ' of π the composition $\sigma' \circ \tau : X \to Z$ is continuous.*

Proof: Let $\sigma' : Y \to Z$ be any section of π and $\sigma = \sigma' \circ \tau$. Consider a closed subset C of Z. Since π is closed, $\pi(C)$ is closed and therefore also $\tau^{-1}(\pi(C))$. $\sigma^{-1}(C)$ is a subset of $\tau^{-1}(\pi(C))$. The difference consists of isolated points, whence also $\sigma^{-1}(C)$ is closed. QED.

3 Proof of Theorem 2

We consider the language

$$L = \{p_i \mid i < \omega\} \cup \{P_\alpha \mid \alpha < 2^\omega\} \cup \{E_\alpha \mid \alpha < 2^\omega\}$$

with unary predicates p_i and P_α and binary relations E_α. Fix a family $(X_\alpha)_{\alpha<2^\omega}$ of infinite subsets of ω which are pairwise almost disjoint.

Consider the (incomplete) theory T^* which says that

a) The p_i are pairwise disjoint and each of them has exactly two elements

b) p_i is a subset of P_α if $i \in X_\alpha$ and otherwise disjoint from P_α.

c) The intersection of two different P_α and P_β is the union of the (finitely many) p_i with $i \in X_\alpha \cap X_\beta$.

d) E_α is an equivalence relation on P_α, where it has two classes and cuts every p_i contained in P_α in two pieces.

T^* is incomplete since it does not tell us how the E_α interact on the intersections of the P_α. But it is already clear that each completion has Morley rank 2 and degree 1. This is because the (parametrically) definable sets are, up to finite number of elements, Boolean combinations of the E_α–classes, which are infinite and almost disjoint.

To describe a completion T we construct a model M of T^*. We have not much choice for the p_i and the P_α: We let ω be the universe of M and define $p_i(M) = \{2i, 2i+1\}$ and

$$P_\alpha(M) = \bigcup_{i \in X_\alpha} p_i(M).$$

Fix a list $(f_\alpha)_{\alpha<2^\omega}$ of all choice functions which pick an element from each p_i:

$$f_\alpha(i) \in p_i(M).$$

We complete the construction of M by choosing, for each α, the two classes of E_α in such a way that Axiom d) is satisfied and both classes contain infinitely many elements of

$$\{f_\alpha(i) \mid i \in X_\alpha\}.$$

It is clear from the construction that M has an automorphism which swaps the elements of each p_i. This shows that the only $\emptyset$–definable sets are Boolean combinations of the p_i and the P_α. Also it is clear that $M = \mathrm{acl}(\emptyset)$.

Now we show that there is no continuous section $\sigma : \mathrm{S}(\emptyset) \to \mathrm{S}(M)$. We need some notation.

Let $p_i^* \in \mathrm{S}(\emptyset)$ denote the complete type axiomatized by $p_i(x)$, $P_\alpha^* \in \mathrm{S}(\emptyset)$ the unique non–algebraic type containing $P_\alpha(x)$.

For each element a of M let q_a be the type of a over M. Let C_α^1 and C_α^2 be the two classes of E_α in P_α. These are two strongly minimal sets and we denote by e_α^1 and e_α^2 the corresponding strongly minimal types in $\mathrm{S}(M)$.

Now suppose that σ is a continuous section. Then each $\sigma(p_i)$ is q_a for some $a \in p_i(M)$. So there is an $\alpha < 2^\omega$ such that

$$\sigma(p_i) = q_{f_\alpha(i)}$$

for all i.

Let, for $j = 1, 2$, I^j be the set of all $i \in X_\alpha$ with $f_\alpha(i)$ in C_α^j. I^1 and I^2 form a partition of X_α and both sets are infinite by construction.

The sequence $(p_i^*)_{i \in I^1}$ converges (in $\mathrm{S}(\emptyset)$) to P_α^*, the sequence

$$(\sigma(p_i^*))_{i \in I^1} = (q_{f_\alpha(i)})_{i \in I^1}$$

converges in $\mathrm{S}(M)$ towards e_α^1. Whence $\sigma(P_\alpha^*) = e_\alpha^1$. In the same way it follows that $\sigma(P_\alpha^*) = e_\alpha^2$, which is a contradiction.

References

[1] J. Donald Monk and Robert Bonnet, editors. *Handbook of Boolean Algebras*, volume 3. North–Holland, 1989.

[2] Philipp Rothmaler. *Introduction to Model Theory.* Gordon & Breach Science Publishers, 2000.

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