Superindecomposable modules over valuation domains*

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In [4] I stated without detailed proof that modules over valuation domains had bounded width, which in turn implies that there are no superindecomposables modules. In recent work Fuchs reported on the construction of superindecomposables over valuation domains.[1] Indeed, it is not true that modules over valuation domains have always bounded width.

Theorem 1 *Let R be a valuation domain whose value (semi) group embeds a dense linear order. Then there is an R-module with unbounded width.*

Corollary 2 *If furthermore R is countable there is an superindecomposable R-module.*

The purpose of this note is to give a proof above theorem.

Let R be a valuation domain. An easy to prove version of the Elementarteilersatz shows that every primitive positive formula $\phi(x)$ is equivalent to a conjunction of formulas

$$\mathrm{E}(r,s)(x) = (sr \text{ divides } sx),$$

where s and r are in R. Let Γ be the valuation semigroup of R and $v(r) = \rho$, $v(s) = \sigma$ be the values of r and s. Clearly the formula $\mathrm{E}(r,s)$ depends only on ρ and σ. Thus we write $\mathrm{E}(\rho,\sigma)$ instead.

We order the set of pp-formulas by

$$\phi \leq \psi \iff \phi(M) \subset \psi(M) \text{ for all } R\text{-modules } M.$$

Clearly $\mathrm{E}(\rho,\infty)(M) = E(0,\sigma)(M) = M$ for all M. Thus we identify all formulas $\mathrm{E}(\rho,\infty)$ and $\mathrm{E}(0,\sigma)$ and call them $\mathrm{E}(\infty)$.

We give $\Gamma \times \Gamma \cup \{\infty\}$ the following partial order with largest element ∞.

$$(\rho,\sigma) \leq (\rho',\sigma') \iff \rho' \leq \rho \text{ and } \sigma \leq \sigma'.$$

Clearly $p \leq q$ implies $\mathrm{E}(p) \leq \mathrm{E}(q)$.

*TEXed version: June 2005

[1] See [2] , [3] and [1, XIII Theorem (5.11)] (Added January 2013)

Lemma 3

$$\mathrm{E}(p_1) \wedge \ldots \mathrm{E}(p_n) \leq \mathrm{E}(q) \;\Leftrightarrow\; p_i \leq q \textit{ for some } i.$$

This clarifies the structure of the partial order of pp-Formulas.

PROOF: The $\Rightarrow$ is clear. For $\Leftarrow$ assume that for $p_i \leq q_i$ for no i. Write $p_i = (\rho_i, \sigma_i)$ and $q = (\rho, \sigma)$. Then there are $\gamma < \rho$ and $\delta > \sigma$ such that for all i

$$\rho_i \leq \gamma \text{ or } \delta \leq \sigma_i.$$

Choose r in R with value γ. Let I be the ideal of all elements of at least value $\gamma + \delta$. Set $M = R/I$. Then $r + I$ belongs to all $\mathrm{E}(p_i)(M)$ but not to $\mathrm{E}(q)(M)$. This proves the lemma.

To prove the theorem we fix a dense linear order Q without last and first element in Γ. Let Φ be the set of all finite conjunctions of formulas $\mathrm{E}(p)$ where the p belongs to $Q \times Q$. Φ is a subset of the lattice of all pp-formulas, closed under infimum (and in fact is also closed under supremum). We show that Φ does not have any proper linearly ordered interval. This proves that the lattices of all pp-formulas is of unbounded width: a pair ϕ/ψ ($\phi, \psi \in \Phi$) of minimal width would be linearly ordered in Φ.

Thus assume that $\psi = \mathrm{E}(p_1) \wedge \mathrm{E}(p_2) \wedge \ldots$ and $\phi = \mathrm{E}(q_1) \wedge \mathrm{E}(q_2) \wedge \ldots$ belong to Φ and that $\psi < \phi$. Since $\phi \not\leq \psi$, we can assume that $q_i \not\leq p_1$ for all i. Write $q_i = (\rho_i, \sigma_i)$ and $p_1 = (\rho, \sigma)$.

Since for all i

$$\rho \not\leq \rho_i \text{ or } \sigma_i \not\leq \sigma,$$

we find γ and δ in Q such that $\gamma < \rho$, $\sigma < \delta$ and $\rho_i < \gamma$ or $\delta < \sigma_i$ for all i.

Set $s = (\gamma, \sigma)$ and $t = (\rho, \delta)$. We have: $s \not\leq t$, $t \not\leq s$, $p_1 \leq s$, $p_1 \leq t$ and for all i $q_i \not\leq s$ and $q_i \not\leq t$. This shows that $\phi \wedge \mathrm{E}(s)$ and $\phi \wedge \mathrm{E}(t)$ lie between ϕ and ψ but are nor comparable.

References

[1] L. Fuchs and L. Salce. *Modules over Non-Noetherian Domains*, volume 84 of *Mathematical Surveys and Monographs*. American Mathematical Society, 2001.

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[3] L. Salce. Valuation domains with superdecomposable pure injective modules. In *Abelian Groups*, volume 146 of *Lecture Notes in Pure Applied Math.*, pages 241–245. Marcel Decker, 1993.

[4] Martin Ziegler. Model theory of modules. *Ann. Pure Appl. Logic*, 26:149–213, 1984.