A Remark on Sums of Units

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Dedicated to Rüdiger Göbel on the occasion of his 60th birthday.

Abstract

For every $n \geq 2$ we construct a factorial domain R for which n is minimal with the property that every element can be written as the sum of at most n units.

For any ring R let $\mathrm{u}(R)$ be the smallest number n such that every element can be written as the sum of at most n units. If no such n exists, set $\mathrm{u}(R) = \infty$.

Peter Vamos computed in [Va] $\mathrm{u}(R)$ for various rings and found examples with values $1, 2, 3, \infty$. We will show that all finite values occur for factorial domains. For a slightly different definition of unit sum number see [GPS].

Everything will follow from the following proposition.

Proposition 1 *Let R be an integral domain, a a non–zero element of R and n a natural number ≥ 2. Then R is contained in a domain R' with the following properties*

1. *a is the sum of n units in R'.*
2. *If an element of R is the sum of $k < n$ units in R', it is the sum of k units in R.*

We call a domain with property 2. an n–***extension*** of R.

Proof: Consider the polynomial ring $P = R[x_1, \ldots, x_{n-1}]$. Let S be the multiplicative monoid generated by $x_1, \ldots, x_{n-1}$ and $w = -x_1 - \cdots - x_{n-1} + a$. R' will be the quotient ring P_S. Clearly, a is a sum of n units in R':

$$a = x_1 + \cdots + x_{n-1} + w.$$

Now assume that $r \in R$ is a sum of $k < n$ units in R'. The units in R' have the form ust^{-1} for an R–unit u and elements s, t of S (because of the special form

Mathematics Subject Classification(2000) 13B30, 16U60 of the generators of S). Hence for R–units $u_1, \ldots, u_k$ and elements $s_0, \ldots, s_k$ of S we have

$$rs_0 - u_1 s_1 - \cdots - u_k s_k = 0.$$

We write $s_i = \mu_i w^{m_i}$ for monomials μ_i. If we denote by f the polynomial

$$f(x_1, \ldots, x_{n-1}, x_n) = r\mu_0 x_n^{m_0} - u_1\mu_1 x_n^{m_1} - \cdots - u_k \mu_k x_n^{m_k}$$

from $R[x_0, \ldots, x_{n-1}, x_n]$, we have $f(x_1, \ldots, x_{n-1}, w) = 0$ and it follows that f is a multiple of $x_n - w = x_1 + \ldots + x_n - a$. On the other hand, f contains at most n monomials. Hence, by the next lemma, $f = 0$ and we have

$$f(1, \ldots, 1) = r - u_1 - \cdots - u_k = 0.$$

Thus r is the sum of k units in R.

Lemma 2 *Let R be an integral domain and a a non–zero element of R. If the polynomial $f \in R[x_1, \ldots, x_n]$ is non–zero and divisible by $x_1 + \cdots + x_n - a$, it contains more than n monomials.*

PROOF: Write $f = g \cdot (x_1 + \cdots + x_n - a)$. Let m be the total degree of g. For each i let μ_i a monomial from g which has total degree m and maximal degree in x_i. Then the monomials $\mu_1 x_1, \ldots, \mu_n x_m$ all occur in f. On the other hand all monomials of g which have minimal total degree survive (multiplied by a) in f. If follows that f contains at least $n + 1$ monomials.

Theorem 3 *Each integral domain R has an n–extension R' with $\mathrm{u}(R') \leq n$.*

Proof: Choose a well ordering (a_α) of the elements of R. We construct an ascending chain of domains R_α starting with $R_0 = R$. We choose $R_{\alpha+1}$ by the proposition as an n–extension of R_α where a_α becomes a sum of n units. For limit ordinals we take R_λ to be the union of the earlier R_α. The union of this chain is an n-extension of R in which every element of R is a sum of n units. If we iterate this process countably many times and take the union of the resulting chain of extensions we find the desired R'.

Corollary 4 *For each $n \geq 2$ there is a factorial domain R with $\mathrm{u}(R) = n$*

PROOF: If we apply the theorem to the ring of integers to obtain an n–extension R with $\mathrm{u}(R) \leq n$. Since the integer n is not the sum of fewer than n units, we have $\mathrm{u}(R) = n$. If we rewind the proof, we see that R is a quotient ring of the polynomial ring over the integers with infinitely many variables, hence factorial.

References

[Va] P.Vamos, 2–good rings, talk at the Festkolloquium and Conference on Algebra, Model Theory and Theoretical Physics celebrating Rüdiger Göbel's 60th Birthday in Essen, February 2001.

[GPS] B.Goldsmith, S.Pabst, A.Scott: Unit Sum Numbers of Rings and Modules, in: Quart. Journ. Math. Oxford (2), 49 (1998), 331-344.